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Some remarks on the restriction of operators to coordinate subspaces

by

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The present paper contains some results which complement the research done in [B-T1], [B-T2], [K] and [L].

We consider below $q \times n$ matrices $A = (a_{i,j})_{i=1,\ j=1}^{q\ \ \ n}$, as linear operators acting from a q-dimensional space ℓ_p^q to an n-dimensional space ℓ_r^n, for various values of $1 \leq p, r \leq \infty$.

In the papers mentioned above and in other articles, the authors obtain results concerning the possibility of "improving" the properties of the operator A by restricting its range in the corresponding coordinate space $\mathbb{R}^n$. In the language of matrices, one considers the possibility of selecting in A a submatrix A' of size $q \times m$, with $m \leq n$, which has additional properties. For instance, one such situation studied below is when A' is selected as to have the smallest possible norm among all the submatrices of A of the given size $q \times m$. Another case of interest is when A' is selected as to minimize the expression

$$M(A') = \max_{x \neq 0} \frac{||x||}{||A'_x||}.$$

The first case (i.e. when A' has the smallest possible norm) is subsequently used below in order to improve in the case of square matrices of small rank, a suppression theorem proved in [B-T2]. The other case is connected with a problem appearing naturally in function theory, namely to find for a given orthonormal system $\{\varphi_i(x)\}_{i=1}^q$ in the space L_2 (X, Σ, μ) a finite set of points $\{x_j\}_{j=1}^m \subset X$ of smallest possible cardinality m such that the discrete system

$$\{(\varphi_i(x_1), \varphi_i(x_2), \ldots, \varphi_i(x_m))\}_{i=1}^q$$

of vectors in $\mathbb{R}^m$ is "close" to being an orthogonal system.

We begin with a result on flat matrices, whose proof is based on a slight modification of the argument used to prove Proposition 1.6 in [B-T2], which in turn relies on a method developed in [G-Z].

THEOREM 1. *There exists a constant $D < \infty$ such that, whenever $n \geq D$, $1 \leq q \leq n$, A is a linear operator from ℓ_2^q to ℓ_2^n, and $\{\xi_i\}_{i=1}^n$ is a sequence of $\{0,1\}$-valued independent random variables of mean δ for some $0 < \delta < 1$ over some probability space (Ω, Σ, μ), then*

$$\int_\Omega ||R_{\sigma(\omega)}A||_{2\to 1} d\mu \leq D(\delta n^{1/2} + (\delta q)^{1/2})||A||_{2\to 2},$$

where, for $\omega \in \Omega$, $\sigma(\omega) = \{1 \le i \le n,\ \xi_i(\omega) = 1\}$.

PROOF: Suppose that $Ae_i = \sum_{j=1}^{n} a_{i,j} e_j$, $1 \le i \le q$. Then, for $\omega \in \Omega$ and $x = (x_i)_{i=1}^q \in \ell_2^q$, we have that

$$R_{\sigma(\omega)} Ax = \sum_{j=1}^{n} \xi_j(\omega) \left(\sum_{i=1}^{q} a_{ij} x_i \right) e_j .$$

Hence,

$$I = \int_\Omega ||R_\sigma(\omega)A||_{2\to 1} d\mu = \int_\Omega \sup_{x\in\ell_2^q,\ ||x||_2\le 1} \sum_{j=1}^{n} \xi_j(\omega) \left| \sum_{i=1}^{q} a_{i,j} x_i \right| d\mu \le$$

$$\le \delta \int_\Omega \sup_{x\in\ell_2^q,\ ||x||_2\le 1} \sum_{j=1}^{n} \left| \sum_{j=1}^{q} a_{i,j} x_i \right| d\mu + \int_\Omega \sup_{x\in\ell_2^q,\ ||x||_2\le 1} \left| \sum_{j=1}^{n} (\xi_j(\omega) - \delta) \left| \sum_{i=1}^{q} a_{ij} x_i \right| \right| d\mu \le$$

