Analysis Mathematica, 2 (1976), 249—266

On Weyl's multipliers for almost everywhere convergence of orthogonal series

B. S. KAŠIN

The aim of this paper is to prove the theorems stated in [4].

Let $\{\varphi_n(x)\}_{n=1}^{\infty}$ be an orthonormal set of functions (ONS) on the segment [0, 1]. A sequence of numbers λ_n ($\lambda_1 \geqq 0$, $\lambda_{n+1} > \lambda_n$) is called a *Weyl multiplier* (WM) for almost everywhere (a. e.) convergence of series with respect to this set, if the inequality

$$(1) \qquad \sum_{n=1}^{\infty} c_n^2 \lambda_n < \infty$$

implies the a. e. convergence on [0, 1] of the series

$$(2) \qquad \sum_{n=1}^{\infty} c_n \varphi_n(x).$$

The WM $\{\lambda_n\}$ is called a *precise Weyl multiplier* (PWM) if for every sequence $\beta_n = o(\lambda_n)$ there exists a series (2) diverging on a set of positive measure, whereas

$$\sum_{n=1}^{\infty} c_n^2 \beta_n < \infty.$$

The classical theorem of D. E. Menšov and G. Rademacher (see, e.g., [2, Chapter V, § 3]) states that $\lambda_n = \log^2 n$ is a WM for the a. e. convergence of series with respect to any ONS.

Menšov also constructed an ONS for which $\lambda_n = \log^2 n$ is a PWM. The set constructed by Menšov was not uniformly bounded, and for a long time it had not been clear of what order a WM for series with respect to a uniformly bounded ONS must be.

In [5] Kolmogorov and Menšov constructed an ONS $\{\varphi_n(x)\}$ such that

1) $|\varphi_n(x)| = 1$ for $x \in [0, 1]$ and $n = 1, 2, \ldots$;

2) every sequence β_n, such that $\beta_n = o(\log n)$, is not a WM for the a. e. convergence of series with respect to $\{\varphi_n(x)\}$.

Received October 31, 1975.

In 1938 MENŠOV proved (see [6]) that for any $K>1$ there exists an ONS $\{\varphi_n(x)\}$ such that

1) $|\varphi_n(x)|\leqq K$ for $x\in[0,1]$ and $n=1,2,\dots$;

2) the sequence $\{\log^2 n\}$ is its PWM.

The extreme case $K=1$ is interesting. In [9] TANDORI proved that for every $\varepsilon>0$ there exists an ONS $\{\varphi_n(x)\}$, $|\varphi_n(x)|\equiv 1$, $n=1, 2, \dots$, and a series (2) such that

1) $\sum_{n=3}^{\infty} c_n^2 \log n (\log\log n)^{1-\varepsilon} < \infty$;

2) the series (2) diverges a. e. after a certain rearrangement of its terms.

The lessening of the order of magnitude of the WM for the orthonormal sets $\{\varphi_n(x)\}$, $|\varphi_n(x)|\equiv 1$, would have had important effects in number theory and in the theory of orthogonal series itself. It follows from Theorem 1, however, that such a lessening is impossible.

Theorem 1. *There exists an* OSN $\{\varphi_n(x)\}$, $|\varphi_n(x)|\equiv 1$, *on the segment* $[0, 1]$ *such that the sequence* $\lambda_n=\log^2 n$ *is its* PWM.

Theorem 1 is easily deduced (as shown at the end of this paper) from the following lemma with standard reasoning.

Fundamental lemma. *There exist absolute constants* $C_1>0$, $C_2>0$ *and an integer* $M\geqq 1$ *such that for any positive integer* q *there exists a set of functions* $\{\varphi_j(x)\}_{j=1}^{2^q}$ *defined on the segment* $[0, 2M+1]$ *and satisfying the requirements*

1) $|\varphi_j(x)|=1$ *for* $x\in[0,2M+1]$ *and* $j=1,2,\dots,2^q$;

2) $\int_0^{2M+1} \varphi_i(x)\varphi_j(x)\,dx = 0;\quad i\neq j,\quad 1\leqq i,j\leqq 2^q$;

3) *there exist* $N_1(x)$ *and* $N_2(x)$ *such that*

$$\mu\left\{x: \left|\sum_{j=N_1(x)}^{N_2(x)} \varphi_j(x)\right| > C_1\sqrt{n}\log n\right\} \geqq C_2,$$

where $n=2^q$.

To prove the fundamental lemma we need the following auxiliary statements.

Lemma 1 (MENŠOV [6, p. 104]). *Let* $\{a_{ij}\}_{i,j=1}^n$ *be a real square matrix of order* n *and* $p\in(0,n)$ *be an integer. Let* $\beta_p=\max_{|i-j|=p}|a_{ij}|$. *Then on every segment* $[c, d]$ *such that* $d-c>2\beta_p$ *one can construct a set of functions* $\{\varphi_s(x)\}_{s=1}^n$ *such that*

1) $|\varphi_s(x)| = 1$ *for* $x \in [c, d]$ *and* $1 \leqq s \leqq n$;

2) $\int_c^d \varphi_i(x)\varphi_j(x)\,dx = -a_{ij}$ *for* $|i-j| = p$, $1 \leqq i, j \leqq n$;

3) $\int_c^d \varphi_i(x)\varphi_j(x)\,dx = 0$ *for* $i \neq j$, $|i-j| \neq p$, $1 \leqq i, j \leqq n$.

Lemma 2. *Let* $\{\varphi_i(x)\}_{i=1}^n$ *be a set of functions defined on the segment* [0, 1] *for which a sequence* $\{\gamma_p\}_{p=1}^{n-1}$ *with the properties*

1) $\left|\int_0^1 \varphi_i(x)\varphi_j(x)\,dx\right| \leqq \gamma_{|i-j|}$ *for every* $1 \leqq i < j \leqq n$,

2) $\sum_{p=1}^{n-1} \gamma_p < M$

exists. Then the functions $\varphi_i(x)$ *can be extended to the segment* $[1, 2M+1]$ *in such a way that*

1) $\int_0^{2M+1} \varphi_i(x)\varphi_j(x)\,dx = 0$, $1 \leqq i, j \leqq n$, $i \neq j$;

2) $|\varphi_i(x)| = 1$ *for* $x \in (1, 2M+1]$ *and* $i = 1, 2, \ldots, n$.

