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ON HOMOGENEOUS POLYNOMIALS OF SEVERAL VARIABLES ON THE COMPLEX SPHERE

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ABSTRACT. This article gives a generalization of a theorem of Ryll and Wojtaszczyk on the existence of a sequence of homogeneous polynomials P_N, $N = 1, 2, \dots$, in d variables with degree $P_N = N$ for which

$$\|P_N\|_{L^2(S^d)} \geq c_d \|P_N\|_{C(S^d)} > 0,$$

where S^d is the sphere in d-dimensional complex space.

Bibliography: 11 titles.

Let $\mathbf{C}^d$ be d-dimensional complex space ($\mathbf{C}^1 = \mathbf{C}$), B^d the open unit ball in $\mathbf{C}^d$ and $S^d \equiv \partial B^d$, i.e.,

$$z = (z_1, \dots, z_d) \in S^d \Leftrightarrow |z|^2 \equiv \sum_{j=1}^{d} |z_j|^2 = 1; \qquad z_j \in \mathbf{C}.$$

Further, let $E(d, N)$ denote a set of d-tuples of integers:

$$E(d,N) = \left\{ (k_1, \dots, k_d)\colon k_j \geq 0,\ 1 \leq j \leq d,\ \sum_{j=1}^{d} k_j = N \right\}.$$

We note that the number of elements $|E(d,N)| \asymp N^{d-1}$ if d is fixed and $N \to \infty$. Finally, let $C(S^d)$ and $L^2(S^d)$ be the spaces of functions continuous on S^d and those square summable with respect to the natural normalized measure on S^d, which is invariant under rotations.

In this note we establish several properties of polynomials of the form

$$P(z) = \sum_{(k_1, \dots, k_d) \in E(d,N)} a_{k_1, \dots, k_d} z_1^{k_1} \cdots z_d^{k_d}; \qquad z \in S^d. \tag{1}$$

The questions we consider here are directly connected with the topic of the articles of Ryll and Wojtaszczyk [1] and A. B. Aleksandrov [2].

It is established in [1] that for $d = 2, 3, \dots$ and $N = 1, 2, \dots$ there exists a polynomial $P_N(z)$ of the form (1) such that

$$\|P_N(z)\|_{L^2(S^d)} \leq \|P_N(z)\|_{C(S^d)} \leq K_d \|P_N(z)\|_{L^2(S^d)},$$

where K_d is a constant depending only on d.

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In [2] it is shown that on the basis of this result it is possible to construct nonconstant inner functions in the ball B^d for $d = 2, 3, \ldots$. (A function f which is holomorphic and bounded in B^d is called *inner* if $\lim_{r\to 1} |f(rz)| = 1$ for almost all $z \in S^d$.) The existence of nonconstant inner functions in B^d, $d = 2, 3, \ldots$, was proved earlier by other methods by Aleksandrov [3] and Low [4].

The construction of the polynomials P_N in [1] is based on concepts in the geometry of finite-dimensional normed spaces. Somewhat earlier a geometric approach was applied by the author in [5] and [6] (see also [11]) for the construction of trigonometric polynomials $t_N(x)$ for $N = 1, 2, \ldots$ with the property

$$\|t_N\|_{C(-\pi,\pi)} \asymp \|t_N\|_{L^2(-\pi,\pi)} \asymp N^{-1/2} \sum_{k=-N}^{N} |\hat{t}_N(k)|, \tag{2}$$

$$t_N \in T_N \equiv \left\{ t(x) : t(x) = \sum_{k=-N}^{N} \hat{t}(k) e^{ikx},\ x \in (-\pi, \pi) \right\}.$$