$$\le \delta ||A||_{2\to 1} + \int_\Omega \int_{\Omega'} \sup_{x\in\ell_2^q,\ ||x||_2\le 1} \left| \sum_{j=1}^{n} (\xi_j(\omega) - \xi_j'(\omega')) \left| \sum_{i=1}^{q} a_{i,j} x_i \right| \right| d\mu d\mu',$$

where $\{\xi_j'\}_{j=1}^n$ is a copy of $\{\xi_j\}_{j=1}^n$ over a probability space (Ω', Σ', μ'), independent of (Ω, Σ, μ). It follows that

$$I \le \delta n^{1/2} ||A||_{2\to 2} + 2 \iint_\Omega \sup_{x\in\ell_2^q,\ ||x||_2\le 1} \left| \sum_{j=1}^{n} \varepsilon_j \xi_j(\omega) \left| \sum_{i=1}^{q} a_{i,j} x_i \right| \right| d\mu d\varepsilon.$$

Let now $\{g_j(\omega'')\}_{j=1}^n$ be a sequence of independent Gaussian random variables over a third probability space $(\Omega'', \Sigma'', \mu'')$, independent of (Ω, Σ, μ). Then, by Slepian's lemma (cf. [B-T2], Lemma 1.5) and Cauchy-Schwartz inequality, we conclude that

$$I \le \delta n^{1/2}||A||_{2\to 2} + \sqrt{2\pi}\left(\int_\Omega\int_{\Omega''} \sup_{x\in\ell_2^q,\ ||x||_2\le 1}\left|\sum_{j=1}^n g_j(\omega'')\xi_j(\omega)\left|\sum_{i=1}^q a_{i,j}x_i\right|\right|^2 d\mu d\mu''\right)^{1/2} \le$$

$$\le \delta n^{1/2}||A||_{2\to 2} + 4\sqrt{\pi}\left(\int_\Omega\int_{\Omega''} \sup_{x\in\ell_2^q,\ ||x||_2\le 1}\left|\sum_{j=1}^n g_j(\omega'')\xi_j(\omega)\sum_{i=1}^q a_{i,j}x_i\right|^2 d\mu d\mu''\right)^{1/2} \le$$

$$\le \delta n^{1/2}||A||_{2\to 2} + 4\sqrt{\pi}\left(\int_\Omega\int_{\Omega''} \sum_{i=1}^q\left|\sum_{j=1}^n g_j(\omega'')\xi_j(\omega)a_{i,j}\right|^2 d\mu d\mu''\right)^{1/2} \le$$

$$\le \delta n^{1/2}||A||_{2\to 2} + 4\sqrt{\pi}\left(\int_\Omega \sum_{i=1}^q\sum_{j=1}^n \xi_j(\omega)|a_{i,j}|^2 d\mu\right)^{1/2} \le$$

$$\le \delta n^{1/2}||A||_{2\to 2} + 2\sqrt{\pi\delta}\left(\sum_{i=1}^q\sum_{j=1}^n |a_{ij}|^2\right)^{1/2}.$$

But

$$\left(\sum_{i=1}^q\sum_{j=1}^n |a_{i,j}|^2\right)^{1/2} \le ||A||_{HS} \le q^{1/2}||A||_{2\to 2}$$

where $||A||_{HS}$ denotes as usual the Hilbert-Schmidt of A. This yields that

$$\int ||R_{\sigma(\omega)}A||_{2\to 1}d\mu \le [\delta n^{1/2} + 4(\pi\delta q)^{1/2}]||A||_{2\to 2},$$

thus completing our proof. □

Corollary 2. *There exists a constant $D_1 < \infty$ so that, whenever n is an integer, $1/n < \delta < 1$, $1 \le q \le n$, and A is a linear operator from ℓ_2^q to ℓ_2^n, then there exists a set $\eta \subset \{1, 2, \ldots, n\}$ of cardinality $|\eta| > \delta n/4$ so that*

$$||R_\eta A||_{2\to 2} \le D_1\left(\delta^{1/2} + \left(\frac{q}{n}\right)^{1/2}\right)||A||_{2\to 2}.$$