Lemma 2 follows easily from Lemma 1. In fact, to prove it we subdivide the segment $[1, 2M+1]$ into $n-1$ non-overlapping segments Δ_p, $1 \leqq p \leqq n-1$, of length $|\Delta_p| > \gamma_p$ and put

$$a_{ij} = \int_0^1 \varphi_i(x)\varphi_j(x)\,dx.$$

Then for every integer p we define the set of functions $\{\varphi_j(x)\}_{j=1}^n$ on the segment Δ_p according to Lemma 1. Thus we define the set $\{\varphi_j(x)\}_{j=1}^n$ on $[1, 2M+1]$. It is obvious that for this set the requirement 2) is satisfied and that for any $1 \leqq i < j \leqq n$ such that $j - i = p$ we have

$$\int_0^{2M+1} \varphi_i(x)\varphi_j(x)\,dx = \int_0^1 \varphi_i(x)\varphi_j(x)\,dx + \int_{\Delta_p} \varphi_i(x)\varphi_j(x)\,dx = 0.$$

Thus Lemma 2 is proved.

Lemma 3 (see [7, pp. 16 and 76]). *Let* $\{\chi_i(z)\}_{i=1}^N$ *be a set of independent functions on the segment* [0, 1] *with the mean value of* $\chi_i(z)$ *being equal to* 0 *and* $|\chi_i(z)| \leqq B$ *for* $z \in [0, 1]$ *and* $i = 1, \ldots, N$. *Then for any* $y > 0$, *any positive integer* $k \leqq N$ *and for* *any set of real numbers* $\{a_i\}_{i=1}^N$ *the following inequality holds:*

$$\mu\left\{z : \left|\sum_{i=1}^{k} a_i \chi_i(z)\right| \geqq y\left(\sum_{i=1}^{k} a_i^2\right)^{1/2}\right\} \leqq 2e^{-y^2/2B}. \tag{3}$$

It follows directly from the estimate (3) that for any integer r there exists a constant C_r^1 depending only on B and r such that

$$\mu\left\{z : \left|\sum_{i=1}^{k} a_i \chi_i(z)\right| > C_r^1\left(\sum_{i=1}^{k} a_i^2 \log v\right)^{1/2}\right\} < \frac{1}{v^r}, \tag{4}$$

where v is any integer exceeding 1.

It easily follows from the estimate (4) that for any $r \geqq 1$ there exists a constant C_r^2 such that

$$\mu\left\{z : \left\|\sum_{k=1}^{m} \chi_k(z) e^{ikt}\right\|_C > C_r^2 (m \log v)^{1/2}\right\} \leqq \frac{1}{v^r} \tag{5}$$

for $m < v$ and $m \leqq N$ $(v > 1)$.

In fact, we put $C_r^2 = 4C_{r+5}^1$ where C_{r+5}^1 is the constant from (4). For any trigonometric polynomial $P(x)$ of order not exceeding v the obvious inequality

$$\|P(x)\|_C \leqq 2 \max_{1 \leqq q \leqq v^5} \left|P\left(\frac{2\pi q}{v^5}\right)\right|$$

holds. Hence

$$\mu\left\{z : \left\|\sum_{k=1}^{m} \chi_k(z) e^{ikt}\right\|_C \geqq C_r^2 (m \log v)^{1/2} = 4C_{r+5}^1 (m \log v)^{1/2}\right\} \leqq$$

$$\leqq \sum_{q=1}^{v^5} \mu\left\{z : \left|\sum_{k=1}^{m} \chi_k(z) \exp\left(ik\frac{2\pi q}{v^5}\right)\right| > 2C_{r+5}^1 (m \log v)^{1/2}\right\}.$$

By virtue of (4) the latter sum does not exceed

$$\sum_{k=1}^{v^5} \frac{1}{v^{r+5}} \leqq \frac{1}{v^r},$$

which supplies the estimate (5).

Further, we put $\chi_i(z) \equiv 0$ for $i < 1$ and $i > N$. Let $s \neq 0$ be an integer. Consider the set of functions $\{\chi_i(z)\chi_{i+s}(z)\}_{i=-\infty}^{\infty}$. It is clear that $|\chi_i(z)\chi_{i+s}(z)| < B^2$, by virtue of the requirements of Lemma 3. It is easy to verify that for every s the functions of this set can be divided into two groups so that the functions of each group are independent (e.g., for $s = 1$ the first group contains the functions of the form $\chi_{2k}\chi_{2k+1}$, $-\infty < k < \infty$, while the second one contains the functions of the form $\chi_{2k-1}\chi_{2k}$, $-\infty < k < \infty$).

Therefore we can use the estimate (3), by virtue of which we obtain

$$\mu\left\{z:\left|\sum_{i=1}^{m}\chi_i(z)\chi_{i+s}(z)\right|>C_r^3(m\log\nu)^{1/2}\right\}<\frac{1}{\nu^r},\quad |s|=1,2,\ldots,\nu.$$

Hence we immediately obtain that there exists a constant C_r^4 for which

$$\mu\left\{z:\max_{1\le|s|\le\nu}\left|\sum_{i=1}^{m}\chi_i(z)\chi_{i+s}(z)\right|>C_r^4(m\log\nu)^{1/2}\right\}<\frac{1}{\nu^r}\tag{5'}$$

(e.g. $C_r^4=C_{r+2}^3$).

Lemma 4. *Let $f(t)\in C[0,2\pi]$ and $\{c_n\}_{n=-\infty}^{\infty}$ be the sequence of its Fourier coefficients. Then if $A=\{a_{ij}\}_{i,j=-\infty}^{\infty}$ is the infinite matrix such that $a_{ij}=c_{i-j}$, then*

$$\|A\|=\sup_{\sum_{i=-\infty}^{\infty}x_i^2=\sum_{-\infty}^{\infty}y_j^2=1}\left|\sum_{i,j=-\infty}^{\infty}a_{ij}x_iy_j\right|\le C^0\|f\|_C,$$

where C^0 is an absolute constant.

The statement of Lemma 4 is well known (see, e.g. [1, Theorem 303]).