The existence of such polynomials was established back in 1914 by S. N. Bernstein and has been systematically employed in the theory of trigonometric series. It follows from the results in [5] and [6] that the polynomials t_N with the property (2) can be found in every subspace $L \subset T_N$ of large dimension: $\dim L \geq cN$ with $c > 0$. It is easy to see that the problem of constructing polynomials $t_N(x)$ for $N = 1, 2, \ldots$ with property (2) is equivalent to that of constructing polynomials $P_N = P_N(z)$ with $N = 1, 2, \ldots$ of the form (1) (for $d = 2$) for which

$$\|P_N\|_{C(\partial U^2)} \asymp \|P_N\|_{L^2(\partial U^2)} \asymp N^{-1/2} \sum_{k_1=0}^{N} |a_{k_1, N-k_1}|, \tag{3}$$

where $U^d = \{z \in \mathbf{C}^d : |z_j| < 1,\ 1 \leq j \leq d\}$. (In distinction from the case of the ball, the question of the existence of a sequence of polynomials $P_N(z_1, z_2)$ for $N = 1, 2, \ldots$ for which only the left equivalence in (3) holds is trivial: the monomials $z_1^k z_2^{N-k}$ possess this property.)

After the publication of [5] and [6] it was observed by the author (see, in particular, [7], Russian p. 53, English p. 65) that to construct trigonometric polynomials with certain extremal properties of the type of (2) the following simple result can be successfully applied.

ASSERTION 1. *For any collection of vectors $\{e_i\}_1^m \subset S^n$ with $e_i = \{(e_i)_j\}_1^n$ for $1 \leq i \leq m$ and $m \leq n\rho$ there is a vector $e_0 = \{(e_0)_j\} \in S^n$ with $\operatorname{Im}(e_0)_j = 0$ for $1 \leq j \leq n$ and such that*

$$\left| \sum_{j=1}^{n} (e_0)_j (e_i)_j \right| \leq C_\rho \cdot n^{-1/2}, \qquad 1 \leq i \leq m. \tag{4}$$

Assertion 1 is applied in [8] to obtain higher-dimensional analogs of the results in [5]. In this note we use Assertion 1 to prove the following generalization of the theorem of Ryll and Wojtaszczyk.

THEOREM 1. *For every $\delta > 0$ and $d = 2, 3, \ldots$ there is a constant $K = K(\delta, d)$ such that for any set $\Lambda \subset E(d, N)$ with $|\Lambda| > \delta N^{d-1}$ there is a polynomial*

$$P_\Lambda = P_\Lambda(z) = \sum_{(k_1, \ldots, k_d) \in \Lambda} a_{k_1, \ldots, k_d} z_1^{k_1} \cdots z_d^{k_d}$$

with

$$\|P_\Lambda\|_{L^2(S^d)} \le \|P_\Lambda\|_{C(S^d)} \le K(\delta,d)\|P_\Lambda\|_{L^2(S^d)}.$$

The possibility of applying Assertion 1 to obtain Theorem 1 is based on the following result, which is of independent interest.

ASSERTION 2. *For $d = 2,3,\dots$ and $N = 1,2,\dots$ there exists on the sphere S^d a set of points $\Omega(d,N) = \{z^{(\nu)},\ 1 \le \nu \le \nu(d,N)\}$ such that*

1) $\nu(d,N) \le C_d \cdot N^{d-1}$ *for $N = 1,2,\dots$, and*

2) *for any polynomial of the form* (1)

$$\|P\|_{C(S^d)} \le 2 \max_{z\in\Omega(d,N)} |P(z)|. \tag{5}$$

PROOF OF ASSERTION 2. It is easy to see that it suffices to find a set $\Omega(d,N) \subset S^d$ with $|\Omega(d,N)| \le K_d N^{d-1}$ such that (5) holds for any polynomial of the form (1) such that

$$\|P\|_{C(S^d)} = |P(z^{(0)})|, \quad z^{(0)} = \{z_j^{(0)}\} \in S^d, \quad |z_j^{(0)}| \le |z_d^{(0)}|, \qquad 1 \le j < d. \tag{6}$$

