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DIAMETERS OF SOME FINITE-DIMENSIONAL SETS AND CLASSES OF SMOOTH FUNCTIONS

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Abstract. Estimates of the diameters of certain sets in the Banach spaces $L^q(0, 1)$ and l_q^n are given; in particular, the orders of the diameters $d_n(W_p^r, L^q)$, $p < q$, $r > 1$, are completely determined.

Bibliography: 18 titles.

Introduction and formulation of the basic theorems[1]

Let X be a Banach space and K a compact centrally symmetric subset of X. The quantity

$$d_n(K, X) = \inf \sup_{x \in K} \inf_{y \in L_n} \| x - y \|,$$

where the inf runs over all subspaces L_n of X having dimension $\leqslant n$, is called the *Kolmogorov n-diameter* of the set K in X.

Furthermore, l_p^n denotes the space R^n, equipped with the norm

$$\|x\|_{l_p^n} = \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} & \text{for } 1 \leqslant p < \infty, \\ \max_{1 \leqslant i \leqslant n} |x_i| & \text{for } p = \infty. \end{cases}$$

By B_p^n we denote the unit ball in l_p^n, and by W_p^r ($r \geqslant 1$, $1 \leqslant p \leqslant \infty$) we denote the well-known class of r-smooth functions defined on the segment $[0, 1]$ (when r is an integer, it consists of the functions whose derivatives of order $r - 1$ are absolutely continuous and for which

$$\|f(x)\|_{L^p} + \|f^{(r)}(x)\|_{L^p} \leqslant 1;$$

for the definition of the class W_p^r when r is not an integer, see [18]).

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(1) Some of the theorems in this paper were announced previously in [7].

THEOREM 1. *Let* $1 \leqslant n < m < \infty$. *Then*(2)

$$d_n(B_2^m, l_\infty^m) \leqslant \frac{C}{\sqrt{n}} \cdot \left(1 + \ln \frac{m}{n}\right)^{3/2}.$$

THEOREM 2. *Let* $1 \leqslant p < q \leqslant \infty$, $q > 2$ *and* $rp > 1$. *Then*

$$d_n(W_p^r, L^q(0, 1)) \asymp \begin{cases} n^{-r}, & \text{if } p > 2, \\ n^{-\left(r + \frac{1}{2} - \frac{1}{p}\right)}, & \text{if } p \leqslant 2. \end{cases}$$

THEOREM 3. *Let* $1 \leqslant p < q \leqslant \infty$. *Then*

$$d_n(B_p^{2n}, l_q^{2n}) \asymp \begin{cases} 1, & \text{if } q \leqslant 2, \\ n^{-\frac{1}{2} + \frac{1}{q}}, & \text{if } p \leqslant 2,\ q > 2, \\ n^{-\frac{1}{p} + \frac{1}{q}}, & \text{if } p > 2. \end{cases}$$

The assertion of Theorem 3 for $p = 1$ or $q \leqslant 2$ is not new: it follows at once from the estimates (2) and (3′).

Besides these theorems, which have to do with intersections, we prove the following result:

THEOREM 4. *For any* $n \geqslant 1$ *there exists in the space* R^n *an orthogonal transformation* T *such that*

$$C \cdot \|x\|_{l_2^n} \leqslant \frac{n^{-1/2}}{2} (\|Tx\|_{l_1^n} + \|x\|_{l_1^n}) \leqslant \|x\|_{l_2^n}, \quad x \in R^n.$$

Theorem 2 finishes the solution of the problem of determining the orders of the quantities $d_n(W_p^r, L^q)$ and, in combination with the already known results, implies that for $r > 1$

$$d_n(W_p^r, L^q) \asymp \begin{cases} n^{-r}, & \text{if } p \geqslant q \text{ or } 2 < p < q, \\ n^{-r - \frac{1}{2} + \frac{1}{p}}, & \text{if } p \leqslant 2 < q, \\ n^{-r - \frac{1}{q} + \frac{1}{p}}, & \text{if } p < q \leqslant 2. \end{cases} \tag{1}$$

The first results on the diameters of classes of smooth functions were obtained by Kolmogorov [8] ($p = q = 2$). Stečkin [9] obtained for an estimate of the diameters of W_1^r in L^2 and W_∞^r in L^∞ the equality ($n \leqslant m$)

$$d_n(B_1^m, l_2^m) = \left(1 - \frac{n}{m}\right)^{\frac{1}{2}}. \tag{2}$$

In 1960 Tihomirov calculated the exact values of the diameters $d_n(W_\infty^r, C)$, and then Tihomirov, Babadžanov and Makovoz (see [5] and [10]–[12]) proved the inequalities (1) in the case $p \geqslant q$. For $1 \leqslant p < q \leqslant 2$ the relations (1) were obtained by Ismagilov [13];

(2) The symbols C, C', and B in the following denote various absolute positive constants.

he also observed that the equivalence $d_n(W_p^r, L^q) \asymp n^{-r+1/p-1/q}$ fails for $p = 1, q = \infty$. Before the appearance of the present paper the asymptotic behavior of $d_n(W_p^r, L^q), p < q$, $q > 2$, was known only for $p = 1, r \geqslant 2$ (Gluskin [14]). In [14] it was shown that for an exact estimate of $d_n(W_1^r, C)$ it is sufficient to get a good estimate of the diameter $d_n(B_1^m, l_\infty^m)$. Later, Maïorov [15] carried out this reduction of the problem of determining the order of the quantity $d_n(W_p^r, L^q)$ to the corresponding "finite-dimensional" problem for all p and q $(p < q)$.

