Sbornik: Mathematics **206**:7 980–987 *Matematicheskiĭ Sbornik* **206**:7 95–102
DOI 10.1070/SM2015v206n07ABEH004485

Lunin's method for selecting large submatrices with small norm

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Abstract. Using an approach proposed by Lunin in 1989, upper bounds are found for the norms of large submatrices of a fixed $(N \times n)$-matrix which defines an operator from l_2^n into l_1^N with unit norm.

Bibliography: 15 titles.

Keywords: submatrix, operator norm, Lunin's lemma, Kadison-Singer problem.

This paper is devoted to estimates for the norms of submatrices of a fixed $(N \times n)$-matrix

$$A = \{a_{ij}\}, \qquad i = 1, \dots, N, \quad j = 1, \dots, n. \tag{1}$$

The following notation will be used below:

- $(\,\cdot\,,\,\cdot\,)$ is the scalar product in $\mathbb{R}^n$;
- $\langle n \rangle = \{1, 2, \dots, n\}$;
- $\{e_i\}_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$;
- $[x]$ is the integer part of the number $x \in \mathbb{R}$;
- $\#B$ (or $|B|$) is the cardinal number of the finite set B;
- l_p^n, $1 \leqslant p \leqslant \infty$, is the linear space over $\mathbb{R}$ of vectors $x = \{x_i\}_{i=1}^n$ with the norm

$$\|x\|_p = \|x\|_{l_p^n} = \begin{cases} \left(\displaystyle\sum_{i=1}^n |x_i|^p\right)^{1/p}, & 1 \leqslant p < \infty, \\ \displaystyle\max_{1 \leqslant i \leqslant n} |x_i|, & p = \infty; \end{cases}$$

- B_p^n is the unit ball in l_p^n;
- S^{n-1} is the unit sphere in l_2^n.

Regarding the matrix (1) as an operator from l_p^n into l_q^N we set

$$\|A\|_{(p,q)} = \sup_{x \in B_p^n} \|Ax\|_{l_q^N}.$$

For $p = q = 2$ we set for brevity $\|A\|_{(2,2)} \equiv \|A\|$. For $\Omega \subset \mathbb{N}$ and $k \in \mathbb{N}$, $k \leqslant |\Omega|$, E_Ω^k denotes the system of all k-subsets of Ω. Also let $E_{\langle N \rangle}^k \equiv E_N^k$. If $\omega = \{i_\nu\}_{\nu=1}^k \in E_N^k$,

This work was supported by the Russian Science Foundation under grant no. 14-50-00005.
AMS 2010 *Mathematics Subject Classification*. Primary 46B20.

then $A(\omega)$ is the following $(k \times n)$-submatrix of the matrix A:

$$A(\omega) = \{a_{i_\nu, j},\ \nu = 1, \dots, k,\ j = 1, \dots, n\}. \tag{2}$$

The question of finding a sufficiently large submatrix with small norm of a fixed matrix A with $\|A\| = 1$ arises in a natural way in various problems in analysis, theoretical computer science and their applications. In 1980 this author established the following result.

Theorem A (see [1], [2]). *Given $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that if $C(\varepsilon) \cdot n \leqslant N$, then each $(N \times n)$-matrix A with $\|A\| = 1$ contains an $(n \times n)$-submatrix $A(\omega)$ of the form (2) such that*

$$\|A(\omega)\| \leqslant \varepsilon.$$

In [2], Theorem A was proved by going over to estimates for the $(2,1)$-norms of random submatrices of A and to Grothendieck's factorization theorem for operators from l_2^n into l_1^k. The latter theorem enables us to find estimates for the $(2,2)$-norm of a submatrix when the $(2,1)$-norm of the matrix is known. The question of a sharp estimate for the constant $C(\varepsilon)$ was not answered in [1] and [2]. Note here that it follows immediately from Theorem A for $\varepsilon = 1/2$ that $C(\varepsilon)$ has an upper estimate $C(\varepsilon) \leqslant B \cdot (1/\varepsilon)^d$, where B and d are some absolute constants.

