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SHORT COMMUNICATIONS

On a Special Orthogonal Decomposition of the Space $L^2(0,1)$

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In this paper, we establish the following result.

Theorem. *There exists a decomposition of the space $L^2(0,1)$ into the orthogonal direct sum of subspaces E_1 and E_2:*

$$L^2(0,1) = E_1 \oplus E_2$$

and an absolute constant K such that, for any function $f \in L^2(0,1)$, $\|f\|_2 \le 1$, there exist functions $g_1 \in E_1$, $g_2 \in E_2$ with

$$\|f - g_1\|_\infty \le K, \qquad \|f - g_2\|_\infty \le K.$$

Here and elsewhere, let $\|\cdot\|_p$ denote the norm on the space $L^p(0,1)$, $1 \le p \le \infty$. In addition, we use the following notation:

- $\operatorname{span}\{f_i\}$ is the linear hull of the set of functions $\{f_i\}$;
- D_N is the N-dimensional space of piecewise constant functions:

$$D_N = \left\{ f \in L^2(0,1) : f(x) = \text{const},\ x \in \left(\frac{i-1}{N}, \frac{i}{N}\right),\ 1 \le i \le N \right\},$$

$\{\chi_k^i,\ k = 0,1,\dots,\ i = 1,\dots,2^k,\ \chi_0^0 \equiv 1\}$ is the Haar system (see [1, p. 70]).

The theorem complements the result obtained in [2], [3] (see also [4]), on the existence of an orthogonal decomposition

$$L^2(0,1) = G_1 \oplus G_2,$$

where, for $g \in G_1 \cup G_2$, $\|g\|_1 \ge æ\|g\|_2$, and $æ > 0$ is an absolute constant. In turn, the last result was derived in [2], [3] from the decompositions

$$\begin{gathered} D_{2N} = G_1^{(N)} \oplus G_2^{(N)}, \qquad \dim G_1^N = \dim G_2^{(N)} = N, \\ \|g\|_1 \ge æ\|g\|_2, \quad g \in G_1^{(N)} \cup G_2^{(N)}, \quad N = 1,2,\dots, \quad æ > 0, \end{gathered} \tag{1}$$

constructed in [5].

Remark. Obviously, in the theorem, $K \ge 1$. We can verify that, in fact, the inequality is strict: $K > 1$.

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Proof of the theorem. Let $H_0 = \operatorname{span}\{\chi_0^0, \chi_0^1\}$ and, for $k = 1, 2, \dots$,

$$H_k = \operatorname{span}\{\chi_k^i,\ 1 \le i \le 2^k\}.$$

Obviously, for $k = 1, 2, \dots$, $H_0 \oplus \dots \oplus H_k = D_{2^{k+1}}$. It immediately follows from (1) that there exists an orthogonal decomposition

$$H_k = E_1^{(k)} \oplus E_2^{(k)}$$

such that $æ\|f\|_2 \le \|f\|_1 \le \|f\|_2$, $f \in E_1^{(k)} \cup E_2^{(k)}$. Let

$$E_1 = E_1^{(0)} \oplus E_1^{(1)} \oplus \cdots, \qquad E_2 = E_2^{(0)} \oplus E_2^{(1)} \oplus \cdots .$$

In proving the existence of the decomposition (1) in [3] (see also [4]) it was actually verified that

$$æ'\|f\|_2 \le \|f\|_1 \le \|f\|_2, \qquad f \in E_1 \cup E_2, \quad æ' > 0. \tag{2}$$

For the completeness of exposition, let us present this argument. Let there be given a function f from $L^2(0,1)$ of the form

$$f = \sum_{k=0}^{\infty} f_k, \qquad f_k \in E_k^i, \quad i = 1, 2, \quad k = 0, 1, \dots .$$