Lemma 5. *There exists an absolute constant M_2 such that for any $\gamma\ge3$*

$$\sum_{p=1}^{[\gamma]-2}\frac{1}{p}\left|\log\left(1-\frac{p}{\gamma}\right)\right|\le M_2.$$

Indeed, the function $\Phi(x)=x^{-1}|\log(1-x)|$ is monotonously increasing on $(0,1)$. Therefore

$$\sum_{p=1}^{[\gamma]-2}\frac{1}{p}\left|\log\left(1-\frac{p}{\gamma}\right)\right|=\frac{1}{\gamma}\sum_{p=1}^{[\gamma]-2}\frac{\gamma}{p}\left|\log\left(1-\frac{p}{\gamma}\right)\right|\le\int_0^1\frac{|\log(1-x)|}{x}\,dx=M_2<\infty.$$

Lemma 6. *Let m and m_1, $m<m_1$, be two given positive integers and let $\{\varphi_1(x),\ldots,\varphi_m(x)\}$ and $\{\varphi_{m+1}(x),\ldots,\varphi_{m_1}(x)\}$ be two sets of measurable functions such that $|\varphi_k(x)|=1$ for $x\in[0,1]$ and $k=1,2,\ldots,m_1$. Then there exists a measurable function $u(x)$ such that*

1) $|u(x)|=1$ *for* $x\in[0,1]$,

2) $\int_0^1\varphi_i(x)\big(u(x)\varphi_j(x)\big)\,dx=0$ *for* $1\le i\le m<j\le m_1$.

Proof. Denote by $\bar\varepsilon=\{\varepsilon_k\}_{k=1}^{m_1}$ an arbitrary vector with coordinates $\varepsilon_k=+1$ or -1. Define the set

$$E(\bar\varepsilon)=\{x\in[0,1]:\ \varphi_k(x)=\varepsilon_k\ \text{ for }\ k=1,2,\ldots,m_1\}.$$

It follows from this definition that for every $\bar{\varepsilon}$ all the functions $\varphi_k(x)$ ($k=1, 2, \dots, m_1$) are constant on the set $E(\bar{\varepsilon})$. Therefore any function $u(x)$ such that $|u(x)|\equiv 1$ and

$$\int_{E(\bar{\varepsilon})} u(x)\,dx = 0 \quad \text{for all} \quad \bar{\varepsilon},$$

will satisfy all the requirements of Lemma 6.

Remark 1. Let the set of functions $\{f_j(x)\}_{j=1}^n$ on $[0, 1]$ piece-wise constant on the intervals $\left[\frac{i}{n}, \frac{i+1}{n}\right)$ be described by the matrix $A=\{a_{ij}\}$, i.e.,

$$f_j(x) = f_j\left(\frac{i}{n}\right) = a_{ij} \quad \text{for} \quad \frac{i}{n} \leqq x < \frac{i+1}{n} \quad \text{and} \quad i = 0, 1, \dots, n-1.$$

Then

$$\left\|\sum_{j=1}^{n} \alpha_j f_j(x)\right\|_{L_2} \leqq \|A\| \left(\frac{1}{n}\sum_{j=1}^{n} \alpha_j^2\right)^{1/2}. \tag{6}$$

In fact,

$$\left\|\sum_{j=1}^{n} \alpha_j f_j(x)\right\|_{L_2} = \left(\frac{1}{n}\sum_{i=0}^{n-1}\left(\sum_{j=1}^{n}\alpha_j a_{ij}\right)^2\right)^{1/2} =$$

$$= \frac{1}{\sqrt{n}} \sup_{\sum_{\nu=0}^{n-1}\gamma_\nu^2=1} \sum_{i=0}^{n-1}\sum_{j=1}^{n} a_{ij}\alpha_j\gamma_i \leqq \|A\| \left(\frac{1}{n}\sum_{j=1}^{n}\alpha_j^2\right)^{1/2},$$

which was to be proved.

Proof of the fundamental lemma. Let q be the number in the fundamental lemma. We set

$$n = 2^q, \quad R = [n^{3/5}], \tag{7}$$

where $[x]$ is the integral part of the number x.

By E_j we denote the following set

$$E_j = \left\{x : \left|x - \frac{j}{n}\right| \leqq \frac{R}{n}\right\} \quad \text{for} \quad R \leqq j \leqq n-R;$$

$$E_j = \emptyset \qquad\qquad \text{for} \quad 1 \leqq j \leqq R-1 \quad \text{and} \quad n-R < j \leqq n.$$

It is clear that $E_j \subset [0, 1]$ for $1 \leqq j \leqq n$. First we construct the set of functions $\{\psi_j(x)\}_{j=R}^{n-R}$ such that $\psi_j(x)$ coincides with the required function $\varphi_j(x)$ on the set E_j and $\psi_j(x)=0$ for $x \in E_j$. We define the piece-wise constant function $\psi_j(x)$ with the help of a matrix $C=\{c_{ij}\}$ by putting $\psi_j(x)=\psi_j\left(\frac{i}{n}\right)=c_{ij}$ for $x\in\left[\frac{i}{n}, \frac{i+1}{n}\right)$ and $i=0, 1, \dots, n-1$; $R\leqq j\leqq n-R$. Denote by $\{\beta_p\}_{p=-n}^{n}$ the following sequence of numbers

$$\beta_p = \begin{cases} 0 & \text{for } p \geqq R \text{ and } p < -R, \\ \dfrac{\sqrt{n}}{2(p-\sqrt{n})} & \text{for } -R \leqq p < 0, \\ \dfrac{\sqrt{n}}{2(p+\sqrt{n})} & \text{for } 0 \leqq p < R. \end{cases} \tag{8}$$

We define the matrix $A=\{a_{ij}\}$, $0 \leqq i \leqq n-1$, $R \leqq j \leqq n-R$, by putting $a_{ij}=\beta_{i-j}$. Finally, we define the set of piece-wise constant functions $\{f_j^{(1)}(x)\}_{j=R}^{n-R}$ so that

$$f_j^{(1)}(x) = f_j^{(1)}\left(\frac{i}{n}\right) = a_{ij} \quad \text{for} \quad \frac{i}{n} \leqq x < \frac{i+1}{n} \tag{8'}$$

(it is easy to see that $f_j^1(x)=0$ for $x \notin E_j$). This set of functions resembles the set used by MENŠOV in the paper [6, p. 110]. Here we have modified Menšov's functions so that their supports be of small measure.

By direct calculation we obtain

$$\mu\left\{x \in [0, 1] : \left|\sum_{j=[nx]}^{[nx]+R} f_j^1(x)\right| > C_1 \sqrt{n} \log n\right\} > C_2, \tag{9}$$

where C_1 and C_2 are absolute positive constants.

We also obtain that there exist numbers M^1 and $\gamma_k^{(1)}$, $1 \leqq k \leqq n-1$, for which

$$\begin{gathered} \sum_{p=1}^{n-1} \gamma_p^{(1)} < M^1, \\ \left|\int_0^1 f_i^{(1)}(x) f_j^{(1)}(x)\, dx\right| \leqq \gamma_{|i-j|}^{(1)} \quad \text{for} \quad i \neq j \quad \text{and} \quad R \leqq i, j \leqq n-R. \end{gathered} \tag{10}$$

The proof of the latter statement will be given at the end of this paper in order not to overcomplicate the proof of the fundamental lemma.