An estimate

$$C(\varepsilon) \leqslant B \cdot \varepsilon^{-2}, \qquad B \text{ is an absolute constant}, \tag{3}$$

which is sharp in order, was proved by Lunin in the remarkable note [3], written when its author was not even 19. He obtained (3) as a consequence of the following result.

Theorem B (see [3]). *There exists an absolute constant B such that for each $(N \times n)$-matrix A, $n \leqslant N$, there exists a set $\omega \in E_N^n$ such that*

$$\|A(\omega)\|_{(2,1)} \leqslant B \frac{n}{N} \|A\|_{(2,1)}.$$

The proof of Theorem B was based on the following lemma.

Lemma (see [3]). *There exists a positive absolute constant c_0 such that for each $(N \times n)$-matrix A and each integer $\mu > c_0 n$ it follows from the relation*

$$2\mu + 4(\mu^3 n)^{1/4} \leqslant N$$

that there exists a system $\Omega \subset E_N^\mu$ such that

$$\|A(\Omega)\|_{(2,1)} \leqslant \frac{1}{2}\left(1 + 9\left(\frac{n}{\mu}\right)^{1/8}\right) \cdot \|A\|_{(2,1)}. \tag{4}$$

For each $\delta > 0$, provided that $N/n \geqslant C_1(\delta)$, this lemma of Lunin ensures that each $(N \times n)$-matrix A with $\|A\|_{(2,1)} = 1$ has a $(\mu \times n)$-submatrix with $\mu > N(1/2 - \delta)$ whose $(2,1)$-norm $\leqslant (1/2 + \delta)$. The set ω required in Theorem B was found by the repeated use of this lemma and by iterating (4).

Lunin's note [3] was published in the period between the two papers [4] and [5] by Bourgain and Tzafriri, which were concerned with related questions. In [4] they obtained the following result, which is now widely known as the restricted invertibility theorem.

Theorem C. *For some absolute positive constants c_3 and c_4 and each linear operator $A\colon l_2^n \to l_2^n$ with $\|A(e_i)\|_2 = 1$, $i = 1,\dots,n$, there exists a set $\omega \in E_n^k$, $k \geqslant c_3 n/\|A\|^2$, such that for any numbers $\{a_i\}_{i\in\omega}$,*

$$\biggl\|\sum_{i\in\omega} a_i A(e_i)\biggr\|_2 \geqslant c_4\biggl(\sum_{i\in\omega} a_i^2\biggr)^{1/2}.$$

The proof of Theorem C in [4] also used the transition to estimates for the $(2,1)$-norm. In [5], for estimates of the $(2,1)$-norms of submatrices, some nontrivial ideas in probability theory were also used; this enabled the authors, in particular, to give another proof of the estimate (3). Furthermore, in [5] they put forward a method for estimating the $(2,2)$-norm of a submatrix which does not use the result on $(2,1)$-norms. Using this method, [5] gave a partial solution (under some restrictions on the absolute values of the entries of the $(n\times n)$-matrix A) of the famous old Kadison-Singer problem (see [6]).

Some important results in this area are also due to Rudelson and Vershynin [7], [8].

A new efficient method for finding both upper and lower bounds for the norms of submatrices was proposed by Batson, Spielman and Srivastava [9]. The following result was proved in [9].

Theorem D. *Let A be a matrix of the form (1) and let $u_i = \{a_{ij}\}_{j=1}^n$, $i = 1,\dots,N$, $z_j = \{a_{ij}\}_{i=1}^N$, $j = 1,\dots,n$, be its rows and columns, respectively, where $\{z_j\}_{j=1}^n$ is an orthonormal system in $\mathbb{R}^N$. For each $\delta > 0$ there exist a set $\omega \in E_N^s$, $s \leqslant (1+\delta)n$, and positive numbers $\{\lambda_i\}_{i\in\omega}$ such that for each $v \in \mathbb{R}^n$,*

$$b(\delta)\|v\|_2^2 \leqslant \sum_{i\in\omega} \lambda_i |(v,u_i)|^2 \leqslant B(\delta)\|v\|_2^2, \tag{5}$$

where $b(\delta)$ and $B(\delta)$ are positive constants which only depend on δ.

