

On channels with finite Holevo capacity.

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1 Introduction

As it is well known, the properties of quantum channels differ quite substantially for infinite and finite quantum systems. We consider a nontrivial class of infinite dimensional quantum channels, characterized by finiteness of the Holevo capacity (in what follows the χ -capacity).

The results in [12] imply some general properties of channels with finite χ -capacity such as compactness of the output set and existence of the unique output optimal average state (theorem 1). It is shown that each channel with finite χ -capacity can be uniformly approximated by channels with finite dimensional output system, having close χ -capacities and the output optimal average states (corollary 1).

The class of channels with finite χ -capacity contains subclass of channels with continuous output entropy, called CE-channels. These channels inherit some analytical properties of finite dimensional channels (proposition 1), revealing at the same time essential features of infinite dimensional channels (example 2). So, we may consider the class of CE-channels as an intermediate one and use technical advantages of dealing with such channels to approach general infinite dimensional channels and systems. For example, the channel of this type was used in [5] in considering properties of infinite dimensional entanglement-breaking channels and of the set of separable states.

In [4] the notion of an optimal measure (generalized ensemble) for constrained infinite dimensional channels was introduced and the sufficient condition of its existence was obtained. In this paper we consider a special condition of existence of an optimal measure for unconstrained infinite dimensional channels with finite χ -capacity (theorem 2, corollary 2, remark 1)

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and construct examples of channel with bounded output entropy, for which there exist no optimal measures (examples 1C and 2).

An interesting question is a problem of extension of a quantum channel from the set of all quantum states (=normal states) on $\mathfrak{B}(\mathcal{H})$ to the set of all states (=positive normalized linear functionals) on $\mathfrak{B}(\mathcal{H})$, which is a compact set in the *-weak topology. By using a simple topological observation we obtain a condition of existence of an extension of a channel to nonnormal states (proposition 3), which implies existence of such extension for arbitrary channel with finite χ -capacity (corollary 3). The entropic characteristics of a channel extension are considered as well as the condition of existence of an optimal measure (proposition 4), which shows the meaning of the conditions in theorem 2 (remark 2).

We complete this paper by considering the class of infinite dimensional channel, for which the χ -capacity can be explicitly determined and there exists a simple condition of continuity of the output entropy (proposition 5). The CE-channels of this class demonstrate essential infinite dimensional features such as nonexistence of an input optimal average state and of an optimal measure, discontinuity of the χ -capacity as a function of a channel (example 2).

2 Preliminaries

Let \mathcal{H} be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ - the Banach space of all bounded operators in \mathcal{H} with the cone $\mathfrak{B}_+(\mathcal{H})$ of all positive operators, $\mathfrak{T}(\mathcal{H})$ - the Banach space of all trace-class operators with the trace norm $\|\cdot\|_1$ and $\mathfrak{S}(\mathcal{H})$ - the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of all density operators in \mathcal{H} , which is complete separable metric space with the metric defined by the trace norm. Each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$ [2]. We will also consider the set $\widehat{\mathfrak{S}}(\mathcal{H})$ of all normalized positive functionals on $\mathfrak{B}(\mathcal{H})$, which is a compact subset of $(\mathfrak{B}(\mathcal{H}))^*$ in the *-weak topology. The set $\mathfrak{S}(\mathcal{H})$ can be considered as a *-weak dense subset of $\widehat{\mathfrak{S}}(\mathcal{H})$ [2]. Since the main attention in this paper is focused on $\mathfrak{S}(\mathcal{H})$ rather than $\widehat{\mathfrak{S}}(\mathcal{H})$ we will use the term *state* for elements of $\mathfrak{S}(\mathcal{H})$ while the elements of $\widehat{\mathfrak{S}}(\mathcal{H}) \setminus \mathfrak{S}(\mathcal{H})$ will be called *nonnormal states*.

Let $\mathcal{H}, \mathcal{H}'$ be a pair of separable Hilbert spaces which we shall call correspondingly input and output space. A channel Φ is a linear positive trace preserving map from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ such that the dual map $\Phi^* : \mathfrak{B}(\mathcal{H}') \mapsto \mathfrak{B}(\mathcal{H})$

(which exists since Φ is bounded [3]) is completely positive.

For arbitrary set \mathcal{A} let $\text{co}(\mathcal{A})$ and $\overline{\text{co}}(\mathcal{A})$ be the convex hull and the convex closure of the set \mathcal{A} correspondingly and let $\text{Ext}(\mathcal{A})$ be the set of all extreme points of the set \mathcal{A} [8].

We will use the following compactness criterion for subsets of states: *a closed subset \mathcal{K} of states is compact if and only if for any $\varepsilon > 0$ there is a finite dimensional projector P such that $\text{Tr}\rho P \geq 1 - \varepsilon$ for all $\rho \in \mathcal{K}$* [9],[4].

Speaking about continuity of a particular function on some set of states we mean continuity of the restriction of this function to this set.

Arbitrary finite collection $\{\rho_i\}$ of states in $\mathfrak{S}(\mathcal{H})$ with corresponding set of probabilities $\{\pi_i\}$ is called *ensemble* and is denoted by $\{\pi_i, \rho_i\}$. The state $\bar{\rho} = \sum_i \pi_i \rho_i$ is called *the average state* of the ensemble. Following [4] we treat an arbitrary Borel probability measure μ on $\mathfrak{S}(\mathcal{H})$ as *generalized ensemble* and the barycenter of the measure μ defined by the Pettis integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \rho \mu(d\rho)$$

as *the average state* of this ensemble. In this notations the conventional ensembles correspond to measures with finite support. For arbitrary closed subset \mathcal{A} of $\mathfrak{S}(\mathcal{H})$ we denote by $\mathcal{P}(\mathcal{A})$ the set of all probability measures supported by the set \mathcal{A} [14]. In what follows an arbitrary ensemble $\{\pi_i, \rho_i\}$ is considered as a particular case of probability measure and is also denoted by μ .