Consider the set of numbers $\Delta = \{\delta_i\}_{i=-n}^{n}$ such that

1) $\delta_i = 0$ for $i \geqq R$, $i < -R$;

2) δ_i takes one of the following values

$$\delta_i = \begin{cases} -1-\beta_i \\ 1-\beta_i \end{cases} \text{for } -R \leqq i < R. \tag{11}$$

Since $-\frac{1}{2} \leqq \beta_i \leqq \frac{1}{2}$ (see (8)),

$$|\delta_i| \leqq \frac{3}{2} \quad \text{for} \quad -n \leqq i \leqq n. \tag{12}$$

Denote by $B(\Delta)$ the matrix $B(\Delta)=\{b_{ij}\}$, $0\leqq i\leqq n-1$, $R\leqq j\leqq n-R$, where $b_{ij}=\delta_{i-j}$ and by $B'(\Delta)$ the matrix $B'(\Delta)=\{b'_{ij}\}$, $0\leqq i\leqq n-1$, $R\leqq j\leqq n-R$, where

$$b'_{ij}=\begin{cases}\delta_{i-j} & \text{for } i\leqq j,\\ 0 & \text{for } i>j.\end{cases} \tag{12'}$$

It easily follows from the definition of the matrix $A+B(\Delta)$ that for every set Δ the modulus of the entry of the matrix with indices (i, j) is equal to 1 for $-R\leqq i-j<R$.

We choose the required set using probabilistic considerations. We denote by $\{f_j^{(2)}(x, \Delta)\}_{j=R}^{n-R}$ the set of piece-wise constant functions defined as follows

$$f_j^{(2)}(x, \Delta)=f_j^{(2)}\left(\frac{i}{n}, \Delta\right)=b_{ij} \quad \text{for} \quad \frac{i}{n}\leqq x<\frac{i+1}{n}, \quad R\leqq j\leqq n-R. \tag{13}$$

It is obvious that $f_j^{(2)}(x, \Delta)=0$ for $x\notin E_j$. Observe that $|f_j^{(1)}(x)+f_j^{(2)}(x)|$ is equal to 1 for $x\in E_j$ and is equal to 0 almost everywhere outside E_j.

Consider the set of independent functions $\{\chi_k(z)\}_{k=-n}^{n}$ defined on the segment $[0, 1]$ with zero mean value and such that $\chi_k(z)$ takes only two values, namely the ones taken by δ_k (see (11)). It is obvious that $\chi_k(z)\equiv 0$ for $|k|>R$. By virtue of (12) we have $|\chi_k(z)|\leqq 3/2$ for all k. At every point $z\in[0, 1]$ the set $\{\chi_k(z)\}$ defines some set Δ, i.e. $\chi_k(z)=\delta_k$, $-n\leqq k\leqq n$. We denote this set by $\Delta(z)$.

Let C^0 be the constant given by Lemma 4. By virtue of Lemma 4 and the estimate (5) we obtain

$$\begin{aligned}\mu\{z: \|B'(\Delta(z))\|\geqq C^0 C_r^2 (R\log n)^{1/2}\}\leqq\\ \leqq\mu\left\{z: \left\|\sum_{k=-R}^{0}\chi_k(z)e^{ikt}\right\|_C\geqq C_r^2(R\log n)^{1/2}\right\}\leqq\frac{1}{n^r}.\end{aligned} \tag{14}$$

If $i\neq j$, then

$$\int_0^1 f_i^{(2)}(x, \Delta(z))f_j^{(2)}(x, \Delta(z))\,dx=$$

$$=d_{i-j}=\begin{cases}0 & \text{for } |p|>2R \quad (\text{where } p=i-j),\\ \dfrac{1}{n}\sum\limits_{k=-R}^{R}\delta_k\delta_{k+p}=\dfrac{1}{n}\sum\limits_{k=-R}^{R}\beta_{k-p}\chi_k(z) & \text{for } |p|\leqq 2R.\end{cases}$$

Using the estimate (5′), we obtain

$$\mu\left\{z: \max_{\substack{|i-j|\leqq 2R\\ i\neq j}}\left|\int_0^1 f_i^{(2)}(x, \Delta)f_j^{(2)}(x, \Delta)\,dx\right|\geqq C_r^4\frac{(2R\log n)^{1/2}}{n}\right\}\leqq\frac{1}{n^r} \quad \text{for} \quad n>1. \tag{15}$$

Further

$$\int_0^1 f_j^{(2)}(x, \Delta) f_i^{(1)}(x)\,dx = \begin{cases} 0 & \text{for } |p| = |i-j| > 2R, \\ \dfrac{1}{n}\displaystyle\sum_{k=-R}^{R} \delta_k \beta_{k-p} = \dfrac{1}{n}\sum_{k=-R}^{R} \beta_{k-p}\chi_k(z) & \text{for } |p| \leqq 2R. \end{cases}$$

Since

$$\left(\sum_{k=-n}^{n} \beta_k^2\right)^{1/2} \leqq \left(\frac{n}{2}\sum_{j=\sqrt{n}}^{\infty} \frac{1}{j^2}\right)^{1/2} \leqq n^{1/4},$$

by virtue of (4) and the inequality $1 \leqq |p| \leqq n$, we obtain

$$\mu\left\{z : \max_{p,j}\left|\int_0^1 f_j^{(2)}(x, \Delta) f_{j+p}^{(1)}(x)\,dx\right| > C_r^{(1)} \frac{n^{1/4}\log^{1/2} n}{n}\right\} \leqq \frac{n^2}{n^r}. \tag{16}$$

We set $r=3$ in the inequalities (14)—(16). Then it follows from these inequalities that there exist a point $z_0 \in [0, 1]$, a set $\Delta_0 = \Delta(z_0)$ and an absolute constant C^5 such that

$$\begin{aligned} &1)\ \|B'(\Delta_0)\| \leqq C^5 (R\log n)^{1/2}, \\ &2)\ \max_j \left|\int_0^1 f_j^{(2)}(x, \Delta_0) f_{j+p}^{(2)}(x, \Delta_0)\,dx\right| \leqq \gamma_p^{(2)} = \begin{cases} 0 & \text{for } |p| > 2R, \\ \dfrac{C^5 (R\log n)^{1/2}}{n} & \text{for } |p| \leqq 2R; \end{cases} \\ &3)\ \max_j \left|\int_0^1 f_j^{(2)}(x) f_{j+p}^{(1)}(x)\,dx\right| \leqq \gamma_p^{(3)} = \begin{cases} 0 & \text{for } |p| > 2R, \\ \dfrac{C^5 n^{1/4}\log^{1/2} n}{n} & \text{for } |p| \leqq 2R. \end{cases} \end{aligned} \tag{17}$$