First, we verify that $\|f\|_2 \le C\|f\|_{3/2}$. Indeed, by the classical Paley–Marcinkiewicz theorem (see [1, Theorem 3.9]) applied to the case $p = 3/2$, we have

$$\|f\|_{3/2} \ge c\left\|\left(\sum_{k=0}^{\infty} f_k^2\right)^{1/2}\right\|_{3/2}, \qquad c > 0;$$

hence, using the well-known inequality

$$\left\|\left(\sum_{k=0}^{\infty} f_k^2\right)^{1/2}\right\|_{3/2} \ge \left(\sum \|f_k\|_{3/2}^2\right)^{1/2}$$

valid for any set of functions in $L^{3/2}(0,1)$, and estimate (1), we can write

$$\|f\|_{3/2} \ge c\left(\sum_{k=0}^{\infty} \|f_k\|_1^2\right)^{1/2} \ge cæ\left(\sum_{k=0}^{\infty} \|f_k\|_2^2\right)^{1/2} = cæ\|f\|_2.$$

Combining the last relation and the inequalities $\|f\|_1^{1/3}\|f\|_2^{2/3} \ge \|f\|_{3/2}$, we obtain estimate (2).

In view of the duality of the Kolmogorov and Gelfand widths (see, for example, [6, p. 147]), it follows from (2) that, for any function $F \in D_{2^{k+1}}$, $k = 0, 1, \dots$, there exist functions

$$g_j \in E_j^{(0)} \oplus \dots \oplus E_j^{(k)}, \qquad j = 1, 2,$$

with

$$\|F - g_1\|_\infty \le \widetilde{K}\|F\|_2, \qquad \|F - g_2\|_\infty \le \widetilde{K}\|F\|_2 \tag{3}$$

($\widetilde{K}$ is an absolute constant).

Let us verify that the spaces E_1, E_2 satisfy the requirements of the theorem. For an arbitrary function $f \in L^2(0,1)$, $\|f\|_2 \le 1$, consider its Fourier–Haar series

$$f = a_0^0\chi_0^0 + \sum_{k=0}^{\infty} \sum_{i=1}^{2^k} a_k^i \chi_k^i. \tag{4}$$

For an increasing sequence of natural numbers k_ν, $\nu = 1, 2, \dots$, also consider the grouped series (4):

$$f = \sum_{\nu=1}^{\infty} w_\nu, \qquad \text{where} \quad w_1 = a_0^0 + \sum_{k=0}^{k_1} \sum_{i=1}^{2^k} a_k^i \chi_k^i, \quad w_\nu = \sum_{k=k_{\nu-1}+1}^{k_\nu} \sum_{i=1}^{2^k} a_k^i \chi_k^i, \ \nu > 1.$$

Choose the sequence $\{k_\nu\}$ we choose increasing so rapidly that

$$\sum_{\nu=1}^{\infty} \|w_\nu\|_2 < 2. \tag{5}$$

Since $w_\nu \in D_{2^{k_\nu+1}}$, in view of (3), there exist

$$g_\nu^{(j)} \in E_j^{(0)} \oplus E_j^{(1)} \oplus \dots \oplus E_j^{(k_\nu)}, \qquad j = 1, 2,$$

with

$$\|w_\nu - g_\nu^{(j)}\|_\infty \le \widetilde{K} \|w_\nu\|_2, \qquad j = 1, 2, \quad \nu = 1, 2, \dots . \tag{6}$$

Then

$$\|g_\nu^{(j)}\|_2 \le \|w_\nu\|_2 + \|w_\nu - g_\nu^{(j)}\|_2 \le (\widetilde{K} + 1)\|w_\nu\|_2, \qquad j = 1, 2.$$

Therefore, for $j = 1, 2$, the series

$$\sum_{\nu=1}^{\infty} g_\nu^{(j)}$$

converge absolutely in $L^2(0,1)$ to the functions $g_j \in E_j$. In this case (see (6)), for $j = 1, 2$,

$$\|f - g_j\|_\infty \le \sum_{\nu=1}^{\infty} \|w_\nu - g_\nu^{(j)}\|_\infty \le \widetilde{K} \sum_{\nu=1}^{\infty} \|w_\nu\|_2 < 2\widetilde{K} \equiv K,$$

which proves the assertion. □

The subspace E_1, E_2 constructed in the theorem can be regarded as a pair of "state" spaces such that there exists a sufficiently good uniform approximation of each "state" from E_1 by some "state" from the "virtual" space E_2. In this connection, it would be interesting to find out whether the best possible constant K in the theorem has any physical meaning.

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