For a matrix with orthonormal columns whose rows have equal l_2^n-norms it follows directly from Theorem D that it contains a submatrix $A(\omega)$ of the form (2) with $|\omega| = k \leqslant (1+\delta)n$ such that for each $v \in R^n$

$$\|Av\|_{l_2^k} \geqslant b(\delta)\biggl(\frac{n}{N}\biggr)^{1/2}\|v\|_{l_2^n}.$$

The question of the existence of such submatrices was originally asked in [10]; as concerns applications, see [11] and [12]. The problem of an unweighted analogue of (5) is much more complicated in this case. The paper [13] by Marcus, Speilman and Srivastava, devoted to the solution of the Kadison-Singer problem, opened up an approach to this question. More precisely, using, in particular, some ideas from [9], the following result was established in [13], which — as Weaver [14] showed — solves the Kadison-Singer problem in the affirmative.

Theorem E. *In the hypotheses of Theorem D assume that* $\|u_i\|_2 \leqslant \delta$, $i = 1, \dots, N$. *Then there exists a decomposition*

$$\langle N \rangle = \omega_1 \cup \omega_2, \qquad \omega_1 \cap \omega_2 = \varnothing, \tag{6}$$

such that

$$\|A(\omega_1)\| \leqslant \frac{1}{\sqrt{2}} + \delta \quad \textit{and} \quad \|A(\omega_2)\| \leqslant \frac{1}{\sqrt{2}} + \delta.$$

The difference of Theorem E from the preceding results is as follows: under some natural restrictions on the matrix A it enables us to decompose A into two submatrices with considerably smaller norms. Since $\max(|\omega_1|, |\omega_2|) \geqslant N/2$ (see (6)), for small δ Theorem E gives us an upper bound for the norm of the larger submatrix of A which is close to sharp. In a certain sense, this estimate is similar to Lunin's lemma. In this connection it is natural to attempt to apply the method in [3] to establishing analogues of Theorem E. In this direction we can prove the following.

Theorem 1. *For each pair of numbers* (δ, δ'), *where* $0 < \delta < 1/6$ *and* $0 < \delta' < 1 - 6\delta$, *there exists a constant* $C(\delta, \delta')$ *such that for each* $(N \times n)$*-matrix* A *with* $\varkappa(A) \equiv N/n > C(\delta, \delta')$, $\langle N \rangle$ *has a decomposition into disjoint parts*

$$\langle N \rangle = \omega_1 \cup \omega_2 \cup \Delta$$

such that

$$\|A(\omega_1)\|_{(2,1)} \leqslant \left(\frac{1}{2} + \varkappa^{-\delta}\right)\|A\|_{(2,1)}, \qquad \|A(\omega_2)\|_{(2,1)} \leqslant \left(\frac{1}{2} + \varkappa^{-\delta}\right)\|A\|_{(2,1)},$$

$$|\Delta| \leqslant N\varkappa^{-\delta'}, \qquad |\omega_1| = |\omega_2| = \frac{N - |\Delta|}{2}.$$

In our opinion, the potential of the method in [3] has not yet been exhausted, and it is deplorable that survey papers in the area under consideration give no information about this note.

Proof of Theorem 1. Let A be a fixed matrix and let $u_i \in \mathbb{R}^n$, $i = 1, \dots, N$, be its rows. Also let $v(A)$ be some linear combination of the u_i with coefficients ± 1 which has the maximum norm $\|\cdot\|_2$. Note that

$$\|v(A)\|_{l_2^n} = \|A\|_{(2,1)}. \tag{7}$$

For $\omega \subset \langle N \rangle$ we set

$$v(\omega) = v(A(\omega)).$$

Fix a pair of numbers (c, d) with $0 < c < d/2 < 1/2$ such that

$$d - 2c > \delta' \quad \text{and} \quad \min\{c, 1 - d\} > 2\delta. \tag{8}$$

To do this we can set $c = 1 - d = 2\delta + \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Then $0 < c < 1/3 < d/2$, $d < 1 - 2\delta < 1$ and $d - 2c = 1 - 3c > \delta'$.

