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SHORT COMMUNICATIONS

On the Uniform Approximation of the Partial Sum of the Dirichlet Series by a Shorter Sum

J. Bourgain[1*] and B. S. Kashin[2]**

[1] *Institute for Advanced Study, School of Mathematics, Princeton*

[2] *Steklov Mathematical Institute, Russian Academy of Sciences*

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1. INTRODUCTION AND THE STATEMENT OF THE MAIN RESULT

In this paper, we prove a theorem whose formulation and method of proof can be attributed to the group of results on estimates of widths obtained by the use of functional analysis techniques and probability theory. The main reason for the consideration of the theorem is the analogy with statements having important applications in number theory (see, for example, [1, Chap. 3]). In addition, note that, a few years ago, Karatsuba expressed his interest in general results concerning the uniform approximation of the partial sum of the Dirichlet series by a shorter sum.

Below we use the usual notation: $\mathbb{R}$, $\mathbb{Z}$ are the sets of real numbers and integers, respectively, $\mathbb{C}^N$, $N = 1, 2, \dots$, is N-dimensional complex space. For $x = \{x_i\}_{i=1}^N \in \mathbb{C}^N$ and $1 \le p < \infty$,

$$\|x\|_{L_p^N} = \left(\frac{1}{N}\sum_{i=1}^N |x_i|^p\right)^{1/p}, \qquad \|x\|_{L_\infty^N} = \max_{1\le i\le N} |x_i|.$$

Let $\operatorname{dist}_E(f, L)$ denote the distance from an element f to a set L in a normed space E and $|\Lambda|$ is the number of elements in a finite set Λ. Finally, $\operatorname{span}(\{\varphi_i\}_{i\in\Lambda})$ is the linear hull of the system of elements $\{\varphi_i\}_{i\in\Lambda}$ in a linear space.

Theorem. *Suppose that $T \in \mathbb{R}$, $T \ge 1$, $N \in \mathbb{Z}$, $N \ge 1$, $T > (\log N)^{10}$, and $\rho \in (0, 1)$. There exists a subset of integers*

$$\Lambda \subset \mathbb{Z} \cap \left[0, N\left(1 + \frac{(\log N)^7}{T}\right)\right]$$

such that $|\Lambda| \le \rho \min\{N, T\}$ and, for any polynomial $f(t)$ of the form

$$f(t) = \sum_{n=0}^N a_n n^{it}$$

with

$$\sum_{n=0}^N |a_n|^2 \le 1, \qquad \frac{1}{2T}\int_{-T}^T |f(t)|^2\, dt \le 1,$$

*E-mail: bourgain@math.ias.edu
**E-mail: kashin@mi.ras.ru

there exists a polynomial

$$g(t) = \sum_{n\in\Lambda} b_n n^{it}$$

with

$$\sup_{|t|\le T/2} |g(t) - f(t)| \le C[\log(N+T)]^{18}\rho^{-1/2}$$

(*here and elsewhere, C is an absolute constant*).

2. ON THE PROOF OF THE THEOREM

More than thirty years ago, it was established [2] in the theory of widths that certain k-dimensional subspaces $\mathbb{R}^N$ give a very good approximation to the Euclidean ball in $\mathbb{R}^N$ in the uniform metric even for N much greater than k. In connection with the problem of trigonometric widths introduced in [3], it is natural to investigate the existence of such examples among "coordinate" subspaces, i.e., in our case, of subspaces generated by a collection of elements of a discrete trigonometric system. The first result concerning the existence of "coordinate" trigonometric subspaces in $\mathbb{R}^N$ of dimension $k \le \gamma N$, where $\gamma < 1$, is an absolute constant and $N = 2, 3, \dots$, giving a good approximation to the Euclidean ball in the uniform metric was obtained by Bourgain and improved by Talagrand [4]. The order of the approximation furnished by such subspaces is worse only by a logarithmic multiplier as compared to subspaces of general form. Recently, Guédon, Mendelson, Pajor, and Tomczak-Jaegermann [5] obtained a general result on the existence of "coordinate" subspaces of small dimension with good approximation properties generated by elements of an arbitrary uniformly bounded orthonormal basis in L_2^N. The scheme of proof from [5] is used to prove the following statement.

Proposition. *Suppose we are given the collection of elements $\{\varphi_i\}_{i=1}^n \subset L_2^N$ with $\|\varphi_i\|_{L_\infty^N} \le K$, $i = 1, 2, \dots, n$. For any m, $1 \le m \le n$, there exists a subset $\Lambda \subset \{1, \dots, n\}$ such that $|\Lambda| = m$ and*

$$\sup_{\{a_i\}_{i=1}^n,\, \sum |a_i|^2 \le 1} \operatorname{dist}_{L_\infty^N}\left(\sum_{i=1}^n a_i\varphi_i, \operatorname{span}(\varphi_i, i\in\Lambda)\right) \le C\cdot K(\log N)^{7/2}\sqrt{\frac{n}{m}}.$$

Note that, in this proposition, whose proof also uses Theorem 6 from [6], the orthogonality of the system $\{\varphi_i\}$ is not required.

In addition to the methods associated with the geometry of Banach spaces, we also use techniques of harmonic analysis in the proof of the theorem.

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