We remark that by virtue of (17) and (7) we have

$$\sum_{p=-n}^{n} \gamma_p^{(2)} \leqq R\frac{2C^5 R^{1/2}\log^{1/2} n}{n} < \frac{CR^{3/2}\log^{1/2} n}{n} < Cn^{-1/12} \tag{18}$$

and

$$\sum_{p=-n}^{n} \gamma_p^{(3)} \leqq \frac{2C^5 R n^{1/4}\log^{1/2} n}{n} < Cn^{-1/10}. \tag{19}$$

Let $\chi(E)$ be the characteristic function of the set E. Then

$$\sum_{j=[nx]}^{[nx]+R} f_j^{(2)}(x, \Delta_0) = \sum_{j=R}^{n-R} f_j^{(2)}(x, \Delta_0)\chi\left(\left(0, \frac{j}{n}\right)\right) \equiv \sum_{j=R}^{n-R} f_j^{(3)}(x),$$

where (see (12′), (13))

$$f_j^{(3)}(x) = f_j^{(3)}\left(\frac{i}{n}\right) = b'_{ij} \quad \text{for} \quad x \in \left[\frac{i}{n}, \frac{i+1}{n}\right).$$

Therefore, by virtue of (6) and (17)

$$\left\| \sum_{j=[nx]}^{[nx]+R} f_j^{(2)}(x, \Delta_0) \right\|_{L_2} \leqq C^5 R^{1/2} \log^{1/2} n \leqq C^5 n^{2/5}. \tag{20}$$

Now we define the set of functions $\{\psi_j(x)\}$ by putting

$$\psi_j(x) = f_j^{(1)}(x) + f_j^{(2)}(x, \Delta_0) \quad \text{for} \quad x \in [0, 1] \quad \text{and} \quad R \leqq j \leqq n-R.$$

Using Čebyšev's inequality

$$\mu\{x \in [0, 1] : f(x) > y\} \leqq \frac{1}{y^2} \int_0^1 f^2(x)\, dx,$$

we obtain from (9) and (20) that

$$\mu \left\{ x : \left| \sum_{j=[nx]}^{[nx]+R} \psi_j(x) \right| > C^6 \sqrt{n} \log n \right\} > C^7 > 0. \tag{21}$$

By virtue of (10) and (17) for $i \neq j$ we have

$$\left| \int_0^1 \psi_i(x) \psi_j(x)\, dx \right| = \left| \int_0^1 (f_i^{(1)} f_j^{(1)} + f_i^{(1)} f_j^{(2)} + f_i^{(2)} f_j^{(2)} + f_i^{(2)} f_j^{(1)})\, dx \right| \leqq$$

$$\leqq \gamma_p^{(1)} + \gamma_p^{(3)} + \gamma_p^{(2)} + \gamma_{-p}^{(3)} = \gamma_p^{(0)} \quad \text{where} \quad p = i - j.$$

Besides (see (10), (18) and (19)),

$$\sum_{1 \leqq |p| \leqq n-2R} \gamma_p^0 < M \quad (M \text{ being an absolute constant}). \tag{22}$$

Obviously, we may choose M an integer here. Now we construct the required set of functions $\{\varphi_j(x)\}_{j=1}^n$ $(n = 2^q)$ on the segment $[0, 1]$ by putting

$$\varphi_j(x) = \begin{cases} r_{q+j}(x) & \text{for} \quad x \in [0, 1], 1 \leqq j < R \quad \text{and} \quad n-R < j \leqq n, \\ r_{q+j}(x) & \text{for} \quad x \notin E_j \qquad\qquad \text{and} \quad R \leqq j \leqq n-R, \\ \psi_j(x) & \text{for} \quad x \in E_j \qquad\qquad \text{and} \quad R \leqq j \leqq n-R, \end{cases}$$

where $r_k(x)$ is the kth Rademacher function.

It is clear that $|\varphi_j(x)| = 1$ for $x \in [0, 1]$ and $j = 1, 2, \ldots, n$. Since the functions $\psi_j(x)$ are piece-wise constant it is easy to see that for every $1 \leqq i < j \leqq n$

$$\int_0^1 \varphi_i(x) \varphi_j(x)\, dx = \begin{cases} \int_0^1 \psi_i(x) \psi_j(x)\, dx & \text{for} \quad R \leqq i < j \leqq n-R, \\ 0 & \text{for} \quad \text{other} \quad i \neq j. \end{cases} \tag{23}$$

If $x \in \left[\frac{i}{n}, \frac{i+1}{n}\right)$, then by the definition of $\varphi_j(x)$ we have $\varphi_j(x) = \psi_j(x)$ for all j such that $|j - i| < R$, since for such j the set E_j contains the point x. Therefore

$$\sum_{j=[nx]}^{[nx]+R} \varphi_j(x) = \sum_{j=[nx]}^{[nx]+R} \psi_j(x) \quad \text{for} \quad x \in \sum_{k=R}^{n-R} E_k,$$

and consequently (see (21))

$$\mu\left\{x : \left|\sum_{j=[nx]}^{[nx]+R} \varphi_j(x)\right| > C^6 \sqrt{n}\log n\right\} > C^{(7)}. \tag{24}$$

If we put $\gamma_p=\gamma_p^{(0)}$ for $1\leqq|p|\leqq n-2R$ and $\gamma_p=0$ for other p's, then, as it follows from the relations (22), (23) and from Lemma 2, the functions $\{\varphi_j(x)\}_{j=1}^{n}$ can be extended to the segment $[0, 2M+1]$ in such a way that

1) $|\varphi_j(x)|=1$ for $x\in[0, 2M+1]$ and $j=1,2,\dots;$

2) $\int\limits_0^{2M+1} \varphi_i(x)\varphi_j(x)\,dx=0$ for $i\neq j$.

Since the estimate (24) also holds, the constructed set satisfies all the requirements of the fundamental lemma.