We set $v_j = z_j/z_d$ for $1 \le j \le d-1$. Then for a polynomial (1) and $z \in S^d$ we have the equality

$$\begin{aligned} |P(z)| &= |f_P(v_1,\dots,v_{d-1})| \\ &\equiv \left| \left(1 + \sum_{j=1}^{d-1} |v_j|^2\right)^{-N/2} \sum_{(k_1,\dots,k_{d-1})} b_{k_1,\dots,k_{d-1}} v_1^{k_1}\cdots v_{d-1}^{k_{d-1}} \right|, \end{aligned} \tag{7}$$

where $b_{k_1,\dots,k_{d-1}} = a_{k_1,\dots,k_d}$ and the summation in (7) runs over all tuples of integers $\{k_j\}_1^{d-1}$ with $k_j \ge 0$ and $\sum k_j \le N$.

Thus (see also (6)) it suffices to find a set of points $\Omega' = \Omega'(d,N) \subset U^{d-1}$ with $|\Omega'| \le K_d \cdot N^{d-1}$ such that for any function $f_P(v_1,\dots,v_{d-1})$ of the form (7) with

$$\|f_P\|_{C(\mathbf{C}^{d-1})} = f(v_1^0,\dots,v_{d-1}^0), \qquad |v_j| \le 1,\ 1 \le j \le d-1, \tag{8}$$

the inequality

$$\|f_P\|_{C(\mathbf{C}^{d-1})} \le 2 \cdot \max_{(v_1,\dots,v_{d-1})\in\Omega'} |f_P(v_1,\dots,v_{d-1})|$$

holds. (Then for every point $(v_1,\dots,v_{d-1}) \in \Omega'$ we define a point $z \in S^d$:

$$|z_d| = \left(1 + \sum_{j=1}^{d-1} |v_j|^2\right)^{-1/2}, \qquad \operatorname{Arg} z_d = 0,\ z_j = z_d v_j,\ 1 \le j \le d-1;$$

the set $\Omega(d,N)$ consists of the collection of these points.) In turn, the existence of the set $\Omega'(d,N)$ follows immediately from the following fact:

For every $d = 2,3,\dots$ there exists a constant $\varepsilon = \varepsilon_d > 0$ such that for any function $f_P(v_1,\dots,v_{d-1})$ with property (8) *we have*

$$|f_P(v_1,\dots,v_{d-1})| \ge \frac{1}{2}|f_P(v_1^0,\dots,v_{d-1}^0)|, \quad \text{if } \left(\sum_{j=1}^{d-1} |v_j - v_j^0|^2\right)^{1/2} \le \varepsilon N^{-1/2}. \tag{$*$}$$

It follows from $(*)$ that as $\Omega'(d,N)$ we may take any set which constitutes an $\varepsilon N^{-1/2}$-net in U^{d-1}.

We prove $(*)$. Without loss of generality we assume that

$$\|f_P\|_{C(\mathbf{C}^{d-1})} = |f(v_1^0, \ldots, v_{d-1}^0)| = f(v_1^0, \ldots, v_{d-1}^0) = 1. \tag{9}$$

Note first of all that for any complex numbers v_j and ξ_j with $1 \le j \le d-1$ we have

$$\left(1 + \sum_{j=1}^{d-1} |v_j + \xi_j|^2\right)\left(1 + \sum_{j=1}^{d-1} |v_j - \xi_j|^2\right) \le \left(1 + \sum_{j=1}^{d-1} [|v_j|^2 + |\xi_j|^2]\right)^2 . \tag{10}$$

We consider the polynomial

$$q(\xi_1, \ldots, \xi_{d-1}) = R(v_1^0 + \xi_1, \ldots, v_{d-1}^0 + \xi_{d-1}), \tag{11}$$

where the polynomial $R(v_1, \ldots, v_{d-1})$ denotes the numerator of the right side of (7). Further, let

$$Q(\xi_1, \ldots, \xi_{d-1}) = q(\xi_1, \ldots, \xi_{d-1}) q(-\xi_1, \ldots, -\xi_{d-1}). \tag{12}$$