Consider the functionals

$$\chi_{\Phi}(\mu) = \int H(\Phi(\rho) \|\Phi(\bar{\rho}(\mu))) \mu(d\rho) \quad \text{and} \quad \hat{H}_{\Phi}(\mu) = \int H(\Phi(\rho)) \mu(d\rho).$$

In [4] (proposition 1 and the proof of the theorem) it is shown that both these functionals are lower semicontinuous on $\mathcal{P}(\mathfrak{S}(\mathcal{H})) = \mathcal{P}$ and

$$\chi_{\Phi}(\mu) = H(\Phi(\bar{\rho}(\mu))) - \hat{H}_{\Phi}(\mu) \tag{1}$$

for arbitrary μ such that $H(\Phi(\bar{\rho}(\mu))) < +\infty$.

If $\mu = \{\pi_i, \rho_i\}$ then

$$\chi_{\Phi}(\{\pi_i, \rho_i\}) = \sum_{i=1}^n \pi_i H(\Phi(\rho_i) \|\Phi(\bar{\rho})) \quad \text{and} \quad \hat{H}_{\Phi}(\{\pi_i, \rho_i\}) = \sum_{i=1}^n \pi_i H(\Phi(\rho_i)).$$

In what follows we will use the *decrease coefficient* $\text{dc}(\sigma)$ of a state σ defined in [12] by

$$\text{dc}(\sigma) = \inf\{\lambda > 0 \mid \text{Tr}\sigma^\lambda < +\infty\} \in [0, 1].$$

We will use the notion of the *H-convergence* of a sequence of states $\{\rho_n\}$ to a state ρ_0 defined by the condition $\lim_{n \rightarrow +\infty} H(\rho_n \parallel \rho_0) = 0$.

In [12] the properties of the χ -capacity as a function of a set of states were explored and the notion of the optimal average state $\Omega(\mathcal{A})$ of a set \mathcal{A} with finite χ -capacity was introduced. The set of states \mathcal{A} with finite χ -capacity is called regular if one of the two following conditions hold:

- $H(\Omega(\mathcal{A}))$ is finite and $\lim_{n \rightarrow +\infty} H(\rho_n) = H(\Omega(\mathcal{A}))$ for arbitrary sequence $\{\rho_n\}$ of states in $\text{co}(\mathcal{A})$ *H*-converging to the state $\Omega(\mathcal{A})$;
- the function $\rho \mapsto H(\rho \parallel \Omega(\mathcal{A}))$ is continuous on the set $\overline{\mathcal{A}}$.

In a sense these conditions are the minimal continuity requirements which guarantee the "good" properties of the χ -capacity (see theorems 2 and 3 in [12]). The simplest sufficient condition of regularity of a set \mathcal{A} is given by the inequality $\text{dc}(\Omega(\mathcal{A})) < 1$ (theorem 2E in [12]), but this condition is not necessary and there exist regular sets, consisting of states with infinite entropy.

The all notations used in this paper coincide with the notations accepted in [12].

3 General properties

Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be a channel with finite χ -capacity defined by

$$\bar{C}(\Phi) = \sup_{\{\pi_i, \rho_i\}} \chi_\Phi(\{\pi_i, \rho_i\}) = \sup_{\{\pi_i, \rho_i\}} \sum_i \pi_i H(\Phi(\rho_i) \parallel \Phi(\bar{\rho})). \quad (2)$$

In [4] it is shown that

$$\bar{C}(\Phi) = \sup_{\mu \in \mathcal{P}} \chi_\Phi(\mu) = \sup_{\mu \in \mathcal{P}} \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho) \parallel \Phi(\bar{\rho}(\mu))) \mu(d\rho). \quad (3)$$

According to [10] a sequence of ensembles $\{\{\pi_i^n, \rho_i^n\}\}_n$ such that

$$\lim_{n \rightarrow +\infty} \sum_i \pi_i^n H(\Phi(\rho_i^n) \parallel \Phi(\bar{\rho}_n)) = \bar{C}(\Phi)$$

is called *approximating sequence* for the channel Φ while any partial limit of the corresponding sequence of the average states $\{\bar{\rho}_n = \sum_i \pi_i^n \rho_i^n\}$ is called *input optimal average state* for the channel Φ .

By the compactness argument for a finite dimensional channel there exists at least one input optimal average state, coinciding with the average state of the optimal ensemble for this channel [13]. For an infinite dimensional channel with finite χ -capacity existence of an input optimal average state is a question depending on a channel (example 1), which is closely related to the question of existence of an optimal measure for this channel (theorem 2, corollary 2).

By the definitions the χ -capacity of a channel Φ coincides with the χ -capacity of the output set $\Phi(\mathfrak{S}(\mathcal{H}))$ of this channel. Thus theorems 1 and 2D in [12] imply the following observation.

Theorem 1. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be a channel with finite χ -capacity. The set $\Phi(\mathfrak{S}(\mathcal{H}))$ is a relatively compact subset of $\mathfrak{S}(\mathcal{H}')$. There exists the unique state $\Omega(\Phi)$ in $\mathfrak{S}(\mathcal{H}')$ such that*

$$H(\Phi(\rho) \parallel \Omega(\Phi)) \leq \bar{C}(\Phi), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

The state $\Omega(\Phi)$ lies in $\overline{\Phi(\mathfrak{S}(\mathcal{H}))}$. For arbitrary approximating sequence of ensembles $\{\{\pi_i^n, \rho_i^n\}\}_n$ the corresponding sequence $\{\Phi(\bar{\rho}_n)\}_n$ of images of their average states H -converges to the state $\Omega(\Phi)$.