In what follows, without loss of generality we assume that $\varkappa(A) \equiv \varkappa = N/n$ is sufficiently large and $\|A\|_{(2,1)} = 1$. Let

$$\alpha = \varkappa^{-c}, \qquad l = [N\varkappa^{-d}],$$

where we have chosen c and d above (see (8)).

Now we look at the subsets $\omega\subset\langle N\rangle$ such that $|\omega|=l$ and $\|A(\omega)\|_{(2,1)}\geqslant\alpha$. Among all possible systems of disjoint subsets with these properties we find a system U with largest cardinal number $\#U=q$. Then for

$$\widetilde{\Delta}=\bigcup_{\omega\in U}\omega \tag{9}$$

we have

$$|\widetilde{\Delta}|=ql, \tag{10}$$

$$q^{1/2}\alpha\leqslant\|A\|_{(2,1)}=1. \tag{11}$$

To justify the left-hand inequality in (11), note (see also (7)) that

$$1\geqslant\|A(\widetilde{\Delta})\|_{(2,1)}\geqslant\operatorname*{Aver}_{\varepsilon_\omega=\pm1}\biggl\|\sum_{\omega\in U}\varepsilon_\omega v(\omega)\biggr\|_2=\biggl(\sum_{\omega\in U}\|v(\omega)\|_2^2\biggr)^{1/2}\geqslant\alpha q^{1/2}.$$

Here $\mathrm{Aver}_{\varepsilon(\omega)}f$ denotes the mean value of the function f (which depends on the choice of signs) over all possible systems of signs $\varepsilon(\omega)=\pm1$.

It follows from (11) that $q\leqslant\alpha^{-2}$ and

$$\#\widetilde{\Delta}+1=ql+1\leqslant\alpha^{-2}l+1=\varkappa^{-2c}[N\varkappa^{-d}]+1\leqslant N\varkappa^{-(d-2c)}+1\leqslant N\varkappa^{-\delta'}. \tag{12}$$

If $N-\#\widetilde{\Delta}$ is even, then we set $\Delta=\widetilde{\Delta}$, otherwise we set $\Delta=\widetilde{\Delta}\cup g$, where g is an arbitrary number in $\langle N\rangle\setminus\widetilde{\Delta}$. In either case it follows from (12) that

$$|\Delta|\leqslant N\varkappa^{-\delta'}. \tag{13}$$

Let $\Omega=\langle N\rangle\setminus\Delta$ and $|\Omega|=2\mu$. We look at the set E_Ω^μ of all μ-subsets of Ω. For $\omega\in E_\Omega^\mu$ we set $\overline{\omega}=\Omega\setminus\omega$. Also let

$$\beta=\min_{\omega\in E_\Omega^\mu}\max\bigl\{\|A(\omega)\|_{(2,1)},\|A(\overline{\omega})\|_{(2,1)}\bigr\}. \tag{14}$$

Our aim is to find an upper estimate for β.

Setting $\rho=\varkappa^{-1}$ and using the standard estimate for the cardinal number of an ε-net on a Euclidean sphere, we take a 0.9ρ-net Λ on the sphere S^{n-1} (with respect to the l_2^n-norm) that has cardinal number $\leqslant(3/\rho)^n$. Bearing in mind that ρ is sufficiently small, for each $z\in S^{n-1}$ we obtain that there exists $y\in\Lambda$ such that the angle between the vectors z and y is less than ρ:

$$\angle\{z,y\}<\rho.$$

Hence for some pair (y_1,y_2), where $y_1\in\Lambda$ and $y_2\in\Lambda$,

$$\#\bigl\{\omega\in E_\Omega^\mu:\angle\{v(\omega),y_1\}<\rho,\ \angle\{v(\overline{\omega}),y_2\}<\rho\bigr\}\geqslant\biggl(\frac{\rho}{3}\biggr)^{2n}C_{2\mu}^\mu. \tag{15}$$