Then (see (7) and (9))

$$q^2(0, \ldots, 0) = Q(0, \ldots, 0) = \left(1 + \sum_{j=1}^{d-1} |v_j^0|^2\right)^N \equiv V^N,$$

$$|q(\xi_1, \ldots, \xi_{d-1})| \le \left(1 + \sum_{j=1}^{d-1} |v_j^0 + \xi_j|^2\right)^{N/2} \tag{13}$$

and

$$|Q(\xi_1, \ldots, \xi_{d-1})| \le \left(1 + \sum_{j=1}^{d-1} |v_j^0 + \xi_j|^2\right)^{N/2} \left(1 + \sum_{j=1}^{d-1} |v_j^0 - \xi_j|^2\right)^{N/2} .$$

Therefore (see (10)) if $\sum_1^{d-1} |\xi_j|^2 \le N^{-1}$, then

$$|Q(\xi_1, \ldots, \xi_{d-1})| \le \left(1 + \sum_{j=1}^{d-1} |v_j^0|^2 + \frac{1}{N}\right)^N \le eV^N. \tag{14}$$

Further, let $\xi_j = x_{2j-1} + ix_{2j}$ for $1 \le j \le d-1$; then the function

$$F(x) \equiv \operatorname{Re} Q(\xi_1, \ldots, \xi_{d-1}), \qquad x = (x_1, \ldots, x_{2(d-1)}),$$

is a harmonic function of the variables x_j for $1 \le j \le 2(d-1)$. Consequently by Poisson's formula (see [9], Chapter II, §1, Corollary 1.11) for $(\xi_1, \ldots, \xi_{d-1}) \in rB^{d-1}$ with $r = N^{-1/2}$ we have

$$F(x) = r^{2(d-1)-2} \int_\Sigma F(rs) \frac{r^2 - |x|^2}{|x - rs|^{2(d-1)}}\, ds,$$

where $|x| = (\sum_{j=1}^{2(d-1)} x_j^2)^{1/2}$, Σ is the unit sphere in $R^{2(d-1)}$, $s \in \Sigma$, and the integration is performed with respect to normalized Lebesgue measure on Σ. In particular (see (13))

$$V^N = \operatorname{Re} Q(0, \ldots, 0) = \int_\Sigma F(rs)\, ds; \tag{15}$$

here (see (14)) for any $s \in \Sigma$ we have

$$|F(rs)| \le eV^N. \tag{16}$$

It is easy to see that

$$\left| r^{2d-4}\frac{r^2-|x|^2}{|x-rs|^{2d-2}} - 1\right| < \frac{1}{3e}, \qquad |x| < \varepsilon_0 r,\ \varepsilon_0 = \varepsilon_0(d) > 0;$$

thus (see also (15) and (16)) for $|x| < \varepsilon_0 r$ we have

$$\begin{aligned} |F(x)| &= \left| \int_\Sigma F(rs)\,ds - \int_\Sigma \left(1 - r^{2d-4}\frac{r^2-|x|^2}{|x-rs|^{2d-2}}\right) F(rs)\,ds \right| \\ &\geq \operatorname{Re} Q(0,\dots,0) - \sup_{s\in\Sigma} |F(rs)| \cdot \frac{1}{3e} \geq \frac{2}{3} V^N. \end{aligned}$$