Lemma is completely proved *under the assumption that the estimate* (10) *holds.* Now we prove that estimate. For every j, $R\leqq j\leqq n-R$ we set

$$g_j^{(1)}(x)=\frac{\sqrt{n}}{2(nx-j-\sqrt{n})},\quad g_j^{(2)}(x)=\frac{\sqrt{n}}{2(nx-j+\sqrt{n})}$$

and

$$g_j(x)=\begin{cases} g_j^{(1)}(x) & \text{for } -R\leqq nx-j<0,\\ g_j^{(2)}(x) & \text{for } 0\leqq nx-j\leqq R,\\ 0 & \text{for } |nx-j|>R.\end{cases}$$

As a matter of fact, these are functions constructed by MENŠOV (see [6, p. 110]), but we have modified them so that their supports be of small measure.

The function $g_j(x)$ does not differ much from the function $f_j^{(1)}(x)$. In fact, let k be an integer, then

$$g_j\left(\frac{k}{n}\right)=\begin{cases} 0 & \text{for } |k-j|>R,\\ \dfrac{\sqrt{n}}{2(k-j-\sqrt{n})} & \text{for } -R\leqq k-j<0,\\ \dfrac{\sqrt{n}}{2(k-j+\sqrt{n})} & \text{for } 0\leqq k-j\leqq R.\end{cases} \tag{25}$$

Therefore by virtue of (8), (8′) and (25)

$$g_j\left(\frac{k}{n}\right)=f_j^{(1)}\left(\frac{k}{n}\right)\quad \text{for } k-j\neq R. \tag{26}$$

It is easy to verify that for $k-j\neq R$ and $\frac{k}{n}\leqq x<\frac{k+1}{n}$

$$\left|g_j(x)-g_j\left(\frac{k}{n}\right)\right| \leqq \frac{1}{\sqrt{n}}. \tag{27}$$

Since the functions $f_j^{(1)}(x)$ are piece-wise constant, the relations (26) and (27) supply the following estimate for the function $D_j(x)=f_j^{(1)}(x)-g_j(x)$

$$|D_j(x)| \leqq \frac{1}{\sqrt{n}} \quad \text{for} \quad j=R,\dots,n-R. \tag{28}$$

Suppose that there exists a sequence $\{\gamma_p\}$ such that

(29)

1) $\left|\int\limits_0^1 g_i(x)g_j(x)\,dx\right| \leqq \gamma_{|i-j|}$ for all $i\neq j$, $R\leqq i,j\leqq n-R$;

2) $\sum\limits_{p=1}^{n-1}\gamma_p \leqq M_1$,

where M_1 is an absolute constant.

Then it is easy to deduce (10) from (29). In fact, for $0<|i-j|\leqq 2R$ we have

$$\left|\int\limits_0^1 f_i^{(1)}f_j^{(1)}\,dx\right| = \left|\int\limits_0^1 (g_i+D_i)(g_j+D_j)\,dx\right| \leqq$$

$$\leqq \left|\int\limits_0^1 g_ig_j\,dx\right| + \left|\int\limits_0^1 g_iD_j\,dx\right| + \left|\int\limits_0^1 g_jD_i\,dx\right| + \left|\int\limits_0^1 D_iD_j\,dx\right|.$$

By virtue of (28) we obtain

$$\left|\int\limits_0^1 D_iD_j\,dx\right| \leqq \frac{1}{n}.$$

For every i and j the estimate

$$\left|\int\limits_0^1 g_jD_i\,dx\right| \leqq \frac{1}{\sqrt{n}}\int\limits_0^1 |g_j|\,dx \leqq \frac{C\log n}{n}$$

holds (see (28) and (8′)).

It follows from the definition of the functions $f_j^{(1)}(x)$ that

$$\int\limits_0^1 f_j^{(1)}f_i^{(1)}\,dx = 0 \quad \text{for} \quad |i-j|>2R. \tag{30}$$

Thus

$$\left|\int_0^1 f_i^{(1)} f_j^{(1)}\,dx\right| \leqq \begin{cases} \gamma_{|i-j|}+\dfrac{1}{n}+\dfrac{2C\log n}{n} & \text{for } 1 \leqq |i-j| \leqq 2R, \\ 0 & \text{for } |i-j| > 2R \end{cases}$$

and, consequently, there exist numbers $\gamma_p^{(1)}$ such that

$$\left|\int_0^1 f_i^{(1)} f_j^{(1)}\,dx\right| \leqq \gamma_{|i-j|}^{(1)} \quad \text{for all} \quad R \leqq i, j \leqq n-R \quad \text{and} \quad i \neq j,$$

where

$$\sum_{p=1}^{n-1} \gamma_p^{(1)} \leqq \sum_{p=1}^{2R}\left(\gamma_p+\frac{1}{n}+\frac{2C\log n}{n}\right) \leqq M_1+\frac{4CR\log n}{n} < M.$$

This completes the proof of the estimate (10).

Now we have to prove the estimate (29). To do this we estimate the numbers

$$\alpha_{ij} = \int_0^1 g_i(x) g_j(x)\,dx.$$

We consider three cases.

Case 1. $|i-j|>2R$. By the definition of the functions $g_i(x)$

(31) $$\alpha_{ij} = 0.$$

Case 2. $1 \leqq |i-j| \leqq R$. Let, for example, $i>j$. In this case

$$\alpha_{ij} = \int_{(i-R)/n}^{j/n} g_i^{(1)} g_j^{(1)}\,dx + \int_{j/n}^{i/n} g_i^{(1)} g_j^{(2)}\,dx + \int_{i/n}^{(j+R)/n} g_i^{(2)} g_j^{(2)}\,dx.$$

Note that

(32) $$4\int_{(i-R)/n}^{j/n} g_i^{(1)} g_j^{(1)}\,dx = \frac{\sqrt{n}\cdot\sqrt{n}}{n^2}\int_{(i-R)/n}^{j/n} \frac{dx}{\left[x-\left(\dfrac{i+\sqrt{n}}{n}\right)\right]\left[x-\left(\dfrac{j+\sqrt{n}}{n}\right)\right]} = \frac{1}{i-j}\log\frac{i-j+\sqrt{n}}{\sqrt{n}}+\frac{1}{i-j}\log\frac{j-i+R+\sqrt{n}}{R+\sqrt{n}}.$$

By similar calculation we obtain

(32′) $$4\int_{j/n}^{i/n} g_i^{(1)} g_j^{(2)}\,dx = -\frac{2}{i-j+2\sqrt{n}}\log\frac{i-j+\sqrt{n}}{\sqrt{n}}.$$

Further, the function $g_j^{(1)}(x)$ is the translate of the function $g_i^{(1)}(x)$ along the x-axis by $\dfrac{i-j}{n}$ (a similar relation holds for the functions $g_j^{(2)}(x)$ and $g_i^{(2)}(x)$).