If there exists an input optimal average state ρ_* for the channel Φ then $\Phi(\rho_*) = \Omega(\Phi)$.

The χ -capacity of the channel Φ can be defined by the expression¹

$$\bar{C}(\Phi) = \inf_{\sigma \in \mathfrak{S}(\mathcal{H}')} \sup_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi(\rho) \parallel \sigma) = \sup_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi(\rho) \parallel \Omega(\Phi)).$$

According to [10] the state $\Omega(\Phi)$ is called *the output optimal average state* for the channel Φ .

Below we consider examples of channels with finite χ -capacity with no input optimal average state, for which the output optimal average states are explicitly determined and play important role in studying of these channels.

¹By corollary 9 in [12] the infimum in this expression can be over the subset of $\overline{\Phi(\mathfrak{S}(\mathcal{H}))}$ consisting of states invariant for all automorphism α of $\mathfrak{S}(\mathcal{H}')$ such that $\alpha(\overline{\Phi(\mathfrak{S}(\mathcal{H}))}) \subseteq \overline{\Phi(\mathfrak{S}(\mathcal{H}))}$.

For a finite dimensional channel Φ the state $\Omega(\Phi)$ is an image of the average state of any optimal ensemble for this channel, which implies

$$\Omega(\Phi) \in \Phi(\mathfrak{S}(\mathcal{H})) \quad \text{and} \quad \bar{C}(\Phi) \leq H(\Omega(\Phi)).$$

For an infinite dimensional channel Φ with finite χ -capacity these relations do not hold in general (see example 1C below). The first relation follows from existence of at least one input optimal average state for the channel Φ while a sufficient condition of the second one can be expressed in terms of the spectrum of the state $\Omega(\Phi)$ (see proposition 2 below).

The simplest examples of infinite dimensional channels with finite χ -capacity are channels from an infinite dimensional quantum system into a finite dimensional one. Following [10] such channels will be called IF-channels. Theorem 1 implies, in particular, that each channel with finite χ -capacity can be uniformly approximated by IF-channels.

Corollary 1. *The channel $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ has finite χ -capacity if and only if there exists a sequence $\{\Phi_n : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}'_n), \mathcal{H}'_n \subseteq \mathcal{H}'\}$ of IF-channels such that*

$$\lim_{n \rightarrow +\infty} \sup_{\rho \in \mathfrak{S}(\mathcal{H})} \|\Phi_n(\rho) - \Phi(\rho)\|_1 = 0 \quad \text{and} \quad \sup_n \bar{C}(\Phi_n) < +\infty.$$

The sequence $\{\Phi_n\}$ can be chosen in such a way that

$$\lim_{n \rightarrow +\infty} \bar{C}(\Phi_n) = \bar{C}(\Phi) \quad \text{and} \quad \lim_{n \rightarrow +\infty} \Omega(\Phi_n) = \Omega(\Phi).$$

Proof. If the above sequence $\{\Phi_n\}$ exists then the χ -capacity of the channel Φ is finite due to lower semicontinuity of the χ -capacity as a function of a channel [10].

If the χ -capacity of the channel Φ is finite then by theorem 1 the set $\Phi(\mathfrak{S}(\mathcal{H}))$ is relatively compact. Let $\{P_n\}$ be a sequence of finite rank projectors in \mathcal{H}' strongly converging to $I_{\mathcal{H}'}$. The compactness criterion implies $\lim_{n \rightarrow +\infty} \inf_{\rho \in \mathfrak{S}(\mathcal{H})} \text{Tr} \Phi(\rho) P_n = 1$. Hence the sequence of IF-channels

$$\Phi_n(\rho) = P_n \Phi(\rho) P_n + (\text{Tr}(I_{\mathcal{H}'} - P_n) \Phi(\rho)) \tau_n$$

from $\mathfrak{S}(\mathcal{H})$ into $\mathfrak{S}(P_n(\mathcal{H}') \oplus \mathcal{H}''_n)$, where τ_n is a pure state in some finite dimensional subspace \mathcal{H}''_n of $\mathcal{H}' \ominus P_n(\mathcal{H}')$, uniformly converges to the channel Φ . Since for each n the channel Φ_n can be represented as a composition of

the channel Φ and the channel $\rho \mapsto P_n \rho P_n + (\text{Tr}(I_{\mathcal{H}'} - P_n)\rho) \tau_n$, the monotonicity property of the relative entropy implies $\bar{C}(\Phi_n) \leq \bar{C}(\Phi)$ for all n . This and lemma 4 in [12] imply the limit expressions in the second part of the corollary. \square

The above corollary shows that the class of channels with finite χ -capacity is sufficiently close to the class of IF-channels. Nevertheless channels of this class demonstrate many features of essential infinite dimensional channels (see examples 1 and 2 below).

There exists a class of infinite dimensional channels with finite χ -capacity which is the most close to the class of IF-channels.

Definition 1. *A channel $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ is called CE-channel if the restriction of the quantum entropy to the set $\Phi(\mathfrak{S}(\mathcal{H}))$ is continuous.*

The results in [12] provide different sufficient conditions of the CE-property of a channel (see proposition 2 below). In particular, proposition 10 in [12] implies that the class of CE-channels is closed under tensor products: *If $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi : \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ are CE-channels then the channel $\Phi \otimes \Psi : \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \mapsto \mathfrak{S}(\mathcal{H}' \otimes \mathcal{K}')$ is a CE-channel.*

The χ -capacity $\bar{C}(\Phi)$ of the channel Φ can be defined as the least upper bound of the χ -function of the channel Φ defined by [4]

$$\chi_{\Phi}(\rho) = \sup_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\rho)) = \sup_{\mu \in \mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\sigma) \| \Phi(\rho)) \mu(d\sigma),$$

where $\mathcal{P}_{\{\rho\}}$ is the set of all probability measures on $\mathfrak{S}(\mathcal{H})$ with the barycenter ρ .