But then (see (10))

$$\begin{aligned} &|q(\xi_1,\dots,\xi_{d-1}) q(-\xi_1,\dots,-\xi_{d-1}) \geq \frac{2}{3} V^N \\ &\geq \frac{1}{2}\left(1 + \sum_{j=1}^{d-1} |v_j^0 + \xi_j|^2\right)^{N/2} \left(1 + \sum_{j=1}^{d-1} |v_j^0 - \xi_j|^2\right)^{N/2}, \end{aligned}$$

if $(\sum |\xi_j|^2)^{1/2} \leq \varepsilon_1 N^{-1/2}$, where $\varepsilon_1 = \varepsilon_1(d)$ is sufficiently small ($0 < \varepsilon_1 \leq \varepsilon_0$), and consequently (see (13))

$$q(\xi_1,\dots,\xi_{d-1}) \left(1 + \sum_{j=1}^{d-1} |v_j^0 + \xi_j|^2\right)^{-N/2} \geq \frac{1}{2},$$

provided $(\sum |\xi_j|^2)^{1/2} \leq \varepsilon_1 N^{-1/2}$. Thus the relation (*) and hence Assertion 2 are proved.

REMARK. It follows from (*) that for any polynomial of the form (1)

$$|P(z_1,\dots,z_d)| \geq \tfrac{1}{2} \|P\|_{C(S^d)}$$

if $\sum_1^d |z_j^{(0)} - z_j|^2 \leq \varepsilon_d N^{-1}$, $\varepsilon_d > 0$, and equality (6) holds at the point $(z_1^0,\dots,z_d^0)$.

PROOF OF THEOREM 1. We use the pairwise orthogonality of the monomials $z_1^{k_1} \cdot \dots \cdot z_d^{k_d}$ on the sphere S^d (see [10]). By virtue of this for any polynomial P of the form (1) we have

$$\|P\|_{L^2(S^d)} = \sum_{(k_1,\dots,k_d)\in E(d,N)} |a_{k_1,\dots,k_d}|^2 \frac{k_1!\cdots k_d!}{(N+d-1)!}. \tag{17}$$

We note further that for $(z_1,\dots,z_d) \in S^d$

$$1 = \left(\sum_{j=1}^d |z_j|^2\right)^N = \sum_{(k_1,\dots,k_d)\in E(d,N)} |z_1|^{2k_1}\cdots|z_d|^{2k_d} \frac{N!}{k_1!\cdots k_d!}. \tag{18}$$

Let $p_1,\dots,p_{j_0}$ for $j_0 = |\Lambda| \geq \delta N^{d-1}$ be the monomials $p = z_1^{k_1}\cdots z_d^{k_d}$ for $(k_1,\dots,k_d) \in \Lambda$ indexed in any order, and let $A(p) = N![k_1! \cdot \dots \cdot k_d!]^{-1}$.

For every point $z^{(\nu)} \in \Omega(d,N)$ (see Assertion 2) we define a vector $e_\nu = \{(e_\nu)_j\}_1^{j_0}$:

$$(e_\nu)_j = p_j(z^{(\nu)})\{A(p_j)\}^{1/2} \cdot \{(N+1)\cdots(N+d-1)\}^{1/2}, \qquad 1 \leq j \leq j_0.$$

Then (see (18))

$$\sum_{j=1}^{j_0} |(e_\nu)_j|^2 \leq C_d' N^{d-1}, \qquad \nu = 1,2,\dots,|\Omega(d,N)|.$$

Thus by Assertion 1 and the bound $|\Omega(d,N)| \le C_d N^{d-1}$ there is a vector $e_0 = \{(e_0)_j\}_{j=1}^{j_0} \in S^{j_0}$ such that

$$\left|\sum_{j=1}^{j_0}(e_0)_j(e_\nu)_j\right| \le C_\rho, \qquad 1 \le \nu \le |\Omega(d,N)|,\ \rho = \rho(d). \tag{19}$$

We set

$$P_\Lambda(z) = \sum_{j=1}^{|\Lambda|}(e_0)_j\{A(p_j)(N+1)\cdots(N+d-1)\}^{1/2}p_j(z).$$

Then (see Assertion 2 and (19)) we have $\|P_\Lambda\|_{C(S^d)} \le 2C_\rho$ and (see (17)) $\|P_\Lambda\|_{L^2(S^d)} = [(d-1)!]^{1/2}$. Theorem 1 is proved.

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