The "finite-dimensional" problem of estimating the diameters $d_n(B_p^m, l_q^m), p < q$, also has independent interest. A sufficiently accurate estimate for $d_n(B_p^m, l_q^m)$ was known only for $1 \leqslant p < q \leqslant 2$ and for $1 = p < q \leqslant \infty$. In the first case it follows directly from (2), and in the second case it is a consequence of the following result of the author (see [16]):

$$d_n(B_1^m, l_\infty^m) \leqslant \frac{C_\lambda}{\sqrt{n}}, \quad m^\lambda \leqslant n \leqslant m, \quad \lambda > 0. \tag{3}$$

We mention that for application to an estimate of the diameters $d_n(W_1^r, C)$ for $r \geqslant 2$ it is even sufficient to use the earlier estimate of Ismagilov [13]:

$$d_n(B_1^m, l_\infty^m) \leqslant \frac{C\sqrt{m}}{n}. \tag{3'}$$

For a proof of Theorem 2 we use the following obvious corollary of Theorem 1:

COROLLARY 1. *For $m \geqslant n$ and $1 \leqslant p \leqslant 2$*

$$d_n(B_p^m, l_\infty^m) \leqslant \frac{C}{\sqrt{n}} \cdot \left(1 + \ln \frac{m}{n}\right)^{3/2}. \tag{4}$$

For application of Corollary 1 the power of the factor $(1 + \ln(m/n))$ appearing in (4) is not of importance to us, and we shall not concern ourselves with a determination of the exact value of this power; we mention only (see [17], and also the estimate (3)) that for $p = 1$

$$d_n(B_1^m, l_\infty^m) \leqslant \frac{C}{\sqrt{n}} \cdot \left(1 + \ln \frac{m}{n}\right)^{1/2}.$$

PROOF OF THEOREM 1. It suffices for us to prove Theorem 1 only in the case when

$$\sqrt{n}\left(1 + \ln \frac{m}{n}\right)^{-1} > C, \tag{5}$$

where the constant C is arbitrarily large, since if (5) does not hold, then the theorem follows from the obvious estimate $d_n(B_2^m, l_\infty^m) \leqslant 1$.

Let $A' = \{a_{ij}\}_{i=1,j=1}^{n\ \ m}$ be a matrix with n rows and m columns $(n < m)$. We denote by e_i $(1 \leqslant i \leqslant m)$ the columns of the matrix A'.

An important point in the proof of the theorem is the construction of a matrix A' having the following two properties:

*) Any n columns $e_{i_1}, \ldots, e_{i_n}$ of A' are linearly independent.

**) For any set $e_{i_1}, \ldots, e_{i_{n+1}}$ $(1 \leqslant i_k \leqslant m)$ the coefficients in the expansion

$$e_{i_{n+1}} = \sum_{k=1}^{n} \lambda_k e_{i_k}$$

satisfy the inequality ($\lambda = \{\lambda_1, \ldots, \lambda_n\}$)

$$\frac{\|\lambda\|_{l_2^n}}{\|\lambda\|_{l_1^n}} \cdot \left(1 + \frac{1}{\|\lambda\|_{l_2^n}}\right) \leqslant \frac{C}{\sqrt{n}}\left(1 + \ln\frac{m}{n}\right)^{3/2}. \tag{6}$$

We prove Theorem 1 under the assumption that a matrix A' satisfying $*$) and $**$) has been constructed. For $x \in R^m$ and $1 \leqslant i \leqslant m$ we let $(x)_i$ denote the ith coordinate of the vector x.

We consider the n-dimensional subspace $L \subset R^m$ spanned by the row vectors $\{y_j\}_1^n$ of A', and we show that for any point $z \in B_2^m$ there is an element $y \in L$ such that

$$\|z - y\|_{l_\infty^m} \leqslant \frac{C}{\sqrt{n}}\left(1 + \ln\frac{m}{n}\right)^{3/2}. \tag{7}$$

We make use of the following well-known corollary of Helly's theorem on the intersection of convex sets (for a proof see [2], §1): if $y_1', \ldots, y_n'$ and z are vectors in R^m, $m > n$, then for the distance in the metric of l_∞^m from z to the subspace generated by $y_1', \ldots, y_n'$ to be less than or equal to ρ_0 it is necessary and sufficient that for any set $i_1, \ldots, i_{n+1}$, $1 \leqslant i_k \leqslant m$, there is a linear combination $\Sigma_1^n \beta_r y_r'$ such that

$$\left|\left(z - \sum_{r=1}^{n} \beta_r y_r'\right)_{i_k}\right| \leqslant \rho_0, \quad 1 \leqslant k \leqslant n+1. \tag{8}$$

We now choose an arbitrary set of columns $\{e_{i_k}\}_{k=1}^{n+1}$ of A'. Moreover, let

$$e_{i_{n+1}} = \sum_{k=1}^{n} \lambda_k e_{i_k}, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{9}$$

We choose a nonsingular matrix $\{b_{rj}\}_{r,j=1}^n$ such that

$$\sum_{j=1}^{n} (b_{rj} y_j)_{i_k} = \begin{cases} 0, & r \neq k \\ 1, & r = k \end{cases} \quad (1 \leqslant r,\ k \leqslant n) \tag{10}$$

(this is possible, since, by the property $*$) of the matrix A', $\det\{(y_j)_{i_k}\} \neq 0$). We now define in the space L a new basis $\{y_r'\}_1^n$, setting

$$y_r' = \sum_{j=1}^{n} b_{rj} y_j.$$

By (10),

$$(y_r')_{i_k} = \begin{cases} 0, & r \neq k \\ 1, & r = k \end{cases} \quad (1 \leqslant r,\ k \leqslant n).$$

We determine the values of the quantities $(y_r')_{i_{n+1}}$. Using (9) and (10), we have

$$(y'_r)_{i_{n+1}} = \left(\sum_{j=1}^{n} b_{rj} y_j\right)_{i_{n+1}} = \sum_{j=1}^{n} b_{rj} (y_j)_{i_{n+1}} = \sum_{j=1}^{n} b_{rj} \left(\sum_{k=1}^{n} \lambda_k (y_j)_{i_k}\right)$$
$$= \sum_{k=1}^{n} \lambda_k \left(\sum_{j=1}^{n} b_{rj} y_j\right)_{i_k} = \sum_{k=1}^{n} \lambda_k (y'_r)_{i_k} = \lambda_r.$$

Consequently, in the $m \times n$ matrix $\widetilde{A}$ that determines the basis $\{y'_r\}_1^n$ the following $(n+1) \times n$ matrix is cut out by the columns with the numbers $i_1, \ldots, i_{n+1}$: the first n columns of it form an identity matrix, and the $(n+1)$th column is the column $\lambda = \{\lambda_1, \ldots, \lambda_n\}$.