Proof: As usual, we use the Grothendieck factorization in the following way. Since, for large n,

$$\mu\left\{\omega\in\Omega; \left|\sum_{i=1}^n(\xi_i - \delta)\right| > \delta n/2\right\}$$

is near 0, we can easily find a point $\omega_0 \in \Omega$ and a constant D depending on D_1 so that the set $\sigma = \sigma(\omega_0)$ has cardinality $|\sigma| > \delta n/2$ and

$$||R_\sigma A||_{2\to 1} \leq D(\delta n^{1/2} + (\delta q)^{1/2})||A||_{2\to 2}.$$

Since any operator from $\ell_\infty^{|\sigma|}$ into ℓ_2^q can be factorized through $\ell_2^{|\sigma|}$, one deduces for the operator $A^* R_\sigma : \ell_\infty^{|\sigma|} \to \ell_2^2$ that there exists $U : \ell_2^{|\sigma|} \to \ell_2^q$ with

$$||U||_{2\to 2} \leq \sqrt{2} D \left(\delta^{1/2} + \left(\frac{q}{n}\right)^{1/2} \right) ||A||_{2\to 2}$$

and a diagonal operator $V : \ell_\infty^{|\sigma|} \to \ell_2^{|\sigma|}$ so that $A^* R_\sigma = UV$ and

$$||V||_{\infty\to 2} = \left(\sum_{i\in\sigma} |v_i|^2 \right)^{1/2} \leq \left(\frac{\delta n}{2} \right)^{1/2},$$

where $Ve_i = v_i e_i;\ i \in \sigma$.

Hence, $R_\sigma A = V^* U^*$ so if we put

$$\eta = \{ i \in \sigma \ ;\ |v_i| < \sqrt{2} \}$$

then, for any $x = (x_i)_{i=1}^q \in \ell_2^q$, we have that

$$||R_\eta Ax||_2 = ||R_\eta V^* U_x^*||_2 \leq \sqrt{2}||U^* x||_2 \leq 2D \left(\delta^{1/2} + \left(\frac{q}{n}\right)^{1/2} \right) ||x||_2,$$

i.e.

$$||R_\eta A||_{2\to 2} \leq 2D \left(\delta^{1/2} + \left(\frac{q}{n}\right)^{1/2} \right) ||A||_{2\to 2}.$$

Furthermore,

$$2(|\sigma| - |\eta|) \leq \sum_{i\in\sigma\sim\eta} |v_i|^2 \leq \delta n/2$$

i.e.

$$|\eta| \geq \delta n/4.$$

□

We shall apply now Theorem 1 to the study of square matrices of small rank. First, we prove the following result.

THEOREM 3. *There exists a constant $D_2 < \infty$ so that, whenever $n \geq D_2$, $0 < \delta < 1$, $\{\xi_j\}_{j=1}^n$ is a sequence of $\{0,1\}$-valued independent random variables of mean δ over some probability space (Ω, Σ, μ), $\{\xi_j'\}_{j=1}^n$ a copy of $\{\xi_j\}_{j=1}^n$ over a probability space (Ω', Σ', μ'), independent of (Ω, Σ, μ), $1 \leq q \leq n$, $A : \ell_2^n \to \ell_2^q$ with $||A||_{2\to 2} \leq 1$ and $B : \ell_2^q \to \ell_2^n$ with $||B||_{2\to 2} \leq 1$, then the operator $C = BA : \ell_2^n \to \ell_2^n$ satisfies*

$$\int_\Omega \int_{\Omega'} ||R_{\sigma(\omega)} C R_{\sigma(\omega')}||_{\infty\to 1} d\mu d\mu' \leq D_2(\delta^2 n + \delta^{3/2}(qn)^{1/2}).$$