It was shown in [11] that the convex closure of the output entropy is defined by

$$\hat{H}_{\Phi}(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho)) \mu(d\rho) \leq +\infty.$$

In contrast to the χ -function the infimum over all measures in $\mathcal{P}_{\{\rho\}}$ in the definition of the \hat{H} -function does not coincide in general with the infimum over all measures in $\mathcal{P}_{\{\rho\}}$ with finite support [11].

The χ -function and the \hat{H} -function of an arbitrary channel are nonnegative lower semicontinuous concave and convex functions correspondingly [11]. The following proposition shows, in particular, that the χ -function and the \hat{H} -function of a CE-channel have properties similar to the properties of these

functions of a finite dimensional channel.

Proposition 1. *Let Φ be channel with finite χ -capacity. Then*

$$\chi_{\Phi}(\rho) \leq \bar{C}(\Phi) - H(\Phi(\rho) \parallel \Omega(\Phi)), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

If Φ is a CE-channel then the functions χ_{Φ} and \hat{H}_{Φ} are continuous on $\mathfrak{S}(\mathcal{H})$. Moreover

$$\chi_{\Phi}(\rho) = H(\Phi(\rho)) - \hat{H}_{\Phi}(\rho) \quad \text{and} \quad \hat{H}_{\Phi}(\rho) = \inf_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i)).$$

Proof. The inequality for the χ -function follows from corollary 1 in [10]. The continuity assertion and the representations for the χ -function and the \hat{H} -function are corollaries of proposition 7 and proposition 5 in [11]. \square

The following proposition shows the special role of the output optimal average state.

Proposition 2. *Let Φ be a channel with finite χ -capacity.*

If $\text{dc}(\Omega(\Phi)) < 1$ then $\sup_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi(\rho)) < +\infty$ and $\bar{C}(\Phi) \leq H(\Omega(\Phi))$.

If $\text{dc}(\Omega(\Phi)) = 0$ then Φ is a CE-channel.

Proof. Suppose $\text{dc}(\Omega(\Phi)) < 1$. By theorem 2E in [12] the entropy is bounded on the set $\bar{\Phi}(\mathfrak{S}(\mathcal{H}))$. By theorem 1 for arbitrary approximating sequence of ensembles $\{\{\pi_i^n, \rho_i^n\}\}_n$ the corresponding sequence $\{\Phi(\bar{\rho}_n)\}_n$ of images of their average states H -converges to the state $\Omega(\Phi)$. By proposition 2 in [12] the condition $\text{dc}(\Omega(\Phi)) < 1$ implies

$$\bar{C}(\Phi) = \lim_{n \rightarrow +\infty} \chi_{\Phi}(\{\pi_i^n, \rho_i^n\}) \leq \lim_{n \rightarrow +\infty} H(\Phi(\bar{\rho}_n)) = H(\Omega(\Phi)).$$

Suppose $\text{dc}(\Omega(\Phi)) = 0$. By theorem 2E in [12] the entropy is continuous on the set $\bar{\Phi}(\mathfrak{S}(\mathcal{H}))$. \square

The assertions of the above proposition are illustrated by the below example 1, in which the family of channels with different decrease coefficient of the output optimal average state is considered.

According to [4] a measure μ_* in $\mathcal{P}(\text{Ext}\mathfrak{S}(\mathcal{H}))$, at which the supremum in definition (3) is achieved, is called *optimal measure (optimal generalized ensemble)* for the channel Φ .

The notion of an optimal measure is a generalization of the notion of an optimal ensemble for a finite dimensional channel. In [13] it is shown that each optimal ensemble is characterized by the maximal distance property.

The first part of the following theorem generalizes this observation to the case of infinite dimensional channel.

Theorem 2. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be a channel with finite χ -capacity.*

If there exists an optimal measure μ_ for the channel Φ then its barycenter $\bar{\rho}(\mu_*)$ is an input optimal average state for the channel Φ and the following "maximal distance property" holds*

$$H(\Phi(\rho) \parallel \Omega(\Phi)) = \bar{C}(\Phi) \text{ for } \mu_* - \text{almost all } \rho \text{ in } \mathfrak{S}(\mathcal{H}).$$

If there exists an input optimal average state ρ_ for the channel Φ and the set $\Phi(\mathfrak{S}(\mathcal{H}))$ is regular then there exists an optimal measure μ_* for the channel Φ such that $\bar{\rho}(\mu_*) = \rho_*$.*

Proof. Since every probability measure can be weakly approximated by a sequence of measures with finite support lower semicontinuity of the functional χ_Φ implies that the barycenter $\bar{\rho}(\mu_*)$ of any optimal measure μ_* for the channel Φ is an input optimal average state for the channel Φ . The maximal distance property follows from the definition of an optimal measure and theorem 1.

Suppose the set $\Phi(\mathfrak{S}(\mathcal{H}))$ is regular and $\{\mu_n = \{\pi_i^n, \rho_i^n\}\}_n$ is an approximating sequence of ensembles for the channel Φ such that the corresponding sequence $\{\bar{\rho}(\mu_n)\}_n$ converges to the state ρ_* . By convexity and lower semicontinuity of the relative entropy we may assume that each ensemble in this sequence consists of pure states. Since the sequence $\{\bar{\rho}(\mu_n)\}_n$ is relatively compact the sequence $\{\mu_n\}_n$ is relatively weakly compact by proposition 2 in [4] and hence it contains weakly converging subsequence. So, we may assume that the sequence $\{\mu_n\}_n$ weakly converges to a particular measure μ_* supported by pure states due to theorem 6.1 in [14].