Let $z \in B_2^m$. For $1 \leqslant r \leqslant n$ we set

$$\beta_r = (z)_{i_r} + \left(-\frac{\sum_{k=1}^{n} \lambda_k (z)_{i_k}}{\|\lambda\|_{l_1^n}} + \frac{(z)_{i_{n+1}}}{\|\lambda\|_{l_1^n}} \right) \operatorname{sign} \lambda_r.$$

We estimate the quantities $|(z - \Sigma_1^n \beta_r y'_r)_{i_k}|$, $1 \leqslant k \leqslant n+1$. Using (9) and (6) for $1 \leqslant k \leqslant n$, we have

$$\left|\left(z - \sum_{r=1}^{n} \beta_r y'_r\right)_{i_k}\right| \leqslant \frac{\left|\sum_{k=1}^{n} \lambda_k (z)_{i_k}\right|}{\|\lambda\|_{l_1^n}} + \frac{|(z)_{i_{n+1}}|}{\|\lambda\|_{l_1^n}} \leqslant \frac{\left(\sum_{k=1} \lambda_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} (z)_{i_k}^2\right)^{\frac{1}{2}}}{\|\lambda\|_{l_1^n}}$$
$$+ \frac{|(z)_{i_{n+1}}|}{\|\lambda\|_{l_1^n}} \leqslant \frac{\|\lambda\|_{l_2^n}}{\|\lambda\|_{l_1^n}} + \frac{1}{\|\lambda\|_{l_1^n}} \leqslant \frac{C}{\sqrt{n}} \left(1 + \ln \frac{m}{n}\right)^{3/2}. \tag{11}$$

For $k = n+1$ it is easy to verify that

$$\left(z - \sum_{r=1}^{n} \beta_r y'_r\right)_{i_{n+1}} = 0. \tag{11'}$$

Since the set of columns $\{e_{i_k}\}_{k=1}^{n+1}$ was chosen arbitrarily, the estimate (7) follows from (11), (11′), and the above corollary of Helly's theorem. Thus, Theorem 1 follows from the existence of a matrix A' satisfying the conditions *) and **).

For the construction of such a matrix A' we shall need several auxiliary statements.

LEMMA 1. *For any integer n and any $\alpha > 0$ it is possible to find a set of vectors $\Omega_n(\alpha) = \{z_i\}_1^k$ with $z_i \in S^n$,*(3) *$1 \leqslant i \leqslant k$, such that $k \leqslant (C \cdot \alpha^{-1})^n$ and for any $y \in S^n$ there is a number i for which*

$$\|y - z_i\|_{l_2^n} \leqslant \alpha.$$

Without regard to the size of the constant C, Lemma 1 is easy to prove directly; to save space we refer to [6], where the question of the size of C is considered.

Suppose that we are given integers q and m $(1 \leqslant q \leqslant m)$ and a number $\alpha > 0$. In

(3) By S^n we denote the unit sphere in l_2^n.

R^m we define a system of vectors $\Omega_m(q, \alpha)$ in the following way.

There exist C_m^q q-dimensional subspaces L of R^m that are defined as follows:

$$x=\{x_i\}_{i=1}^m \in L \leftrightarrow x_{i_1}=x_{i_2}= \dots =x_{i_{m-q}}=0 \quad (1 \leqslant i_1 < \dots < i_{m-q} \leqslant m).$$

On the unit euclidean sphere of each such subspace L we define for a given number α a system of vectors $\Omega_q(\alpha)$ satisfying the condition of Lemma 1. The union of all the vectors of these systems gives the set $\Omega_m(q, \alpha)$. It is clear that the number of elements in $\Omega_m(q, \alpha)$ is not greater than $C_m^q \cdot (C \cdot \alpha^{-1})^q$.

LEMMA 2. *For any bilinear form* $A(x, y) = \Sigma_{i,j=1}^n a_{ij}x_iy_j$ $(x = \{x_i\}; y = \{y_j\})$ *we have*

$$\sup_{\|x\|_{l_2^n}=\|y\|_{l_2^n}=1} A(x, y) \equiv \|A\| \leqslant 2 \cdot \sup_{x,y\in\Omega_n(1/6)} A(x, y),$$

where the set $\Omega_n(1/6)$ *is determined by the number* $\alpha = 1/6$ *in Lemma* 1 (*therefore* $|\Omega_n(1/6)| \leqslant C^n$).

Indeed, let $\|A\| = A(x_0, y_0), x_0, y_0 \in S^n$. Using the property of the system of vectors $\Omega_n(1/6)$ (see Lemma 1), we find vectors $y \in \Omega_n(1/6)$ such that $\|x - x_0\|_{l_2^n} \leqslant 1/6$ and $\|y - y_0\|_{l_2^n} \leqslant 1/6$. Then

$$\begin{aligned} A(x, y) &= A(x_0+(x-x_0), y_0+(y-y_0)) = A(x_0, y_0)+A(x_0, y-y_0) \\ &+A(x-x_0, y_0)+A(x-x_0, y-y_0) \geqslant A(x_0, y_0) \\ &-\|A\|\cdot\frac{1}{6}-\|A\|\frac{1}{6}-\|A\|\cdot\frac{1}{36} \geqslant \|A\|\cdot\frac{1}{2}. \end{aligned}$$

The lemma is proved.

LEMMA 3 (see [1], p. 217 and [3], Chapter III, §5, Theorem 8). *If* $P(x) = \Sigma_1^l c_k r_k(x)$ *is any polynomial in the Rademacher system, then the following assertions are true*:

1) *There exists an absolute constant* $C_0 > 0$ *such that*

$$\mu\left\{x \in [0, 1]: |P(x)| \geqslant C_0\left(\sum_{k=1}^l c_k^2\right)^{1/2}\right\} \geqslant C_0.$$

2) *For any* $y \geqslant 0$

$$\mu\left\{x \in [0, 1]: |P(x)| \geqslant y\left(\sum_{k=1}^l c_k^2\right)^{1/2}\right\} \leqslant 2 \cdot e^{-\frac{y^2}{2}}.$$

By $|E|$ we denote the number of elements in any finite set E, and by $N(x)$, $x = \{x_i\} \in R^n$, we denote the set of all numbers i, $1 \leqslant i \leqslant n$, such that $x_i \neq 0$.

LEMMA 4. *If* $\{a_i\}_1^n$ *is a set of real numbers,*

$$\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} = v, \qquad \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} |a_i|\right) \leqslant \gamma \cdot v,$$

then for any $t \geqslant 1$ there is a set of integers E_t with $E_t \subset [1, n]$, $|E_t| \leqslant n \cdot (2t\gamma)^2$, such that

$$\sum_{i \in E_t} a_i^2 > t \sum_{i \notin E_t} a_i^2.$$

PROOF. Without loss of generality, we can suppose that

$$\sum_{i=1}^{n} a_i^2 = 1, \qquad n^{-1/2} \cdot \sum_{i=1}^{n} |a_i| \leqslant \gamma.$$

We set

$$E_t = \{i : 1 \leqslant i \leqslant n,\ a_i^2 \geqslant n^{-1} \cdot (2t\gamma)^{-2}\}.$$

Then

$$|E_t| \cdot (2t\gamma)^{-2} n^{-1} \equiv \sum_{i \in E_t} n^{-1} (2t\gamma)^{-2} \leqslant \sum_{i \in E_t} a_i^2 \leqslant 1,$$

and, therefore,

$$|E_t| \leqslant n (2t\gamma)^2.$$

Furthermore,

$$\sum_{i \notin E_t} a_i^2 \leqslant \sum_{i=1}^{n} |a_i| (2t\gamma \sqrt{n})^{-1} \leqslant (2t)^{-1}.$$

Consequently,

$$\sum_{i \in E_t} a_i^2 \geqslant 1 - (2t)^{-1} > \frac{1}{2} \geqslant t \sum_{i \notin E_t} a_i^2.$$

The lemma is proved.