PROOF: As in the proof of Theorem 1, we notice that

$$J = \int_\Omega \int_{\Omega'} \sup_{x \in \ell_\infty^n,\ ||x||_\infty \leq 1} \sum_{j=1}^n \xi_j(\omega) \left| \sum_{i=1}^n c_{i,j} x_i \xi_i(\omega') \right| d\mu d\mu',$$

where $(c_{i,j})_{i,j=1}^n$ is the matrix defined by $Ce_i = \sum_{j=1}^n c_{i,j} e_j$; $1 \leq i \leq n$. Then

$$J \leq \delta J_1 + J_2,$$

where

$$J_1 = \int_{\Omega'} \sup_{x \in \ell_\infty^n,\ ||x||_\infty \leq 1} \sum_{j=1}^n \left| \sum_{i=1}^n c_{i,j} x_i \xi_i(\omega') \right| d\mu'$$

and

$$J_2 = \int_\Omega \int_{\Omega'} \sup_{x \in \ell_\infty^n,\ ||x||_\infty \leq 1} \left| \sum_{j=1}^n (\xi_j(\omega) - \delta) \left| \sum_{i=1}^n c_{i,j} x_i \xi_i(\omega') \right| \right| d\mu d\mu'.$$

In order to estimate J_1, we observe that

$$J_1 = \int_{\Omega'} ||C R_{\sigma(\omega')}||_{\infty\to 1} d\mu' \leq ||B||_{2\to 1} \int_{\Omega'} ||A R_{\sigma(\omega')}||_{\infty \to 2} d\mu' \leq$$
$$\leq n^{1/2} ||B||_{2\to 2} \int_{\Omega'} ||R_{\sigma(\omega')} A^*||_{2\to 1} d\mu'.$$

Hence, by using Theorem 1 above and our assumptions, we conclude that,

$$J_1 \leq n^{1/2} D(\delta n^{1/2} + (\delta q)^{1/2}) = D(\delta n + (\delta q n)^{1/2}).$$

The estimation of J_2 is done exactly as in the proof of Theorem 1. By using the same notation as there, we get that

$$J_2 \leq 2 \int \int_{\Omega} \int_{\Omega'} \sup_{x \in \ell_\infty^n,\ ||x||_\infty \leq 1} \left| \sum_{k=1}^{n} \varepsilon_j \xi_j(\omega) \left| \sum_{i=1}^{n} c_{i,j} x_i \xi_i(\omega') \right| \right| d\mu d\mu' d\varepsilon \leq$$

$$\leq 4(\pi)^{1/2} \int_{\Omega} \int_{\Omega'} \int_{\Omega''} \sum_{i=1}^{n} \xi_i(\omega') \left| \sum_{j=1}^{n} g_j(\omega'') \xi_j(\omega) c_{i,j} \right| d\mu d\mu' d\varepsilon \leq$$

$$\leq 4\delta(2\pi)^{1/2} \sum_{i=1}^{n} \int_{\Omega} \left(\sum_{j=1}^{n} \xi_j(\omega) |c_{i,j}|^2 \right)^{1/2} d\mu \leq$$

$$\leq 4\delta^{3/2}(2\pi)^{1/2} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |c_{i,j}|^2 \right)^{1/2} \leq 4\delta^{3/2} n^{1/2} (2\pi)^{1/2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |c_{i,j}|^2 \right)^{1/2} \leq$$

$$\leq 4\delta^{3/2} n^{1/2} (2\pi)^{1/2} ||C||_{HS} \leq 4\delta^{3/2} n^{1/2} q^{1/2} (2\pi)^{1/2} ||C||_{2\to 2} =$$

$$= 4\delta^{3/2}(2\pi q n)^{1/2},$$

since C is of rank q.

Now, by combining the above estimates, we obtain the result. □

COROLLARY 4. *There exists a constant $C < \infty$ so that, whenever n is an integer, $1/n < \delta < 1$, $1 \leq q \leq n$ and A is a linear operator from ℓ_2^n into itself whose rank is $\leq q$, then*

$$\min_{|\eta'|, |\eta''| \geq \delta n/4} ||R_{\eta'} A R_{\eta''}||_{2\to 2} \leq C \left(\delta + \left(\frac{\delta q}{n} \right)^{1/2} \right).$$