For given n let $\nu_n = \mu_n \circ \Phi^{-1}$ be the image of the measure μ_n under the mapping Φ , so that $\nu_n = \{\pi_i^n, \Phi(\rho_i^n)\}$. It is easy to see that $\{\nu_n\}_n$ is an approximating sequence of ensembles for the set $\overline{\Phi(\mathfrak{S}(\mathcal{H}))}$. Since this set is regular by the condition the arguments from the proof of theorem 3 and lemma 5 in [12] imply existence of a subsequence $\{\nu_{n_k}\}_k$ weakly converging to an optimal measure ν_* for the set $\overline{\Phi(\mathfrak{S}(\mathcal{H}))}$. But $\mu_{n_k} \rightarrow \mu_*$ in the weak topology implies $\mu_{n_k} \circ \Phi^{-1} \rightarrow \mu_* \circ \Phi^{-1}$ in the weak topology. So, we obtain $\mu_* \circ \Phi^{-1} = \nu_*$. Thus μ_* is an optimal measure for the channel Φ . \square

Theorem 2 and theorem 2E in [12] imply the following sufficient condition of existence of an optimal measure.

Corollary 2. *Existence of an input optimal average state for the channel*

Φ implies existence of an optimal measure for the channel Φ provided one of the following conditions holds:

- $\text{dc}(\Omega(\Phi)) < 1$;
- the channel Φ is a CE-channel.

Remark 1. The conditions in theorem 2 and in corollary 2 are essential as it is shown by the examples of channels with finite χ -capacity for which there exist no optimal measures (examples 1C and 2). The meaning of these conditions are considered in remark 2 below. \square

The above general observations are illustrated by the following family of channels, depending on a sequence of positive numbers.

Example 1. Let $\{|n\rangle\}_{n \in \mathbb{N} \cup \{0\}}$ be an orthonormal basis in \mathcal{H}' and $\{q_n\}_{n \in \mathbb{N}}$ be a sequence of numbers in $(0, 1]$, converging to zero. Consider the set $\mathcal{S}_{\{q_n\}}^1$ consisting of pure states

$$\sigma_k = (1 - q_{|k|})|0\rangle\langle 0| + q_{|k|}|k\rangle\langle k| + \text{sign}(k)\sqrt{(1 - q_{|k|})q_{|k|}}(|0\rangle\langle k| + |k\rangle\langle 0|),$$

indexed by the set $\mathbb{Z} \setminus \{0\}$. This set can be considered as a sequence of states converging to the state $|0\rangle\langle 0|$ and its properties were explored in subsection 5.1 in [12].

Consider the channel

$$\Phi_{\{q_n\}}(\rho) = \sum_k \langle k|\rho|k\rangle \sigma_k,$$

where $\{|k\rangle\}_{k \in \mathbb{Z} \setminus \{0\}}$ be an orthonormal basis in \mathcal{H} .

This channel is entanglement-breaking [6],[5] and $\overline{\Phi_{\{q_n\}}(\mathfrak{S}(\mathcal{H}))} = \overline{\text{co}(\mathcal{S}_{\{q_n\}}^1)}$ is a compact subset of $\mathfrak{S}(\mathcal{H}')$. Since the states in the sequence $\mathcal{S}_{\{q_n\}}^1$ are pure $\bar{C}(\Phi_{\{q_n\}}) = \sup_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi_{\{q_n\}}(\rho))$. Let $\lambda_{\{q_n\}}^*$ be the infimum of the set $\left\{ \lambda : \sum_n \exp\left(-\frac{\lambda}{q_n}\right) < +\infty \right\}$ if it is not empty and $\lambda_{\{q_n\}}^* = +\infty$ otherwise. Let

$$F_{\{q_n\}}(\lambda) = \sum_{k=1}^{+\infty} \left(\frac{1}{q_k} - 1 \right) \exp\left(-\frac{\lambda}{q_k}\right)$$

be a decreasing function on $[\lambda_{\{q_n\}}^*, +\infty)$ with the range $(0, +\infty]$.

We will show that:

A) If $\lambda_{\{q_n\}}^* = 0$ then

- $\Phi_{\{q_n\}}$ is a CE-channel;
- there exist the unique optimal measure and the unique input optimal average state for the channel $\Phi_{\{q_n\}}$;
- $\Omega(\Phi_{\{q_n\}}) \in \overline{\Phi_{\{q_n\}}(\mathfrak{S}(\mathcal{H}))}$,
 $\text{dc}(\Omega(\Phi_{\{q_n\}})) = 0$ and $\bar{C}(\Phi_{\{q_n\}}) = H(\Omega(\Phi_{\{q_n\}}))$;

B) If $0 < \lambda_{\{q_n\}}^* < +\infty$ and $F_{\{q_n\}}(\lambda_{\{q_n\}}^*) \geq 1$ then

- $\bar{C}(\Phi_{\{q_n\}}) < +\infty$, but $\Phi_{\{q_n\}}$ is not a CE-channel;
- there exist the unique optimal measure and the unique input optimal average state for the channel $\Phi_{\{q_n\}}$;
- $\Omega(\Phi_{\{q_n\}}) \in \overline{\Phi_{\{q_n\}}(\mathfrak{S}(\mathcal{H}))}$,
 $0 < \text{dc}(\Omega(\Phi_{\{q_n\}})) \leq 1$ ² and $\bar{C}(\Phi_{\{q_n\}}) = H(\Omega(\Phi_{\{q_n\}}))$;