We introduce the following definitions: let $A = \{\epsilon_{ij}\}_{i=1,j=1}^{m\ \ n}$ $(m \geqslant n)$ be a real matrix. We set

$$F(A) = \sup_{\substack{\|x\|_{l_2^m} \leqslant 1,\ |N(x)| \leqslant n \\ \|y\|_{l_2^n} \leqslant 1}} \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_{ij} x_i y_j = \sup_{\|x\|_{l_2^m} \leqslant 1,\ |N(x)| \leqslant n} \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} \varepsilon_{ij} x_i\right)^2\right)^{\frac{1}{2}}. \tag{12}$$

From the definition it is clear that for $m = n$ we have $F(A) = \|A\|$. For $1/n \leqslant \theta \leqslant 1$ we set

$$G(A, \theta) = \inf_{\|x\|_{l_2^m} = 1,\ |N(x)| \leqslant n \cdot \theta} \sum_{j=1}^{n} \left| \sum_{i=1}^{m} x_i \varepsilon_{ij} \right|. \tag{13}$$

LEMMA 5. *For any numbers $\alpha > 0$ and $1/n \leqslant \theta \leqslant 1$ and matrix $A = \{\epsilon_{ij}\}_{i=1,j=1}^{m\ \ n}$,*

$$G(A,\theta)\geqslant \inf_{x\in\Omega_m([n\theta],\alpha)} \sum_{j=1}^{n}\left|\sum_{i=1}^{m} x_i\varepsilon_{ij}\right|-\sqrt{n}F(A)\cdot\alpha.$$

PROOF. Let

$$G(A,\theta)=\sum_{j=1}^{n}\left|\sum_{i=1}^{m} x_i^0\varepsilon_{ij}\right|,\qquad x_0=\{x_i^0\}\in S^m,\qquad |N(x_0)|\leqslant[n\theta].$$

In the set $\Omega_m([n\theta],\alpha)$ we find a vector $x=\{x_i\}_1^m$ such that $\|x-x_0\|_{l_2^m}\leqslant\alpha$ and $|N(x-x_0)|\leqslant[n\theta]\leqslant n$. Then

$$\sum_{j=1}^{n}\left|\sum_{i=1}^{m} x_i\varepsilon_{ij}\right|\leqslant\sum_{j=1}^{n}\left|\sum_{i=1}^{m} x_i^0\varepsilon_{ij}\right|+\sum_{j=1}^{n}\left|\sum_{i=1}^{m}(x_i-x_i^0)\varepsilon_{ij}\right|$$

$$\leqslant G(A,\theta)+\sqrt{n}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}(x_i-x_i^0)\varepsilon_{ij}\right)^2\right)^{1/2}$$

$$\leqslant G(A,\theta)+\sqrt{n}F(A)\cdot\left(\sum_{i=1}^{m}(x_i-x_i^0)^2\right)^{1/2}\leqslant G(A,\theta)+\sqrt{n}F(A)\cdot\alpha.$$

The lemma is proved.

We make use of the following simple estimate for the number of combinations $\mathbf{C}_m^n$:

$$\mathbf{C}_m^n\leqslant C^n\cdot\left(\frac{m}{n}\right)^n. \tag{14}$$

Indeed,

$$\mathbf{C}_m^n=\frac{m\cdots(m-n+1)}{n!}\leqslant\frac{m^n}{n!}.$$

By Stirling's formula, $n!\geqslant n^n\cdot C^{-n}$, which proves (14).

We proceed to the construction of a matrix A' satisfying the conditions *) and **). Here we use probability arguments.

On the set D_{mn} of all $m\times n$ matrices $A=\{\epsilon_{ij}\}_{i=1,j=1}^{m\ \ n}$ with elements equal to ± 1 we introduce a measure that assigns to each matrix A the measure $2^{-m\cdot n}$. Then $\mu D_{mn}=1$.

For $y>0$ let (see (12))

$$f(y)=\mu\{A\in D_{mn}:F(A)\geqslant y\}. \tag{15}$$

By Lemma 2 and the definition of $\Omega_m(n,\alpha)$,

$$f(y)\leqslant\mu\{A\in D_{mn}:\sup_{x\in\Omega_m\left(n,\frac{1}{6}\right),\,y\in\Omega_n\left(\frac{1}{6}\right)} A(x,y)\geqslant 2^{-1}\cdot y\}$$

$$\leqslant\mathbf{C}_m^n\cdot C^n\cdot\sup_{x,y\in S^n}\mu\left\{A\in D_{nn}:\left|\sum_{i,j=1}^{n}\varepsilon_{ij}x_iy_j\right|\geqslant 2^{-1}\cdot y\right\}. \tag{16}$$

We estimate the right-hand side of (16). Since for $x,y\in S^n$

$$\sum_{i=1}^{n} x_i^2=\sum_{j=1}^{n} y_j^2=1,$$

it follows that

$$\sum_{i,j=1}^{n} (x_i y_j)^2=\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{j=1}^{n} y_j^2\right)=1.$$

Let $\{c_k\}_1^{n^2}$ be the numbers $x_i \cdot y_j$ $(1 \leqslant i, j \leqslant n)$, numbered in any order. Then it is easy to see that

$$\mu\left\{A\in D_{nn}:\left|\sum_{i,j=1}^{n} \varepsilon_{ij}x_i y_j\right|\geqslant 2^{-1}y\right\}=\mu\left\{t\in[0,1]:\left|\sum_{k=1}^{n^2} c_k r_k(t)\right|\geqslant 2^{-1}y\right\}. \quad (17)$$

By part 2) in Lemma 3 and the relation $\Sigma_1^{n^2} c_k^2 = 1$, the right-hand side of (17) does not exceed $2 \cdot e^{-y^2/8}$. Finally, for the function $f(y)$ we obtain the estimate (see (16))

$$f(y)\leqslant \mathbf{C}_m^n \cdot C^n \cdot e^{-\frac{1}{8}y^2}. \quad (18)$$

From (18) it follows directly that for $y = C'(n + \ln \mathbf{C}_m^n)^{1/2}$ we have $f(y) \leqslant 1/100$, where C' is a sufficiently large absolute constant.