PROOF: If rank A is $\leq q$ then in the decomposition $A = S_1 + iS_2$, with $S_1 = (A + A^*)/2$ and $S_2 = (A - A^*)/2i$, we have two selfadjoint operators of rank $\leq 2q$. A simple diagonalization argument yields the existence of unitary operators $U_i : \ell_2^n \to \ell_2^n$; $i = 1, 2$, so that $U_i S_i U_i^* = V_i$, where V_i are diagonal operators on ℓ_2^n with real entries and rank $V_i \leq 2q$; $i = 1, 2$. If $V_i = V_i^+ - V_i^-$, with V_i^+ and V_i^- being diagonal operators of rank $\leq 2q$ having non-negative entries only, then

$$S_i = U_i^* V_i^+ U_i - U_i^* V_i^- U_i = (\sqrt{V_i^+} U_i)^* (\sqrt{V_i^+} U_i) - (\sqrt{V_i^-} U_i)^* (\sqrt{V_i^-} U_i); \quad i = 1, 2.$$

Since $\sqrt{V_i^+}U_i$ and $\sqrt{V_i^-}U_i$ are in fact operators from ℓ_2^n into ℓ_2^{2q}, we conclude from Theorem 3 above that, for some constant $D_3 < \infty$,

$$\int_\Omega \int_{\Omega'} ||R_{\sigma(w)} A R_{\sigma(w')}||_{\infty \to 1} d\mu d\mu' \leq D_3(\delta^2 n + \delta^{3/2}(qn)^{1/2}).$$

Hence,

$$\min_{|\sigma'|,|\sigma''| \geq \delta n/2} ||R_{\sigma'} A R_{\sigma''}||_{\infty \to 1} \leq D_3(\delta^2 n + \delta^{3/2}(qn)^{1/2})$$

from which, by using factorization as in the proof of Corollary 3, we deduce that

$$\min_{|\eta'|,|\eta''| \geq \delta n/4} ||R_{\eta'} A R_{\eta''}||_{2 \to 2} \leq C\left(\delta + \left(\frac{\delta q}{n}\right)^{1/2}\right),$$

for a suitable constant $C < \infty$. □

COROLLARY 5. *There exists a constant $C < \infty$ so that, whenever n is an integer, $1/n < \delta < 1$, $1 \leq q \leq n$ and A is linear operator on ℓ_2^n, then*

$$\min_{|\eta'|,|\eta''| \geq \delta n/4} ||R_{\eta'} A R_{\eta''}||_{2 \to 2} \leq C\left(a_q(A)\delta^{1/2} + \delta + \left(\frac{\delta q}{n}\right)^{1/2}\right),$$

where $a_q(A) = \min\{||A - B||_{2 \to 2}, B : \ell_2^n \to \ell_2^n, \text{rank } B \leq q\}$ denotes the q-approximation number of A.

PROOF: This is a consequence of Theorem 3 and the definition of $a_q(A)$. □

COROLLARY 6. *There exists a constant $C_1 < \infty$ such that, whenever n is an integer, $1/n < \delta < 1$, $1 \leq q \leq n$ and A is a linear operator on ℓ_2^n of rank q with 0's on the diagonal, and $||A||_{2 \to 2} \leq 1$, then there exists a set $\eta \subset \{1, 2, \ldots, n\}$, $|\eta| \geq \delta n/4$ for which*

$$||R_\eta A R_\eta||_{2 \to 2} \leq C_1\left(\delta + \left(\frac{\delta q}{n}\right)^{1/2}\right).$$

PROOF: By the decoupling result Proposition 1.9 from [B-T1],

$$\int_\Omega ||R_{\sigma(\omega)} A R_{\sigma(\omega)}||_{\infty \to 1} d\mu \leq 20 \int_\Omega \int_{\Omega'} ||R_{\sigma(\omega)} A R_{\sigma(\omega')}||_{\infty \to 1} d\mu d\mu'.$$

The rest of the proof is exactly as in that of Corollary 4. □

The next topic considered here concerns certain matrices consisting of q orthogonal rows in ℓ_2^n, where q is, in general, much smaller than n. The aim is to find submatrices of size $q \times m$, for different values of m, which are well invertible.

We need first a lemma of a probabilistic nature.