C) If $0 < \lambda_{\{q_n\}}^* < +\infty$ and $F_{\{q_n\}}(\lambda_{\{q_n\}}^*) < 1$ then

- $\bar{C}(\Phi_{\{q_n\}}) < +\infty$, but $\Phi_{\{q_n\}}$ is not a CE-channel;
- there exist no optimal measure and no input optimal average state for the channel $\Phi_{\{q_n\}}$;
- $\Omega(\Phi_{\{q_n\}}) \in \overline{\Phi_{\{q_n\}}(\mathfrak{S}(\mathcal{H}))} \setminus \Phi_{\{q_n\}}(\mathfrak{S}(\mathcal{H}))$,
 $\text{dc}(\Omega(\Phi_{\{q_n\}})) = 1$ and $\bar{C}(\Phi_{\{q_n\}}) > H(\Omega(\Phi_{\{q_n\}}))$;

D) If $\lambda_{\{q_n\}}^* = +\infty$ then $\bar{C}(\Phi_{\{q_n\}}) = +\infty$.

In the cases A and B the χ -capacity, the unique optimal measure, the unique input and output optimal average states are defined by the following expressions (correspondingly):

$$\bar{C}(\Phi_{\{q_n\}}) = \lambda_{\{q_n\}}^1 - \log \pi_{\{q_n\}}^1, \quad \mu_* = \left\{ \frac{\pi_{\{q_n\}}^1}{2q_{|k|}} \exp\left(-\frac{\lambda_n}{q_{|k|}}\right); |k\rangle\langle k| \right\}_{k \in \mathbb{Z} \setminus \{0\}},$$

$$\bar{\rho}(\mu_*) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\pi_{\{q_n\}}^1}{2q_{|k|}} \exp\left(-\frac{\lambda_{\{q_n\}}^1}{q_{|k|}}\right) |k\rangle\langle k|,$$

²In this case $\text{dc}(\Omega(\Phi_{\{q_n\}})) = 1$ if and only if $F_{\{q_n\}}(\lambda_{\{q_n\}}^*) = 1$.

$$\Omega(\Phi_{\{q_n\}}) = \pi_{\{q_n\}}^1 |0\rangle\langle 0| + \pi_{\{q_n\}}^1 \sum_{n=1}^{+\infty} \exp\left(-\frac{\lambda_{\{q_n\}}^1}{q_n}\right) |n\rangle\langle n|,$$

where $\lambda_{\{q_n\}}^1$ is the unique solution of the equation $F_{\{q_n\}}(\lambda) = 1$ and

$$\pi_{\{q_n\}}^1 = \left(1 + \sum_{n=1}^{+\infty} \exp\left(-\frac{\lambda_{\{q_n\}}^1}{q_n}\right)\right)^{-1} = \left(\sum_{n=1}^{+\infty} \frac{1}{q_n} \exp\left(-\frac{\lambda_{\{q_n\}}^1}{q_n}\right)\right)^{-1} \in [0, 1].$$

In the case C the χ -capacity and the output optimal average state are defined by the following expressions (correspondingly):

$$\bar{C}(\Phi_{\{q_n\}}) = \lambda_{\{q_n\}}^* - \log \pi_{\{q_n\}}^*,$$

$$\Omega(\Phi_{\{q_n\}}) = \pi_{\{q_n\}}^* |0\rangle\langle 0| + \pi_{\{q_n\}}^* \sum_{n=1}^{+\infty} \exp\left(-\frac{\lambda_{\{q_n\}}^*}{q_n}\right) |n\rangle\langle n|,$$

where

$$\pi_{\{q_n\}}^* = \left(1 + \sum_{n=1}^{+\infty} \exp\left(-\frac{\lambda_{\{q_n\}}^*}{q_n}\right)\right)^{-1} \in [0, 1].$$

If $\lambda_{\{q_n\}}^* = +\infty$ then $\bar{C}(\Phi_{\{q_n\}}) = \bar{C}(\mathcal{S}_{\{q_n\}}^1) = +\infty$ by the observation in subsection 5.1 in [12].

Suppose $\lambda_{\{q_n\}}^* < +\infty$. Without loss of generality we may assume that the sequence $\{q_n\}$ is nonincreasing. Let

$$H = 0|0\rangle\langle 0| + \sum_{n=1}^{+\infty} q_n^{-1} |n\rangle\langle n| \quad (4)$$

be a \mathfrak{H} -operator in \mathcal{H}' so that $\text{ic}(H) = \lambda_{\{q_n\}}^*$ (see section 3 in [12]). It is easy to see that $\mathcal{S}_{\{q_n\}}^\varepsilon \subseteq \mathcal{K}_{H,1}$ and hence proposition 1b in [12] implies $\bar{C}(\Phi_{\{q_n\}}) = \bar{C}(\mathcal{S}_{\{q_n\}}^1) \leq \bar{C}(\mathcal{K}_{H,1}) = \sup_{\rho \in \mathcal{K}_{H,1}} H(\rho) < +\infty$. We will show that $\bar{C}(\mathcal{S}_{\{q_n\}}^1) = \bar{C}(\mathcal{K}_{H,1})$. In the proof of proposition 1a in [12] the sequence $\{\rho_n\}$ of states defined by formula (9) was constructed and it was shown that $\lim_{n \rightarrow +\infty} H(\rho_n) = \sup_{\rho \in \mathcal{K}_{H,1}} H(\rho)$. For the \mathfrak{H} -operator H defined by (4) the states of the above sequence have the form

$$\rho_n = \left(1 + \sum_{k=1}^n \exp\left(-\frac{\lambda_n}{q_k}\right)\right)^{-1} \left(|0\rangle\langle 0| + \sum_{k=1}^n \exp\left(-\frac{\lambda_n}{q_k}\right) |k\rangle\langle k|\right) \quad (5)$$

for sufficiently large $n \in \mathbb{N}$, where λ_n is the unique solution of the equation

$$1 + \sum_{k=1}^n \exp\left(-\frac{\lambda}{q_k}\right) = \sum_{k=1}^n \frac{1}{q_k} \exp\left(-\frac{\lambda}{q_k}\right). \quad (6)$$