Since (see (14)) $\ln \mathbf{C}_m^n \leqslant C(n + n \cdot \ln(m/n))$, it follows that

$$f(y)\leqslant \frac{1}{100}, \quad y=B\sqrt{n}\left(1+\ln\frac{m}{n}\right)^{1/2}. \quad (19)$$

Further, for $\alpha > 0$, $1/n \leqslant \theta \leqslant 1$ and $z > 0$ we set

$$g(\theta,\alpha,z)=\mu\left\{A\in D_{mn}:\inf_{x\in\Omega_m([n\theta],\alpha)}\sum_{j=1}^{n}\left|\sum_{i=1}^{m} x_i\varepsilon_{ij}\right|\leqslant zn\right\}. \quad (20)$$

We estimate the magnitude of the function $g(\theta, \alpha, z)$; for this, we first estimate for any fixed $x \in S^m$ the measure

$$\mu\left\{A\in D_{mn}:\sum_{j=1}^{n}\left|\sum_{i=1}^{m} x\,\varepsilon_{ij}\right|\leqslant zn\right\}.$$

By part 1) of Lemma 3 we have for each j $(1 \leqslant j \leqslant n)$

$$\mu\left\{A\in D_{mn}:\left|\sum_{i=1}^{m} x_i\varepsilon_{ij}\right|\geqslant C_0\right\}\geqslant C_0.$$

Consequently, for $z < 1/2$

$$\mu\left\{A\in D_{mn}:\sum_{j:\left|\sum_{i=1}^{m}x_i\varepsilon_{ij}\right|\geqslant C_0}1\leqslant zn\right\}$$
$$\leqslant \mathbf{C}_n^{n-[nz]}\cdot\mu\left\{A\in D_{mn}:\left|\sum_{i=1}^{m}x_i\varepsilon_{ij}\right|\leqslant C_0;\ j=1,2,\ \ldots,\ n-[zn]\right\}\tag{21}$$
$$\leqslant \mathbf{C}_n^{n-[nz]}\cdot(1-C_0)^{n-[nz]}\leqslant \mathbf{C}_n^{[zn]}\cdot(1-C_0)^{\frac{n}{2}}.$$

From (21) it follows that for any $x\in S^m$

$$\mu\left\{A\in D_{mn}:\sum_{j=1}^{n}\left|\sum_{i=1}^{m}x_i\varepsilon_{ij}\right|\leqslant C_0zn\right\}\leqslant \mathbf{C}_n^{[zn]}\cdot(\sqrt{1-C_0})^n.\tag{22}$$

Since $\Omega_m([n\theta],\alpha)$ contains not more than $\mathbf{C}_m^{[n\theta]}(C\cdot\alpha^{-1})^{n\theta}$ elements, (22) implies the following estimate for the function $g(\theta,\alpha,z)$ (see (20)):

$$g(\theta,\alpha,C_0z)\leqslant \mathbf{C}_m^{[n\theta]}\cdot(C\cdot\alpha^{-1})^{n\theta}\cdot\mathbf{C}_n^{[zn]}\cdot(\sqrt{1-C_0})^n.\tag{23}$$

We simplify the right-hand side of (23). Since $\lim_{z\to 0}(Cz^{-1})^z=1$, we find a number z_0 such that for $z\leqslant z_0$

$$\left(\frac{C}{z}\right)^z\leqslant\left(1-\frac{1}{2}C_0\right)^{-\frac{1}{2}},$$

where C is the constant from (14). Then, by (14), for $z\leqslant z_0$ we have

$$\mathbf{C}_n^{[zn]}\leqslant(Cz^{-1})^{nz}\leqslant\left(1-\frac{1}{2}C_0\right)^{-\frac{1}{2}n}.\tag{24}$$

Consequently, for $z\leqslant z_0$ (23) can be written as follows:

$$g(\theta,\alpha,C_0\cdot z)\leqslant \mathbf{C}_m^{[n\theta]}\cdot(C\alpha^{-1})^{n\theta}\cdot(\widetilde{C})^n,\tag{25}$$

where $\widetilde{C}=(1-C_0)^{1/2}\cdot(1-C_0/2)^{-1/2}<1$ is an absolute constant.

We now fix the numbers α and θ, setting

$$\alpha=z_0C_0\left(2B\sqrt{1+\ln\frac{m}{n}}\right)^{-1}\tag{26}$$

(here the constant B is the same as in (19) and z_0 is defined by (20)) and

$$\theta=B'\cdot\left(1+\ln\frac{m}{n}\right)^{-1}.\tag{27}$$

We show that if the absolute constant B' is sufficiently small, then (see (25))

$$g(\theta,\alpha,C_0\cdot z)\leqslant \mathbf{C}_m^{[n\theta]}(C\alpha^{-1})^{n\theta}(\widetilde{C})^n\leqslant\frac{1}{10}\quad(\text{for } z\leqslant z_0 \text{ and } n>n_0).$$

Indeed (see (14), and consider also (5)),

$$C_m^{[n\theta]} \leqslant C^{n\theta}\left(\frac{m}{n\theta}\right)^{n\theta} \leqslant C^{n\theta}\cdot \exp\left\{\ln\frac{m}{n\theta}\cdot n\theta\right\}$$
$$\leqslant C^{n\theta}\cdot \exp\left\{\left(\ln\frac{m}{n}-\ln\theta\right)n\theta\right\} \leqslant C^{B'\cdot n}\cdot e^{B'\cdot n}\cdot e^{(-\theta\ln\theta)n} \leqslant u^n,$$

where $u < (\widetilde{C})^{-1/3}$, if B' in (27) is sufficiently small. Further,

$$\left(\frac{C}{\alpha}\right)^{n\theta} \leqslant (C')^{n\theta}\cdot\left(1+\ln\frac{m}{n}\right)^{\frac{n\theta}{2}} \leqslant (C')^{n\theta}\cdot \exp\left\{\ln\left(\ln\left(\frac{m}{n}\right)+1\right)\cdot B'\left(1+\ln\frac{m}{n}\right)^{-1}\right\} \leqslant u^n,$$