LEMMA 7. *Fix $0 < \varepsilon < 1$, $B > 0$ and $0 < \delta < 1/2$, and suppose that q and n are integers so that q divides n and*

$$q \log q < \frac{\varepsilon^2 \delta n}{4(B+1)}.$$

Then, whenever $\{\xi_i\}_{i=1}^n$ is a sequence of $\{0,1\}$-valued independent random variables of mean δ over some probability space (Ω, Σ, μ) and, for $1 \leq j \leq q$, $I_j = \left\{\frac{(j-1)n}{q} + 1, \ldots, \frac{jn}{q}\right\}$, the set

$$\sigma(\omega) = \{1 \leq k \leq n; \xi_k(\omega) = 1\}$$

satisfies the inequality

$$(1-\varepsilon)\delta|I_j| < |\sigma(\omega) \cap I_j| < (1+\varepsilon)\delta|I_j|;$$

for all $1 \leq j \leq q$, with probability $\geq 1 - \frac{2}{qB}$.

PROOF: First notice that, by [B] (see also Lemma 7 in [K-T]),

$$\mu\{\omega \in \Omega; \max_{1 \leq j \leq q} |\sum_{i \in I_j} (\xi_i(\omega) - \delta)| \geq \gamma\} < 2qe^{-\frac{\gamma^2}{4\delta n/q}},$$

for any $0 \leq \gamma \leq \delta n/q$. Hence, with $\gamma = \frac{\varepsilon \delta n}{q}$, it follows that

$$\mu\{\omega \in \Omega; \max_{1 \leq j \leq q} ||\sigma(\omega) \cap I_j| - \delta|I_j|| \geq \varepsilon \delta n/q\} <$$
$$< 2qe^{-\frac{\varepsilon^2 \delta n}{4q}} < 2qe^{-(B+1)\log q} = \frac{2}{q^B},$$

which, of course, completes the proof. □

We present now a result which allows the selection of submatrices of a certain size, which are at the same time bounded and well invertible, from the matrix consisting of the first q Walsh vectors of length n. We have considered here only random selections.

THEOREM 8. *Suppose that $0 < \varepsilon < 1$, $B > 0$ and $0 < \delta < \frac{1}{2}$ are given, and that q and n are integers so that n is a power of 2, q divides n and*

$$q \log q < \frac{\varepsilon^2 \delta n}{4(B+1)}.$$

Let $W_{q,n}$ be the $q \times n$ matrix whose rows consist of the first q Walsh vectors of length n, normalized in ℓ_2^n. Then, with $\sigma(\omega)$ having the same meaning as in Theorem 1 and with probability $\geq 1 - \frac{2}{q^B}$,

$$(1-\varepsilon)^{1/2}\delta^{1/2}||x||_2^2 \leq ||R_{\sigma(\omega)}W_{q,n}x||_2 \leq (1+\varepsilon)^{1/2}\delta^{1/2}||x||_2,$$

for all $x \in \ell_2^q$, and

$$(1-\varepsilon)\delta n < |\sigma(\omega)| < (1+\varepsilon)\delta n.$$

PROOF: Since the first q vectors in the Walsh system of order n are constant on each of the intervals $I_j = \left\{\frac{(j-1)n}{q} + 1, \ldots, \frac{jn}{q}\right\}$, introduced in the previous lemma, it follows that, for any $x \in \ell_2^q$, the vector $W_{q,n}x$ is of the form

$$W_{q,n}x = \sum_{j=1}^{q} c_j \chi_{I_j},$$

where χ_{I_j} denotes the characteristic function of the set $I_j \subset \{1,2,\ldots,n\}$; $1 \leq j \leq n$. Hence, for any $\omega \in \Omega$, and $x \in \ell_2^q$,

$$(+) \qquad ||R_{\sigma(\omega)}W_{q,n}x||_2^2 = \sum_{j=1}^{q} ||R_{\sigma(\omega)\cap I_j}W_{q,n}x||_2^2 = \sum_{j=1}^{q} |c_j|^2|\sigma(\omega)\cap I_j|.$$

Thus, by Lemma 7 above, we have that

$$(1-\varepsilon)\delta^{1/2}||W_{q,n}x||_2 \leq ||R_{\sigma(\omega)}W_{q,n}x||_2 \leq (1+\varepsilon)^{1/2}\delta^{1/2}||W_{q,n}x||_2,$$

with probability $\geq 1 - \frac{2}{q^B}$.

