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Observations on discretization of trigonometric polynomials with given spectrum

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The following notation is used throughout: $\#\Lambda$ is the cardinality of a finite set Λ; $\operatorname{span}\{F\}$ is the linear span of a family F of elements of a linear space; for $\Lambda \subset \mathbb{Z}$

$$T(\Lambda) \equiv \operatorname{span}\{e^{ikx},\ k \in \Lambda\} \subset C(0,2\pi). \tag{1}$$

Let $W = \{f\}$ be a family of complex functions defined on a set Ω. For an arbitrary set $X_m \subset \Omega$ with $\#X_m = m$ let $\|f\|_{X_m} = \max\{|f(x)|,\ x \in X_m\}$ and consider the quantity

$$D(W,m) = \inf_{X_m} \sup_{f \in W} \frac{\sup_{x\in\Omega}|f(x)|}{\|f\|_{X_m}}. \tag{2}$$

We are interested in the situation when W is a subspace of the form (1). The first results on estimates for the quantities (2) were due to Marcinkiewicz and Zygmund (see [1]), who showed, in particular, that

$$D(T(\Lambda_N),4N) \leqslant C_1, \qquad \Lambda_N = \mathbb{Z}\cap[-N,N], \quad N=1,2,\dots \tag{3}$$

(here and below, $C, C_1, C_2, \dots$ are different positive constants). Estimates like (3) for various finite-dimensional function spaces have found diverse applications in analysis. Systems Φ of functions spanning such subspaces were called *quasi-matrix systems* in [2]. To control the uniform norm of Nth-order polynomials constructed from functions in a system Φ, it is far from always sufficient to optimally choose a grid X_m with $m \leqslant CN$ nodes. In a number of important cases significantly more points are needed for this purpose. For instance, this is so for the space of trigonometric polynomials of several variables with spectrum in a hyperbolic cross (see [3]).

Fix a constant $b > 1$ and for $n \in \mathbb{N}$ with $n \geqslant n(b)$ consider sets of positive integers as follows:

$$\mathscr{K} = \{k_j\}_{j=n}^{2n-1}; \quad \frac{k_{j+1}}{k_j} \geqslant b, \quad j=n,\dots,2n-2; \quad \frac{k_j}{k_n} \in \mathbb{N}, \quad j=n,\dots,2n-1. \tag{4}$$

Also, fix a positive integer $\nu \leqslant k_n/n$ and consider the subspace

$$T(\mathscr{K},\nu) = \left\{f\colon f = \sum_{j=n}^{2n-1} p_j(x)e^{ik_jx}\right\}$$

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of $C(0,2\pi)$, where $p_j \in T(\Lambda_\nu)$ for $j = n,\dots,2n-1$. For functions f in $T(\mathscr{K},\nu)$ we have the inequality

$$\|f\|_{C(0,2\pi)} \geqslant c(b) \sum_{j=n}^{2n-1} \|p_j\|_{L^1(0,2\pi)}, \qquad c(b) > 0 \tag{5}$$

(see [3], [4]). Using estimates of type (5) and the method in [5], we establish the following.

Proposition 1. *For $m \geqslant n(2\nu+1)$*

$$D(T(\mathscr{K},\nu),m) \geqslant C_1 n^{1/2} \biggl(\log \frac{em}{(2\nu+1)n}\biggr)^{-1/2}. \tag{6}$$

For subspaces consisting of lacunary polynomials the following result is a consequence of (6).

Corollary. *Let R_1 and R_2 be constants. Then the following is true.*

(a) *The inequality $(T(\mathscr{K},0),m) \leqslant R_1$ can hold only for $m \geqslant ce^{\delta n}$, where $c = c(R_1)$ and $\delta = \delta(R_1)$ are positive constants.*

(b) *For $m \leqslant R_2 n$*

$$D(T(\mathscr{K},0),m) \geqslant c(R_2) n^{1/2}, \qquad c(R_2) > 0. \tag{7}$$

The next result shows that the estimate (7) is sharp.

Proposition 2. *There exist absolute constants C_2 and C_3 such that, for any set $\Lambda \subset \mathbb{Z}$ with $\#\Lambda = n$, $D(T(\Lambda),m) \leqslant C_3 n^{1/2}$ if $m \geqslant C_2 n$.*

Proposition 2 is a consequence of results on the existence of well-conditioned $n \times Cn$ submatrices of an $n \times N$ matrix with orthogonal rows (see [6]–[9]).

The right-hand side of (7) does not exceed $c \log^{1/2} d(T(\mathscr{K},\nu))$, where $d(T(\Lambda)) = \max\{|k_j|, k_j \in \Lambda\}$. A similar bound also holds for polynomials with spectrum in a hyperbolic cross. It is natural to ask how essential this barrier is.

In conclusion we present a result refining (5).

Proposition 3. *Let $\mathscr{K}$ be as in (4), let $\nu \leqslant k_n/n$, and let $f = \sum_{j=n}^{2n-1} p_j(x) e^{ik_j x} \in T(\mathscr{K},\nu)$. Then*

$$\|f\|_{C(0,2\pi)} \geqslant c(b) \biggl\| \sum_{j=n}^{2n-1} |p_j(x)| \biggr\|_{C(0,2\pi)}, \qquad c(b) > 0.$$

The proof of Proposition 3 is elementary but makes essential use of the properties of $\mathscr{K}$. It would be interesting to know to what spectra this result can be generalized.

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