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Dyadic analogues of Hilbert matrices

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The question treated below goes back to my notes of 40 years ago and concerns the properties of a sequence of $2^s \times 2^s$ matrices A_s, $s = 1, 2, \dots$, given by the inductive construction

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A_s = \begin{pmatrix} A_{s-1} & -2^{1-s} E_{s-1} \\ 2^{1-s} E_{s-1} & A_{s-1} \end{pmatrix}, \quad s = 2, 3, \dots,$$

where E_{s-1} is the $2^{s-1} \times 2^{s-1}$ matrix with all entries equal to 1.

The matrices A_s can be regarded as dyadic analogues of the Hilbert matrices (for instance, see [1], p. 170). Letting $\widehat{A}_s$ denote the matrix obtained from A_s by replacing by zeros all the entries above the main diagonal, we obtain a matrix with non-negative entries. Then it is easy to verify that

$$C_1 s \leqslant \|\widehat{A}_s\|_{\mathrm{op}} \leqslant C_2 s, \tag{1}$$

where, in (1) and below, $C_1, C_2, \dots$ are different absolute positive constants and $\|\cdot\|_{\mathrm{op}}$ is the norm of a matrix viewed as an operator in the Euclidean space with the norm $\|\cdot\|_2$.

By analogy with Hilbert matrices, it is natural to try to use the matrices A_s to construct orthogonal series that diverge almost everywhere. However, to do this we must have estimates for the norms of the matrices A_s. In contrast to the case of Hilbert matrices, the operator norms of the matrices A_s grow as $s \to \infty$: if $e_s = (1, \dots, 1) \in \mathbb{R}^{2^s}$, then it can easily be verified that

$$\|A_s(e_s)\|_2 \geqslant C_3 s^{1/2} \|e_s\|_2. \tag{2}$$

It turns out that (2) is a sharp estimate.

Claim 1.

$$\|A_s\|_{\mathrm{op}} \leqslant C_4 s^{1/2}, \qquad s = 1, 2, \dots .$$

Proof. For $p = 1, \dots, s$ we consider the set Ω_p of entries a_{ij} of the matrix A_s that are equal to $\pm 2^{p-s}$. It consists of 2^{p-1} submatrices Q_ν^+ (with even ν, $1 \leqslant \nu \leqslant 2^p$) of order 2^{s-p} with positive entries and 2^{p-1} submatrices Q_ν^- (with odd ν, $1 \leqslant \nu \leqslant 2^p$) of order 2^{s-p} with negative entries. The submatrices Q_ν^+ and Q_ν^- are described, respectively, by the inequalities

$$\begin{aligned} &(\nu-1)\cdot 2^{s-p} + 1 \leqslant i \leqslant \nu \cdot 2^{s-p}, \qquad (\nu-2)\cdot 2^{s-p} + 1 \leqslant j \leqslant (\nu-1)\cdot 2^{s-p}, \\ &(\nu-1)\cdot 2^{s-p} + 1 \leqslant i \leqslant \nu \cdot 2^{s-p}, \qquad \nu \cdot 2^{s-p} + 1 \leqslant j \leqslant (\nu+1)\cdot 2^{s-p}. \end{aligned} \tag{3}$$

Let T_p be the operator in $\mathbb{R}^{2^s}$ given by the matrix with entries

$$(T_p)_{i,j} = \begin{cases} a_{ij} & \text{if } (i,j) \in \Omega_p, \\ 0 & \text{otherwise.} \end{cases}$$

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Clearly, $A_s = \sum_{p=1}^{s} T_p$.

Let $\mathbf{w}_i$, $0 \leqslant i < 2^s$, be the Walsh functions: $\mathbf{w}_0 \equiv 1$, and for $i = 1, 2, \dots$ and $t \in (0,1)$

$$\mathbf{w}_i(t) = \mathbf{w}(t) = \mathbf{r}_{d_1}(t) \cdots \mathbf{r}_{d_q}(t), \qquad 1 \leqslant d_1 < \cdots < d_q \leqslant s, \tag{4}$$

where the $\mathbf{r}_d$ are the Rademacher functions (see [1], p. 150). With each Walsh function $\mathbf{w}_i$ we associate the vector $w_i \in \mathbb{R}^{2^s}$ with components

$$(w_i)_k = \mathbf{w}_i\biggl(\frac{k-1/2}{2^s}\biggr), \qquad 1 \leqslant k \leqslant 2^s,$$

in an obvious way. We consider the action of the operators T_p on the vectors associated with the functions (4). It follows from (3) and (4) and the simplest properties of the Rademacher functions that: a) $T_p(w) = 0$ for $p < d_q$; b) $T_p(w) = r_p w$ for $p = d_q$; c) $T_p(w) = -r_p w$ for $p > d_q$. Hence, for these vectors

$$A_s(w) = -\biggl(\sum_{p=d_q+1}^{s} r_p\biggr) w + r_{d_q} w.$$

Let us expand an arbitrary $f \in \mathbb{R}^{2^s}$ with $\|f\|_2 \leqslant 1$ in the Walsh basis and then group the terms of the expansion in the standard dyadic packets:

$$f = \sum_{i=0}^{2^s-1} a_i w_i = c(f) e_s + \sum_{j=1}^{s} \Delta_j(f).$$

Then

$$A_s(f) = -c(f) \sum_{p=1}^{s} r_p - \sum_{j=1}^{s-1} \biggl(\Delta_j(f) \sum_{p=j+1}^{s} r_p\biggr) + \sum_{j=1}^{s} \Delta_j(f) r_j \equiv \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{5}$$

Here $|c(f)| \leqslant 2^{-s/2}$, and therefore $\|\Sigma_1\|_2 \leqslant \sqrt{s}$. Next it is easy to verify the following equality for scalar products: if $j \neq j'$, then

$$\biggl\langle \Delta_j(f) \sum_{p=j+1}^{s} r_p,\ \Delta_{j'}(f) \sum_{p=j'+1}^{s} r_p \biggr\rangle = 0,$$

which yields the estimate $\|\Sigma_2\|_2^2 \leqslant s$. Finally, $\Sigma_3 = \sum_{i=0}^{2^s-1} b_i w_i$, $b_i = \sum a_h$, where the last sum is taken over all $h < 2^s$ such that $\mathbf{w}_h = \mathbf{w}_i \mathbf{r}_\mu$, $\mu > d_q$ (see (4); if $i = 0$, then $d_q = 0$), which easily shows that $\|\Sigma_3\|_2^2 \leqslant s$. $\square$

Bibliography

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