However, because of the orthonormality of $W_{q,n}$, we have that $||W_{q,n}x||_2 = ||x||_2$, for all $x \in \ell_2^q$, and this completes the proof. □

Theorem 8 is no longer valid if $q \log q$ is large relative to δn. More precisely, we can prove the following result.

PROPOSITION 9. *Suppose that (q_i, n_i, δ_i); $i = 1, 2, \ldots$ are triples such that q_i and n_i are integers,*

$$\text{(i)} \qquad q_i \leq \delta_i n_i \leq \varepsilon_i q_i \log q_i,$$

with $\varepsilon_i \to 0$, *as* $i \to \infty$,

(ii)
$$n_i^{\rho} \leq q_i \leq n_i^{1-\rho}$$

for some $\rho > 0$. *Then, with probability tending to* 1,

$$||R_{\sigma(\omega)} W_{q_i,n_i}||_{2\to 2}/\delta_i^{1/2} \to \infty,$$

as $i \to \infty$.

The proof of this proposition can be obtained by direct computation using the identity (+) above. We do not reproduce the details here.

Proposition 9 shows that even in the simple case of the Walsh matrix, the random choice of $q \times \delta n$ submatrices of $W_{q,n}$ does not give the optimal estimate of the norm which in this case is $\leq \delta^{1/2}$.

It turns out that Theorem 8 remains valid if we replace the system consisting of the first Walsh vectors of length n by any other orthonormal system of q vectors in ℓ_2^n, whose entries are equal in absolute value to $1/\sqrt{n}$. However, in this case, we have to impose a stronger condition on q and n.

The condition below that all the entries of the matrix have the same absolute value can be wekened considerably but we do not present here sharper versions in this direction since the main problem seem to be that the condition below involving $q^2 \log q$ can be improved probably on a large extent.

THEOREM 9. *Suppose that* $0 < \varepsilon < 1/3$, $B > 0$ *and* $0 < \delta < 1/2$ *are given, and that* q *and* n *are integers satisfying the condition*

$$q^2 \log q < \frac{\varepsilon^4 \delta n}{16(B+2)}.$$

Then, whenever $T_{q,n}$ *is a* $q \times n$ *matrix consisting of orthonormal rows whose entries have all absolute values equal to* $1/\sqrt{n}$, *we have, with probability* $\geq 1 - \frac{4}{q^B}$, *that*

$$(1-\varepsilon)\delta^{1/2}||x||_2 \leq ||R_{\sigma(\omega)} T_{q,n} x||_2 \leq (1+\varepsilon)\delta^{1/2}||x||_2,$$

for all $x \in \ell_2^q$, and

$$(1-\varepsilon)\delta n < |\sigma(\omega)| < (1+\varepsilon)\delta n.$$

PROOF: Fix $0 < \varepsilon < 1/3$, $B > 0$, $0 < \delta < 1/2$, q and n as above, and let $T_{q,n} : \ell_2^q \to \ell_2^n$ be an operator whose entries $(a_{i,j})_{i=1,j=1}^{q,\ \ n}$ satisfy $|a_{i,j}| = 1/\sqrt{n}$, for all i and j. Let $\{\xi_j\}_{j=1}^n$ be, as before, a sequence of $\{0,1\}$-valued independent random variables of mean δ over some probability space (Ω, Σ, μ). Then, by [B] or Lemma 7 from [K-T], we get that,

$$\mu\left\{\omega \in \Omega;\ \max_{1\le i\le q}\left|\sum_{j=1}^n n|a_{i,j}|^2(\xi_j(\omega)-\delta)\right| \ge \gamma\right\} = \mu\{\omega\in\Omega;\ ||\sigma(\omega)-\delta n| \ge \gamma\} \le 2e^{-\frac{\gamma^2}{4\delta n}},$$

and

$$\mu\left\{\omega \in \Omega;\ \max_{1\le i\neq h\le q}\left|\sum_{j=1}^n n a_{i,j}a_{h,j}(\xi_j(\omega)-\delta)\right| \ge \gamma\right\} \le 2q^2e^{-\frac{\gamma^2}{4\delta n}},$$

for any choice of $0 \le \gamma \le \delta n$.