The sequence of states defined by (5) lies in $\overline{\text{co}}(\mathcal{S}_{\{q_n\}}^1)$. Indeed, for given n let

$$\pi_k^n = \left(\sum_{k=1}^n \frac{2}{q_{|k|}} \exp\left(-\frac{\lambda_n}{q_{|k|}}\right) \right)^{-1} \frac{1}{q_{|k|}} \exp\left(-\frac{\lambda_n}{q_{|k|}}\right), \quad k \in \mathbb{Z} \setminus \{0\},$$

be a probability distribution. By using equality (6) with $\lambda = \lambda_n$ it is easy to see that $\sum_k \pi_k^n \sigma_k = \rho_n$ and hence $\rho_n \in \overline{\text{co}}(\mathcal{S}_{\{q_n\}}^1)$. This, proposition 1b and theorem 2C in [12] imply

$$\bar{C}(\Phi_{\{q_n\}}) = \bar{C}(\mathcal{S}_{\{q_n\}}^1) = \bar{C}(\mathcal{K}_{H,1}) = \lambda^* - \log \pi^* \quad (7)$$

and

$$\Omega(\Phi_{\{q_n\}}) = \Omega(\mathcal{S}_{\{q_n\}}^1) = \Omega(\mathcal{K}_{H,1}) = \pi^* |0\rangle\langle 0| + \pi^* \sum_{n=1}^{+\infty} \exp\left(-\frac{\lambda^*}{q_n}\right) |n\rangle\langle n|, \quad (8)$$

where $\pi^* = 1 + \sum_{n=1}^{+\infty} \exp\left(-\frac{\lambda^*}{q_n}\right)$ and λ^* is the unique solution of the equation

$$1 + \sum_{k=1}^{+\infty} \exp\left(-\frac{\lambda}{q_k}\right) = \sum_{k=1}^{+\infty} \frac{1}{q_k} \exp\left(-\frac{\lambda}{q_k}\right) \quad (9)$$

if

$$h_*(H) = \frac{\text{Tr} H \exp(-ic(H)H)}{\text{Tr} \exp(-ic(H)H)} = \frac{\sum_{k=1}^{+\infty} \frac{1}{q_k} \exp\left(-\frac{\lambda_{\{q_n\}}^*}{q_k}\right)}{1 + \sum_{k=1}^{+\infty} \exp\left(-\frac{\lambda_{\{q_n\}}^*}{q_k}\right)} \geq h = 1 \quad (10)$$

and $\lambda^* = \lambda_{\{q_n\}}^*$ otherwise. It is easy to see that (9) is equivalent to the equation $F_{\{q_n\}}(\lambda) = 1$ while (10) means the inequality $F_{\{q_n\}}(\lambda_{\{q_n\}}^*) \geq 1$.

The assertion concerning existence of the unique optimal measure μ_* for the channel $\Phi_{\{q_n\}}$ in the cases A and B and the expression for this measure

follows from the observation in subsection 5.1 in [12] (with $\varepsilon = 1$). The barycenter of the measure μ_* is the unique input optimal average state for the channel $\Phi_{\{q_n\}}$. Uniqueness of the input optimal average state follows from uniqueness of the optimal measure, since proposition 1b in [12] and (8) imply regularity of the set $\Phi_{\{q_n\}}(\mathfrak{S}(\mathcal{H})) \subseteq \mathcal{K}_{H,1}$ in the cases A and B and hence theorem 2 shows that each input optimal average state is a barycenter of at least one optimal measure.

In the cases A and B the state $\Omega(\Phi_{\{q_n\}})$ is an image of the barycenter of the optimal measure and hence it lies in $\Phi(\mathfrak{S}(\mathcal{H}))$. To prove nonexistence of an input optimal average state for the channel $\Phi_{\{q_n\}}$ in the case C it is sufficient to show that in this case the state $\Omega(\Phi_{\{q_n\}})$ does not lie in $\Phi(\mathfrak{S}(\mathcal{H}))$.

The set $\Phi(\mathfrak{S}(\mathcal{H}))$ is a σ -convex hull of the set $\mathcal{S}_{\{q_n\}}^1 = \{\sigma_k\}_{k \in \mathbb{Z} \setminus \{0\}}$, so that the assumption $\Omega(\Phi_{\{q_n\}}) \in \Phi(\mathfrak{S}(\mathcal{H}))$ implies existence of a probability distribution $\{\pi_k\}_{k \in \mathbb{Z} \setminus \{0\}}$ such that $\Omega(\Phi_{\{q_n\}}) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \pi_k \sigma_k$. By using the expression for the state $\Omega(\Phi_{\{q_n\}})$ in the case C we obtain from this decomposition that

$$q_n(\pi_{-n} + \pi_n) = \frac{\exp\left(-\frac{\lambda_{\{q_n\}}^*}{q_n}\right)}{1 + \sum_{k=1}^{+\infty} \exp\left(-\frac{\lambda_{\{q_n\}}^*}{q_k}\right)}, \quad \forall n \in \mathbb{N},$$

and hence

$$\sum_{k=1}^{+\infty} \frac{1}{q_k} \exp\left(-\frac{\lambda_{\{q_n\}}^*}{q_k}\right) = 1 + \sum_{k=1}^{+\infty} \exp\left(-\frac{\lambda_{\{q_n\}}^*}{q_k}\right).$$

But this equality means that the equality holds in inequality (16) in [12] for the \mathfrak{H} -operator H defined by (4). By the observation in [12] equality in inequality (16) means inequality $h_*(H) \geq h = 1$ equivalent to the inequality $F_{\{q_n\}}(\lambda_{\{q_n\}}^*) \geq 1$, which contradicts to the definition of the case C.