Using this fact, we easily prove that

$$(32'')\qquad \int_{i/n}^{(j+R)/n} g_i^{(2)} g_j^{(2)}\,dx = \int_{(i-R)/n}^{j/n} g_i^{(1)} g_j^{(1)}\,dx.$$

The latter integral has already been calculated.

Thus by virtue of (32), (32′) and (32″) we obtain

$$4\alpha_{ij} = \frac{2}{i-j}\log\frac{i-j+\sqrt{n}}{\sqrt{n}} - \frac{2}{i-j+2\sqrt{n}}\log\frac{i-j+\sqrt{n}}{\sqrt{n}} + \frac{2}{i-j}\log\frac{j-i+R+\sqrt{n}}{R+\sqrt{n}}.$$

Let

$$\gamma_{i-j}^{(2)} = \left|\frac{2}{i-j}\log\frac{i-j+\sqrt{n}}{\sqrt{n}} - \frac{2}{i-j+2\sqrt{n}}\log\frac{i-j+\sqrt{n}}{\sqrt{n}}\right|,$$

$$\gamma_{i-j}^{(3)} = \frac{2}{i-j}\left|\log\frac{j-i+R+\sqrt{n}}{R+\sqrt{n}}\right|.$$

Then (see (7))

$$\sum_{p=1}^{R}\gamma_p^{(2)} \leqq \sum_{p=1}^{n-1}\gamma_p^{(2)} = \sum_{p=1}^{n-1}\frac{4\sqrt{n}}{p(p+2\sqrt{n})}\log\frac{p+\sqrt{n}}{\sqrt{n}}.$$

MENŠOV showed (see [6, formulae (3.8), (3.11) and (3.14)]) that the latter sum is bounded by an absolute constant, i.e.,

$$\sum_{p=1}^{n-1}|\gamma_p^{(2)}| < M_3.$$

Since

$$\gamma_p^{(3)} = \frac{2}{p}\left|\log\left(1-\frac{p}{R+\sqrt{n}}\right)\right|,$$

by Lemma 5 we obtain

$$\sum_{p=1}^{R}\gamma_p^{(3)} = 2\sum_{p=1}^{R}\frac{1}{p}\left|\log\left(1-\frac{p}{R+\sqrt{n}}\right)\right| \leqq 2M_2.$$

Hence $|\alpha_{ij}|\leqq\gamma_p^{(2)}+\gamma_p^{(3)}=\gamma_p$ for $1\leqq|i-j|\leqq R$, where

$$(33)\qquad \sum_{p=1}^{R}\gamma_p \leqq M_3+2M_2.$$

Case 3. $R<|i-j|\leqq 2R$. In this case

$$4\alpha_{ij} = 4\int_{(i-R)/n}^{(j+R)/n} g_j^{(2)} g_i^{(1)}\,dx = \frac{1}{n}\int_{(i-R)/n}^{(j+R)/n}\frac{dx}{\left(x-\left(\frac{i+\sqrt{n}}{n}\right)\right)\left(x-\left(\frac{j-\sqrt{n}}{n}\right)\right)} =$$

$$= \frac{2}{i-j+2\sqrt{n}}\log\left(\frac{i-j+2\sqrt{n}}{R+\sqrt{n}}-1\right)$$

for $i>j$. Therefore $|\alpha_{ij}|=\gamma_{i-j}$ $(R<i-j\leqq 2R)$, where

$$\gamma_p = \frac{2}{2(p+2\sqrt{n})}\left|\log\left(\frac{p+2\sqrt{n}}{R+\sqrt{n}}-1\right)\right|.$$

Now we estimate the sum

$$S = \sum_{p=R+1}^{2R}\gamma_p = \frac{1}{2}\sum_{p=R+1}^{2R}\frac{1}{p+2\sqrt{n}}\left|\log\left(\frac{p+2\sqrt{n}}{R+\sqrt{n}}-1\right)\right| \leqq$$

$$\leqq \frac{1}{R}\sum_{p=R+1}^{2R}\left|\log\left(\frac{p+2\sqrt{n}}{R+\sqrt{n}}-1\right)\right|.$$

We put $q=2R-p$, then

$$\frac{p+2\sqrt{n}}{R+\sqrt{n}}-1 = 1-\frac{q}{R+\sqrt{n}}.$$

Consequently

$$S \leqq \frac{1}{R}\sum_{q=0}^{R-1}\left|\log\left(1-\frac{q}{R+\sqrt{n}}\right)\right| \leqq \sum_{q=1}^{R-1}\frac{1}{q}\left|\log\left(1-\frac{q}{R+\sqrt{n}}\right)\right|.$$

By Lemma 5 the latter sum does not exceed M_2, i.e.,

$$\sum_{p=R+1}^{2R}\gamma_p \leqq M_2. \tag{34}$$

Combining the estimates of cases 1—3 (see (31), (33) and (34)) we obtain

$$\left|\int_0^1 g_i(x)\,g_j(x)\,dx\right| \leqq \gamma_{|i-j|}$$

for all $i\neq j$ and $R\leqq i, j\leqq n-R$, where $\sum_{p=1}^{n-1}\gamma_p<M_1$. This is the required estimate (29), thus the fundamental lemma is completely proved.

Now we show how the set satisfying the requirements of Theorem 1 is constructed.

Denote by $\{\varphi_m^q(x)\}_{m=1}^{2^q}$ the set of functions satisfying the conditions of the fundamental lemma with the given q and with the variable mapped homothetically onto the segment [0, 1]. We construct the infinite set of functions $\{\psi_n(x)\}_{n=3}^{\infty}$ by putting

$$\psi_n(x) = \varphi_m^q(x) \quad \text{for} \quad n = 2^q+m, \quad 1 \leqq m \leqq 2^q.$$

Using Lemma 6 it is easy to prove that there exists a sequence of functions $f_q(x)$, $q=1, 2, \ldots$, such that $|f_q(x)|\equiv 1$ and the set $\{\varphi_n(x)\}_{n=3}^{\infty}$

$$\varphi_n(x) = f_q(x)\psi_n(x), \quad 2^q < n \leqq 2^{q+1}, \quad q = 1, 2, \ldots,$$

is orthonormal on the segment [0, 1].