Take now $\gamma = \sqrt{4(B+2)\delta n \log q}$ and notice that

$$2q^2e^{-\frac{\gamma^2}{4\delta n}} = 2q^2e^{-(B+2)\log q} = \frac{2}{q^B}.$$

Hence, with probability $\ge 1 - \frac{4}{q^B}$, we have that

$$\max_{1\le i\le q} |\ ||R_{\sigma(\omega)}T_{q,n}e_i||_2^2 - \delta\ | < \frac{\gamma}{n}$$

and

$$\max_{1\le i\neq h\le q} |(R_{\sigma(\omega)}T_{q,n}e_i, R_{\sigma(\omega)}T_{q,n}e_h)| < \frac{\gamma}{n}.$$

Consider the operator $T_{q,n}^*R_{\sigma(\omega)}T_{q,n} : \ell_2^q \to \ell_2^q$, and observe that, with probability $\ge 1 - 4/q^B$, the diagonal $D(\omega)$ of $T_{q,n}^*R_{\sigma(\omega)}T_{q,n}$ satisfies

$$\delta + \frac{\gamma}{n} \ge (D(\omega)e_i, e_i) = ||R_{\sigma(\omega)}T_{q,n}e_i||_2^2 > \delta - \frac{\gamma}{n};\quad 1 \le i \le q,$$

and also

$$||T_{q,n}^*R_{\sigma(\omega)}T_{q,n} - D(\omega)||_2 \le ||T_{q,n}^*R_{\sigma(\omega)}T_{q,n} - D(\omega)||_{HS} \le$$
$$\le \left(\sum_{1\le i\neq h\le q} |(T_{q,n}^*R_{\sigma(\omega)}T_{q,n}e_i, e_h)|^2\right)^{1/2} \le q\frac{\gamma}{n}.$$

Thus, with the same probability as above,

$$||R_{\sigma(\omega)}T_{q,n}x||_2^2 = (T_{q,n}^* R_{\sigma(\omega)}T_{q,n}x, x) \leq \left(\frac{\delta\gamma}{n} + ||D(\omega)||_2\right) ||x||_2^2 \leq$$
$$\leq \left(\delta + \frac{(q+1)\gamma}{n}\right) ||x||_2^2,$$

for $x \in \ell_2^q$. In particular, we get that

$$||R_{\sigma(\omega)}T_{q,n}||_2 \leq \left(\delta + \frac{(q+1)\gamma}{n}\right)^{1/2}.$$

On the other hand, again with the same probability,

$$||(T_{q,n}^* R_{\sigma(\omega)}T_{q,n})^{-1}||_2 = ||(D(\omega) + (T_{q,n}^* R_{\sigma(\omega}T_{q,n} - D(\omega))^{-1}||_2 \leq$$
$$\leq ||D(\omega)^{-1}||_2/(1 - ||D^{-1}(\omega)||_2 \; ||T_{q,n}^* R_{\sigma(\omega)}T_{q,n} - D(\omega)||_2) \leq \frac{1}{\delta - \frac{(q+1)\gamma}{n}}.$$

Furthermore,

$$(T_{q,n}^* R_{\sigma(\omega)}T_{q,n})^{-1}(R_{\sigma(\omega)}T_{q,n})^* R_{\sigma(\omega)}T_{q,n}x = x,$$

for all $x \in \ell_2^q$, and thus,

$$||x||_2 \leq \frac{(\delta + (q+1)\gamma/n)^{1/2}}{(\delta - (q+1)\gamma/n)^{1/2}} ||R_{\sigma(\omega)}T_{q,n}x||_2,$$

i.e.

$$||R_{\sigma(\omega)}T_{q,n}x||_2 \geq \left(\delta^{1/2} - \frac{3(q+1)\gamma}{\delta^{1/2}n}\right) ||x||_2,$$

for all $x \in \ell_2^q$. The proof is then completed once we know that $j(q+1) < \varepsilon^2\delta n$, which follows from the condition imposed on q and n.

The fact that $|\,|\sigma(\omega)| - \delta_n| < \varepsilon$ is immediate. □

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