Let Ψ be the system of subsets ω such that

$$\angle\{v(\omega),y_1\}<\rho,\qquad\angle\{v(\overline{\omega}),y_2\}<\rho,$$

$$\Psi_1=\bigl\{\omega\in\Psi:\|A(\omega)\|_{(2,1)}\geqslant\beta\bigr\},\qquad\Psi_2=\bigl\{\overline{\omega}:\omega\in\Psi,\ \|A(\overline{\omega})\|_{(2,1)}\geqslant\beta\bigr\}.$$

It follows from the definition of β (see (14)) that at least one of the following inequalities holds:

1) $\#\Psi_1 \geqslant \frac{1}{2}\#\Psi$;

2) $\#\Psi_2 \geqslant \frac{1}{2}\#\Psi$.

If 1) holds, then we set $y_0 = y_1$ and $\Psi_0 = \Psi_1$; otherwise we set $y_0 = y_2$ and $\Psi_0 = \Psi_2$. Then, in view of (15), we can say that

$$\#\Psi_0 \geqslant \frac{1}{2} \cdot \left(\frac{\rho}{3}\right)^{2n} C_{2\mu}^{\mu} \quad \text{and} \quad \angle\{v(\omega), y_0\} < \rho, \tag{15$'$}$$

for $\omega \in \Psi_0$.

Let σ be a bijective map of the segment $\langle 2\mu\rangle$ of positive integers onto Ω such that the absolute values of the scalar products $g_\nu = |(u_{\sigma(\nu)}, y_0)|$, $\nu = 1, 2, \dots, 2\mu$, form a nonincreasing sequence. Then

a) for $\omega \in \Psi_0$, $\omega = (i_1, \dots, i_\mu)$, we have

$$\sum_{i\in\omega} g_{\sigma^{-1}(i)} > \beta\cos\rho; \tag{16}$$

b) we have

$$0 \leqslant g_1 + \dots + g_l \leqslant \|A(Q)\| \leqslant \alpha, \tag{17}$$

where $Q = \sigma(\langle l\rangle)$.

The right-hand inequality in (17) holds because U is a maximal system (see (9)), and the left-hand one is obvious.

Let $s = [2\mu/l]$. Then we subdivide Ω into subsets $\Phi_1, \dots, \Phi_{s+1}$ such that

$$\Phi_1 = Q, \qquad |\Phi_j| = l, \quad j = 1, \dots, s,$$

for the (maybe empty) set Φ_{s+1} we have the inequality $|\Phi_{s+1}| < l$, and for any $k \in \Phi_j$ and $k' \in \Phi_{j'}$, $j < j'$, we have

$$g_{\sigma^{-1}(k)} \geqslant g_{\sigma^{-1}(k')}.$$

We shall show that there exists a set $\omega_0 \in \Psi_0$ such that

$$|\omega_0 \cap \Phi_\nu| \leqslant \frac{1+\varepsilon}{2}|\Phi_\nu|, \qquad \nu = 1, \dots, s, \quad \varepsilon = \varkappa^{-\delta}. \tag{18}$$

For fixed ν, $1 \leqslant \nu \leqslant s$, we can estimate the number P_ν of $\omega \in E_\Omega^\mu$ such that

$$|\omega \cap \Phi_\nu| > \frac{1+\varepsilon}{2}|\Phi_\nu|. \tag{19}$$

Here we can assume that

$$\Omega = \langle 2\mu\rangle, \qquad \Phi_\nu = \langle |\Phi_\nu|\rangle.$$

Then with each ω we can uniquely associate the vector $\{\varepsilon_j\}_{j=1}^{2\mu}$ such that $\varepsilon_j = +1$ if $j \in \omega$ and $\varepsilon_j = -1$ if $j \notin \omega$. Then

$$P_\nu \leqslant \#\left\{\{\varepsilon_j\}_{j=1}^{2\mu},\, \varepsilon_j = \pm 1 : \left|\sum_1^{|\Phi_\nu|} \varepsilon_j\right| \geqslant \varepsilon|\Phi_\nu|\right\}.$$

The last quantity has an obvious estimate in terms of the distribution function of the polynomial in the Rademacher system $\sum_{j=1}^{|\Phi_\nu|} r_j(t)$ (see [15]).