To prove that the set $\{\varphi_n(x)\}$ satisfies all the requirements of Theorem 1 it is sufficient for every sequence $\{\beta_n\}$ such that

$$\beta_n = o(\log^2 n), \quad \beta_n \uparrow \infty, \quad n \to \infty, \tag{35}$$

to find a series (2) diverging on a set of positive measure, while

$$\sum_{n=3}^{\infty} c_n^2 \beta_n < \infty.$$

For this purpose we construct the sequence of integers q_j in such a way that $10 \leqq q_1 < q_2 < \dots$ and (see (35))

$$0 < \beta_{2^{q_j}}^{1/2} \leqq \frac{1}{j^3} q_j, \quad j = 1, 2, \dots. \tag{36}$$

We put

$$c_n \leqq \begin{cases} (2^{q_j} \beta_{2^{q_j}})^{-1/2} j^{-1} & \text{for} \quad 2^{q_j-1} < n \leqq 2^{q_j}, \quad j = 1, 2, \dots; \\ 0 & \text{for other} \quad n. \end{cases}$$

By virtue of the estimate (36) and the requirement 3) of the fundamental lemma we easily obtain

$$\mu\left\{x \in [0, 1] : \left| \sum_{n=N_j(x)}^{N'_j(x)} c_n \varphi_n(x) \right| \geqq C_1 j \right\} \geqq C_2,$$

where $2^{q_j-1} < N_j(x) \leqq N'_j(x) \leqq 2^{q_j}$ and $j = 2, 3, \dots$.

It follows from the latter estimate that the series (2) diverges on a set of positive measure. Besides,

$$\sum_{n=3}^{\infty} c_n^2 \beta_n \leqq \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.$$

The proof of Theorem 1 is completed.

Remark 2. While proving Theorem 1 we actually obtained the following intermediate result.

Let the numbers n *and* R *be as in the fundamental lemma* (see (7)). *Then there exists a sequence* $\{\varepsilon_k\}_{k=-R}^{R}$, $\varepsilon_k = \pm 1$, *such that*

$$\begin{aligned} &1) \quad \sum_{k=0}^{R} \varepsilon_k \geqq C_1 \sqrt{n} \log n, \\ &2) \quad \left\| \sum_{k=-R}^{R} \varepsilon_k e^{ikt} \right\|_C \leqq C_2 \sqrt{n}, \end{aligned} \tag{37}$$

where C_1 *and* C_2 *are positive absolute constants.*

It is well known (see [3]) that for every integer there exists a sequence $\{\delta_k\}_{k=1}^m$, $\delta_k=\pm 1$, such that

$$\left\|\sum_{k=1}^{m}\delta_k e^{ikt}\right\|_C \leqq C_3\sqrt{m}. \tag{38}$$

Using (37) and (38) it is not difficult to construct for every given integer r a polynomial $P_r(t)=\sum_{k=1}^{2r}\varepsilon_k e^{ikt}$, $\varepsilon_k=\pm 1$, such that

(39)
1) $\|P_r(t)\|_C \leqq C\sqrt{r}$,

2) the partial sum $S_r(P_r, 0)\geqq C_1\sqrt{r}\log r$.

Denote by $P_1(r, m, t)$ the following trigonometric polynomial

$$P_1(r, m, t) = \operatorname{Re}\left(\frac{1}{\sqrt{2r}}P_r(t)e^{imt}\right) = \sum_{k=1}^{2r}\frac{\varepsilon_k}{\sqrt{2r}}\cos(m+k)t.$$

It follows from (39) that for any integers r and m

$$\|P_1(r, m, t)\|_C \leqq C, \quad S_{m+r}(P_1(r, m, t), 0) \geqq C_1\log r. \tag{40}$$

These properties of the polynomials $P_1(r, m, t)$ are analogous to those of Fejér's polynomials. However, all non-zero coefficients in $P_1(r, m, t)$ have the same modulus. Using these polynomials instead of Fejér's polynomials and repeating the routine reasonings (see [10, p. 477, Theorem (2.1)]) we obtain the following result.

There exists a continuous function $f(x)$ on $[-\pi, \pi]$ such that

1) $f(x) = \sum_{k=1}^{\infty} a_k\cos kx$,

2) $\omega(f, \delta) = O\left(\dfrac{1}{\log\frac{1}{\delta}}\right)$,

3) $|a_k|\downarrow 0$, *as* $k\to\infty$,

4) *the Fourier series of $f(x)$ diverges at the origin.*

This result is a sharpening of a theorem due to Salem (see [8]), who obtained it without requirement 2).

References

[1] Г. Харди, Д. Литтльвуд и Г. Полиа, *Неравенства,* Иностранная литература (Москва, 1948).

[2] С. Качмаж и Г. Штейнгауз, *Теория ортогональных рядов,* Физматгиз (Москва, 1958).

[3] J. P. Kahane, *Séries de Fourier absolument convergentes,* Springer (Berlin—Heildelberg—New York, 1970).

[4] Б. С. Кашин, О множителях Вейля, *Докл. АН СССР,* **222** (1975), 1031—1032.
[5] A. Kolmogoroff et D. Menchoff, Sur la convergence des séries de fonctions orthogonales, *Math., Z.,* **26** (1927), 432—441.
[6] D. Menchoff, Sur les séries des fonctions orthogonales bornées dans leur ensemble, *Матем. сб.*, **3** (1938), 103—120.
[7] В. В. Петров, *Суммы независимых случайных величин,* Наука (Москва, 1972).
[8] R. Salem, On a problem of Smithies, *Nederl. Akad. Wetensch. Indag. Math.,* **16** (1954), 403—407.
[9] K. Tandori, Eine Bemerkung zum Konvergenzproblem der Orthogonalreihen, *Acta Math. Acad. Sci. Hungar.,* **20** (1969), 314—322.
[10] А. Зигмунд, *Тригонометрические ряды.* I, Мир (Москва, 1965).

О множителях Вейля для сходимости почти всюду ортогональных рядов

Б. С. КАШИН

Изучается сходимость почти всюду ортогональных рядов. Основным результатом является следующая теорема.

Теорема. *Существует ортонормированная на отрезке* [0, 1] *система функций* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *такая, что* $|\varphi_n(x)|=1$, $x\in[0, 1]$, $n=1, 2, \ldots$ *и для любой последовательности* $\beta_n=o(\log^2 n)$ *найдется ряд*

$$\sum_{n=1}^{\infty} c_n\varphi_n(x)$$

который расходится на множестве положительной меры, хотя

$$\sum_{n=1}^{\infty} c_n^2\beta_n < \infty.$$

Б. С. КАШИН
СССР, МОСКВА 117333
УЛ. ВАВИЛОВА 42
МАТЕМАТИЧЕСКИЙ ИНСТИТУТ
ИМ. В. А. СТЕКЛОВА АН СССР