where $u < (\widetilde{C})^{-1/3}$, if B' in (27) is sufficiently small. Consequently, for the numbers α and θ defined by (26) and (27) and for $z \leqslant z_0$ (see (25)) we have

$$g(\theta,\alpha,C_0 z) \leqslant (\widetilde{C})^{-\frac{1}{3}n}(\widetilde{C})^{-\frac{1}{3}n}(\widetilde{C})^n \leqslant (\widetilde{C})^{\frac{1}{3}n} \leqslant \frac{1}{10} \tag{28}$$

for $n \geqslant n_0$. From (19) and (28) it follows that for some constants B, B', $0 < z_0 < 1$, $0 < C_0 < 1$, and $n \geqslant n_0$ there exists a matrix $A = \{\epsilon_{ij}\}_{i=1,j=1}^{m\ \ n} \in D_{mn}$ such that

$$\left.\begin{aligned} &1)\ F(A) \leqslant B\cdot\sqrt{n}\left(1+\ln\frac{m}{n}\right)^{\frac{1}{2}},\\ &2)\ \inf_{x\in\Omega_m([n\theta],\alpha)}\sum_{j=1}^{n}\left|\sum_{i=1}^{m}x_i\varepsilon_{ij}\right| \geqslant C_0\cdot z_0\cdot n,\\ &\theta = B'\left(1+\ln\frac{m}{n}\right)^{-1},\quad \alpha = z_0\cdot C_0\cdot\left(2B\sqrt{1+\ln\frac{m}{n}}\right)^{-1}.\end{aligned}\right\} \tag{29}$$

Applying Lemma 5 for this matrix A and the number θ (see also (13)), we get (see (29))

$$\begin{aligned} G(A,\theta) &\geqslant \inf_{x\in\Omega_m([n\theta],\alpha)}\sum_{j=1}^{n}\left|\sum_{i=1}^{m}x_i\varepsilon_{ij}\right| - \sqrt{n}F(A)\cdot\alpha\\ &\geqslant C_0\cdot z_0\cdot n - \sqrt{n}\cdot\sqrt{n}\cdot B\left(1+\ln\frac{m}{n}\right)^{\frac{1}{2}}\cdot z_0\cdot C_0\left(2B\sqrt{1+\ln\frac{m}{n}}\right)^{-1} \geqslant \frac{C_0 z_0}{2}\cdot n.\end{aligned} \tag{30}$$

It is easy to see (see (29) and (30)) that by a very small change in the elements of the matrix $A = \{\epsilon_{ij}\}$ it is possible to get an $m \times n$ matrix $A' = \{\epsilon'_{ij}\}$ for which the following conditions hold: (31)

1) $|\epsilon'_{ij}| \geqslant 1/2$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.

2) $F(A') \leqslant 2B\sqrt{n}(1+\ln(m/n))^{1/2}$, B an absolute constant; obviously we can assume that $B \geqslant 1$.

3) $G(A',\theta) \geqslant Q\cdot n$, $\theta = B'(1+\ln(m/n))^{-1}$, $Q>0$ an absolute constant; obviously we can take $Q \leqslant 1/2$.

4) If e_i $(1 \leqslant i \leqslant m,\ e_i \in R^n)$ is the ith column vector of A', then any set $\{e_{i_k}\}_{k=1}^n$ of n columns forms a linearly independent system of vectors in R^n.

The matrix A' thus constructed is the desired one, i.e. it satisfies the conditions *) and **).

The condition *) holds by (31), part 4). We prove that the condition **) holds. By

(5) we can assume here that

$$\frac{B'}{2}\cdot\left(1+\ln\frac{m}{n}\right)^{-1}\cdot n>1. \tag{32}$$

We choose an arbitrary set $\{e_{i_k}\}_{k=1}^{n+1}$ of $n+1$ columns of the matrix A'. Since $e_i\in R^n$ $(1\leqslant i\leqslant m)$, there is a linear dependence among these vectors:

$$e_{i_{n+1}}=\sum_{k=1}^{n}\lambda_k e_{i_k} \tag{33}$$

[by (31), part 4), the coefficient of $e_{i_{n+1}}$ in (33) cannot be zero].

Let

$$\left(\sum_{k=1}^{n}\lambda_k^2\right)^{1/2}=v,\qquad \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n}|\lambda_k|\right)=u,\qquad \gamma=\frac{u}{v}.$$

We prove the following estimates:

$$\left.\begin{aligned}&\text{a)}\quad v\geqslant\frac{1}{4B}\left(1+\ln\frac{m}{n}\right)^{-1/2},\\ &\text{b)}\quad \gamma\geqslant\frac{1}{32}\cdot\frac{Q^2}{B^2}\cdot\sqrt{\frac{B'}{2}}\left(1+\ln\frac{m}{n}\right)^{-3/2}\left(\frac{v^2}{1+v^2}\right)^{-1},\end{aligned}\right\} \tag{34}$$

where the constants Q, B, and B' are the same as in (31). From (34) we easily get (6) for the vector $\lambda=\{\lambda_1,\dots,\lambda_n\}$ defined by (33). Indeed, if the inequalities (34) have been proved, then

$$\frac{\sqrt{n}\,\|\lambda\|_{l_2^n}}{\|\lambda\|_{l_1^n}}\left(1+\frac{1}{\|\lambda\|_{l_2^n}}\right)=\frac{1}{\gamma}\left(1+\frac{1}{v}\right)$$

$$\leqslant C\left(1+\ln\frac{m}{n}\right)^{\frac{3}{2}}\cdot\frac{v^2}{1+v^2}+C\left(1+\ln\frac{m}{n}\right)^{\frac{3}{2}}\frac{v}{1+v^2}\leqslant C\left(1+\ln\frac{m}{n}\right)^{\frac{3}{2}}.$$

Since the set $\{e_{i_k}\}_{k=1}^{n+1}$ was chosen arbitrarily, the last estimate implies directly the condition **) for the matrix A'. Thus to conclude the proof of Theorem 1 it suffices to prove (34).

Since (see (31), part 1)) $\|e_i\|_{l_2^n}\geqslant\sqrt{n}/2$ $(1\leqslant i\leqslant m)$, it follows (see (33), (12), and (31), part 2)) that

$$\frac{1}{2}\sqrt{n}\leqslant\|e_{i_{n+1}}\|_{l_2}=\left\|\sum_{k=1}^{n}\lambda_k e_{i_k}\right\|_{l_2}\leqslant F(A')\cdot\left(\sum_{k=1}^{n}\lambda_k^2\right)^{\frac{1}{2}}\leqslant 2B\cdot\sqrt{n}\left(1+\ln\frac{m}{n}\right)^{\frac{1}{2}}\cdot v,$$

i.e.

$$v\geqslant\frac{1}{4B}\left(1+\ln\frac{m}{n}\right)^{-\frac{1}{2}},$$

which proves (34), a).