As a result,

$$P_\nu \leqslant 2^{2\mu} \cdot 2\exp\biggl(-\frac{\varepsilon^2}{2}|\Phi_\nu|\biggr).$$

Hence the number P of the sets $\omega \in E_\Omega^\mu$ such that (19) holds for some ν has the estimate

$$2^{2\mu}\frac{4\mu}{l}\exp\biggl(-\frac{\varepsilon^2 l}{2}\biggr). \tag{20}$$

Now we verify that $|\Psi_0| > P$. To do this it is sufficient to show (see (15′) and (20)) that

$$\frac{1}{2}\biggl(\frac{\rho}{3}\biggr)^{2n} C_{2\mu}^{\mu} > 2^{2\mu}\frac{4\mu}{l}\exp\biggl(-\frac{\varepsilon^2 l}{2}\biggr), \tag{21}$$

or, in view of the choice of ρ, ε and l and the inequality $C_{2\mu}^{\mu} \geqslant (1/(3\sqrt{\mu}))2^{2\mu}$, to show that

$$\frac{1}{16\varkappa^d}\exp\biggl(\frac{\varkappa^{-2\delta}N\varkappa^{-d}}{3}\biggr) \geqslant 3\sqrt{N}\exp(2n\ln 3\varkappa). \tag{22}$$

Recalling that $N = n\varkappa$ and $1 - 2\delta - d > 0$, we easily see that (22) holds.

Thus we have established the existence of an $\omega_0 \in \Psi_0$ which satisfies (18). For this set $\omega_0 = (i_1, \dots, i_\mu)$, using the properties of σ and relations (16)–(18) we obtain

$$\begin{aligned}
\beta\cos\rho &\leqslant \sum_{i\in\omega_0} g_{\sigma^{-1}(i)} = \sum_{\nu=1}^{s}\sum_{i\in\Phi_\nu\cap\omega_0} g_{\sigma^{-1}(i)} + \sum_{i\in\Phi_{s+1}} g_{\sigma^{-1}(i)} \\
&\leqslant \sum_{i\in\Phi_1} g_{\sigma^{-1}(i)} + \sum_{\nu=2}^{s}\frac{1+\varepsilon}{2}\sum_{i\in\Phi_{\nu-1}} g_{\sigma^{-1}(i)} + \sum_{i\in\Phi_{s+1}} g_{\sigma^{-1}(i)} \\
&\leqslant \alpha + \frac{1+\varepsilon}{2}\|A\|_{(2,1)} + \frac{\|A\|_{(2,1)}}{s}.
\end{aligned}$$

Thus (see the definition of the parameters α, l, s, ρ and ε),

$$\beta \leqslant \biggl[\varkappa^{-c} + \biggl(\frac{1}{2} + \frac{1}{2}\varkappa^{-\delta}\biggr) + \varkappa^{-d}\biggr](\cos\varkappa^{-1})^{-1}. \tag{23}$$

Bearing in mind that, in view of (8),

$$\varkappa^{-d/2} < \varkappa^{-c} < \varkappa^{-2\delta},$$

and using the inequality $(\cos\varkappa^{-1})^{-1} < 1 + 1/\varkappa^2$, we deduce from (23) that

$$\beta \leqslant \frac{1}{2} + \varkappa^{-\delta}.$$

This proves the assertion of Theorem 1 (see (15)).

The author acknowledges the useful discussions he has had with S. V. Konyagin and K. S. Ryutin.

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Received 24/FEB/15
Translated by N. KRUZHILIN