We now assume that (34), b) does not hold. Then the system of inequalities

$$\begin{cases} 4\gamma^2 t^2 \leqslant \frac{B'}{2}\left(1+\ln\frac{m}{n}\right)^{-1}, \\ \sqrt{t} \geqslant \frac{1}{Q}\cdot 4B\left(1+\ln\frac{m}{n}\right)^{\frac{1}{2}}\left(\frac{v^2}{1+v^2}\right)^{1/2} \end{cases} \tag{35}$$

has a solution $t = t_0$, where

$$\sqrt{t_0} = \frac{1}{Q}\cdot 4B\left(1+\ln\frac{m}{n}\right)^{1/2}\cdot\left(\frac{v^2}{1+v^2}\right)^{1/2}. \tag{35'}$$

Since for $0 < v_1 < v_2$

$$\frac{v_1^2}{1+v_1^2} < \frac{v_2^2}{1+v_2^2},$$

we have (see (34), a), and (31), 2))

$$\left(\frac{v^2}{1+v^2}\right)^{1/2} \geqslant \frac{1}{4B}\cdot\left(1+\ln\frac{m}{n}\right)^{-1/2}\cdot\left(\frac{1}{1+\left(\frac{1}{4B}\right)^2\cdot\left(1+\ln\frac{m}{n}\right)^{-1}}\right)^{1/2}$$
$$\geqslant \frac{1}{4B}\left(1+\ln\frac{m}{n}\right)^{-1/2}\cdot\left(\frac{1}{1+(16)^{-1}}\right)^{1/2}.$$

Using the last inequality and also the inequality (see (31), part 3)) $Q \leqslant 1/2$, we get that

$$\sqrt{t_0} \geqslant \left(\frac{Q}{2}\right)^{-1} \geqslant 1.$$

If we now apply Lemma 4 for this value $t = t_0 \geqslant 1$ to the set of numbers $\{\lambda_k\}_1^n$, then we get that all the numbers $\{i_k\}_1^n$ can be partitioned into two groups E and $\widetilde{E}$ such that:

1) $|E| \leqslant n\cdot 4\gamma^2 t^2$, and consequently (see (35) and (32)) also

$$|E| \leqslant \frac{B'}{2}\left(1+\ln\frac{m}{n}\right)^{-1} n < B'\left(1+\ln\frac{m}{n}\right)^{-1} n - 1; \tag{36}$$

$$2)\ \frac{v^2}{2} \leqslant \sum_{k:i_k\in E}\lambda_k^2 \geqslant t_0\cdot\sum_{k:i_k\in\widetilde{E}}\lambda_k^2 = \left(\frac{4B}{Q}\right)^2\cdot\left(1+\ln\frac{m}{n}\right)\cdot\frac{v^2}{1+v^2}\cdot\sum_{k:i_k\in\widetilde{E}}\lambda_k^2. \tag{37}$$

From (36) and (37) it follows (see also (13) and (31), part 3)) that

$$\left\| e_{i_{n+1}} - \sum_{k:i_k\in E}\lambda_k e_{i_k}\right\|_{l_2^n} \geqslant n^{-\frac{1}{2}}\cdot\left\| e_{i_{n+1}} - \sum_{k:i_k\in E}\lambda_k e_{i_k}\right\|_{l_1^n}$$
$$\geqslant n^{-\frac{1}{2}}\cdot G(A',\theta)\cdot\left(1+\sum_{k:i_k\in E}\lambda_k^2\right)^{\frac{1}{2}} \geqslant Q\sqrt{n}\left(\frac{v^2}{2}+1\right)^{\frac{1}{2}}.$$

Next (see (12), (34) and (31), part 3)), we have

$$\left\|\sum_{k:i_k\in\widetilde{E}}\lambda_k e_{i_k}\right\|_{l_2^n}\leqslant F(A')\cdot\left(\sum_{k:i_k\in\widetilde{E}}\lambda_k^2\right)^{\frac{1}{2}}\leqslant F(A')\cdot\left(\frac{1}{t_0}\sum_{k:i_k\in E}\lambda_k^2\right)^{\frac{1}{2}}$$

$$\leqslant F(A')\cdot\frac{1}{\sqrt{t_0}}\cdot v\leqslant\frac{2B}{4B}\cdot\frac{\left(1+\ln\frac{m}{n}\right)^{\frac{1}{2}}}{\left(1+\ln\frac{m}{n}\right)^{\frac{1}{2}}}\cdot\sqrt{n}\cdot Q\cdot\frac{v}{v}\cdot(1+v^2)^{\frac{1}{2}}$$

$$\leqslant\frac{Q}{2}\sqrt{n}(1+v^2)^{\frac{1}{2}}<Q\sqrt{n}\left(\frac{v^2}{2}+1\right)^{\frac{1}{2}}.$$

Thus we have obtained, beginning from the assumption that (34), b) is false, the result that

$$\left\|e_{i_{n+1}}-\sum_{k:i_k\in E}\lambda_k e_{i_k}\right\|_{l_2^n}\geqslant Q\sqrt{n}\left(\frac{v^2}{2}+1\right)^{\frac{1}{2}}>\left\|\sum_{k:i_k\in\widetilde{E}}\lambda_k e_{i_k}\right\|_{l_2^n},$$

but the last estimate contradicts (33). This contradiction proves (34), b), and hence also the whole of Theorem 1.

PROOF OF THEOREM 3. We use the following simple inequality: for $x\in R^{2n}$ and $p<q$,

$$\|x\|_{l_p}\leqslant(2n)^{\frac{1}{p}-\frac{1}{q}}\cdot\|x\|_{l_q}. \tag{38}$$

We first give upper bounds for the quantities $d_n(B_p^{2n}, l_q^{2n})$.

1) Using Theorem 1 and (38), for $1\leqslant p\leqslant 2$ we have

$$d_n(B_p^{2n}, l_q^{2n})\leqslant d_n(B_2^{2n}, l_q^{2n})\leqslant d_n(B_2^{2n}, l_\infty^{2n})\cdot(2n)^{\frac{1}{q}}\leqslant C\cdot n^{-\frac{1}{2}+\frac{1}{q}}.$$

2) Again using Theorem 1 and (38), for $2<p<\infty$ we have

$$d_n(B_p^{2n}, l_q^{2n})\leqslant(2n)^{\frac{1}{q}}\cdot d_n(B_p^{2n}, l_\infty^{2n})\leqslant(2n)^{\frac{1}{q}}\cdot(2n)^{\frac{1}{2}-\frac{1}{p}}\cdot d_n(B_2^{2n}, l_\infty^{2n})\leqslant C\cdot n^{-\frac{1}{p}+\frac{1}{q}}.$$

We now establish a lower bound.

1) For $p>2$ we use the fact that, by (38), the ball B_p^{2n} contains the set

$$E_{pq}=\{x\in R^{2n}:\|x\|_{l_q}\leqslant(2n)^{-\frac{1}{p}+\frac{1}{q}}\}.$$

Consequently

$$d_n(B_p^{2n}, l_q^{2n})\geqslant d_n(E_{pq}, l_q^{2n})\geqslant(2n)^{-\frac{1}{p}+\frac{1}{q}}d_n(B_q^{2n}, l_q^{2n})=(2n)^{-\frac{1}{p}+\frac{1}{q}}.$$

2) For $1\leqslant p\leqslant 2$ we use, besides (38), also the equality (2):

$$d_n(B_p^{2n}, l_q^{2n}) \geqslant a_n(B_1^{2n}, l_q^{2n}) \geqslant n^{-\left(\frac{1}{2}-\frac{1}{q}\right)} d_n(B_1^{2n}, l_2^{2n}) \geqslant \frac{1}{\sqrt{2}} n^{-\left(\frac{1}{2}-\frac{1}{q}\right)} \quad (q \geqslant 2).$$

These estimates constitute the assertion of Theorem 3.

We mention in addition that for any $\gamma > 1$ the diameter $d_n(B_p^{[\gamma n]}, l_q^{[\gamma n]})$, $1 \leqslant p < q \leqslant \infty$, has the same order (for $n \to \infty$) as $d_n(B_p^{2n}, l_q^{2n})$.

PROOF OF THEOREM 2. Using the results of [15], Theorem 2 follows easily from the estimates obtained here for the diameters $d_n(B_p^m, l_q^m)$.

Using the inequality (see [15])

$$d_n(W_p^r, L^q) \geqslant C_{p,r,q} \cdot n^{-\left(r-\frac{1}{p}+\frac{1}{q}\right)} \cdot d_n(B_p^{2n}, l_q^{2n})$$

and Theorem 3, we get for $q \geqslant 2$ that

$$d_n(W_p^r, L^q) \geqslant \begin{cases} C_{p,r,q} n^{-r}, & \text{if } p > 2, \\ C_{p,r,q} n^{-\left(r+\frac{1}{2}-\frac{1}{p}\right)}, & \text{if } 1 \leqslant p \leqslant 2. \end{cases}$$

Now we estimate $d_n(W_p^r, L^q)$ from above. We first mention that for $p > 2$

$$d_n(W_p^r, L^q) \leqslant d_n(W_p^r, C) \leqslant d_n(W_2^r, C).$$

Therefore it suffices for us to obtain the upper estimates required in Theorem 2 only for $1 \leqslant p \leqslant 2$ and $q = \infty$. If in Theorem 2 of [15] we substitute the estimate for the diameter $d_n(B_p^m, l_\infty^m)$ $(1 \leqslant p \leqslant 2)$ obtained in Corollary 1 of this paper, then for $1 \leqslant p \leqslant 2$ we get

$$d_n(W_p^r, C) \leqslant C_{p,r} \cdot n^{-\left(r+\frac{1}{2}-\frac{1}{p}\right)}.$$

By the above remark, the last inequality concludes the proof of Theorem 2.

Only Theorem 4 remains to be considered. We merely outline a proof of this theorem, since the necessary arguments for the proof have already been used in the proof of Theorem 1. The only new fact, in comparison with Theorem 1, is that in Theorem 4 the averagings are carried out with respect to the group of orthogonal matrices.

Let O^n be the group of orthogonal matrices of order n, and let μ be Haar measure on this group (see [4]). By mX we denote the usual Lebesgue measure of a set X in the sphere S^n $(mS^n = 1)$.

It is obvious that for any j $(1 \leqslant j < n)$ we can choose $n-j$ coordinates $(x)_{i_k}$ of a vector $x = \{(x)_i\}_1^n$ with $\|x\|_{l_1^n} \leqslant \alpha\sqrt{n}$ such that

$$|(x)_{i_k}| \leqslant \frac{\alpha\sqrt{n}}{j} \quad (1 \leqslant k \leqslant n-j). \tag{39}$$

Using (39) (for example, for $j = [n/2]$), we can show that for sufficiently small $\alpha_0 > 0$ and any n we have

$$f(\alpha_0, n) \equiv m\{x \in S^n : \|x\|_{l_1^n} \leqslant \alpha_0\sqrt{n}\} < 2^{-n}. \tag{40}$$

Using (40) and the invariance of the measure μ with respect to shifts, we get that for any

$x \in S^n$

$$\mu \{A \in O^n : \| Ax \|_{l_1^n} \leqslant \alpha_0 \sqrt{n}\} = f(\alpha_0, n) < 2^{-n}. \tag{41}$$

Reasoning just as for the proof of Theorem 1 (see, in particular, Lemma 5), we can get from (41) the existence of a matrix $T \in O^n$ such that if $|N(x)| \leqslant \beta n$, then $\|Tx\|_{l_1^n} \geqslant \alpha\sqrt{n}$, where β and α are positive absolute constants.

Then, using Lemma 4, it is not hard to see that such a matrix T satisfies the requirements of Theorem 4.

We have also the following result, which is close to Theorem 4, but is somewhat simpler:

For any positive number θ there exists a constant $C_\theta > 0$ such that for any $n \geqslant 1$ there is a plane

$$L_{n,\theta} \subset R^n, \quad \dim L_{n,\theta} \geqslant n(1-\theta),$$

such that if $x \in L_{n\theta}$, then

$$C_\theta \| x \|_{l_2^n} \leqslant \frac{1}{\sqrt{n}} \| x \|_{l_1^n} \leqslant \| x \|_{l_2^n}.$$

What is more, if we set

$$L_{n,\theta}^0 = \{x \in R^n : (x)_i = 0 \text{ for } i \geqslant n(1-\theta)+1\},$$

then for sufficiently small $C_\theta > 0$ the measure of those $T \in O^n$ for which the plane $T(L_{n\theta}^0)$ does not satisfy the last assertion is smaller than 2^{-n}.

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