If $\lambda_{\{q_n\}}^* = \text{ic}(H) = 0$ then the above observation implies $\text{dc}(\Omega(\Phi_{\{q_n\}})) = 0$ and by proposition 2 $\Phi_{\{q_n\}}$ is a CE-channel.

Suppose the entropy is continuous on the set $\overline{\text{co}}(\mathcal{S}_{\{q_n\}}^1)$. Consider the sequence of states

$$\rho_n = \left(1 - \sum_{k=1}^n q_k \pi_k\right) |0\rangle\langle 0| + \sum_{k=1}^n q_k \pi_k |k\rangle\langle k|,$$

where $\{\pi_k = q_k^{-1}(\sum_{k=1}^n q_k^{-1})^{-1}\}_{k=1}^n$ is a probability distribution. It is easy to see that this sequence lies in $\text{co}(\mathcal{S}_{\{q_n\}}^1)$ and converges to the pure state $|0\rangle\langle 0|$. By the continuity assumption $\lim_{n \rightarrow +\infty} H(\rho_n) = 0$, which implies

$$\lim_{n \rightarrow +\infty} n(q_k \pi_k (-\log(q_k \pi_k))) = \lim_{n \rightarrow +\infty} n f\left(\sum_{k=1}^n q_k^{-1}\right) = 0, \quad (11)$$

where $f(x) = \log x/x$. Since the function $f(x)$ is decreasing for large x the obvious inequality $\sum_{k=1}^n q_k^{-1} \leq n q_n^{-1}$ and (11) imply $\lim_{n \rightarrow +\infty} \nu_n = 0$, where $\nu_n = n f(n q_n^{-1}) = q_n \log(n q_n^{-1})$. Hence for arbitrary $\lambda > 0$ we have

$$\left(\frac{q_n}{n}\right)^{\frac{\lambda}{\nu_n}} = \exp\left(-\frac{\lambda}{q_n}\right),$$

which implies $\lambda_{\{q_n\}}^* = 0$. \square

The most interesting case of the above example is the case C, in which the channel $\Phi_{\{q_n\}}$ demonstrates essential infinite dimensional features. The example of a sequence $\{q_n\}$ corresponding to this case can be found in [12].

Example 1 can be generalized by considering the set $\mathcal{S}_{\{q_n\}}^\varepsilon$ with arbitrary ε in $[0, 1]$ instead of $\mathcal{S}_{\{q_n\}}^1$ [12].

4 On extension to nonnormal states

Let $\widehat{\mathfrak{S}}(\mathcal{H})$ be the set of all normalized positive functionals on $\mathfrak{B}(\mathcal{H})$, so that $\widehat{\mathfrak{S}}(\mathcal{H}) \subset (\mathfrak{B}(\mathcal{H}))^*$. It is known that $\widehat{\mathfrak{S}}(\mathcal{H})$ is compact in the $*$ -weak topology and that $\mathfrak{S}(\mathcal{H})$ can be considered as a $*$ -weak dense subset of $\widehat{\mathfrak{S}}(\mathcal{H})$ [2]. So, it is natural to explore a possibility to extend an arbitrary channel from $\mathfrak{S}(\mathcal{H})$ to $\widehat{\mathfrak{S}}(\mathcal{H})$. The following proposition provides a simple characterization of the class of channels, which has natural extension to $\widehat{\mathfrak{S}}(\mathcal{H})$.

Proposition 3.³ *A channel $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ can be extended to the mapping $\widehat{\Phi} : \widehat{\mathfrak{S}}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ continuous with respect to the $*$ -weak topology on the set $\widehat{\mathfrak{S}}(\mathcal{H})$ and the trace norm topology on the set $\mathfrak{S}(\mathcal{H}')$ if and only if the set $\Phi(\mathfrak{S}(\mathcal{H}))$ is relatively compact.*

If the above extension $\widehat{\Phi}$ exists then $\overline{\Phi(\mathfrak{S}(\mathcal{H}))} = \widehat{\Phi}(\widehat{\mathfrak{S}}(\mathcal{H}))$.

³This simple observation seems to be a corollary of a general result in the functional analysis. The author would be grateful for any references.

Proof. If $\widehat{\Phi}$ is the above extension of the channel Φ then $\widehat{\Phi}(\widehat{\mathfrak{S}}(\mathcal{H}))$ is compact as an image of a compact set under a continuous mapping. Since $\mathfrak{S}(\mathcal{H})$ is a $*$ -weak dense subset of $\widehat{\mathfrak{S}}(\mathcal{H})$ we have $\overline{\widehat{\Phi}(\mathfrak{S}(\mathcal{H}))} = \widehat{\Phi}(\widehat{\mathfrak{S}}(\mathcal{H}))$.

Suppose the set $\overline{\widehat{\Phi}(\mathfrak{S}(\mathcal{H}))}$ is compact. Since $\mathfrak{B}(\mathcal{H})^* = \mathfrak{I}(\mathcal{H})^{**}$ there exists the linear mapping $\Phi^{**} : \mathfrak{B}(\mathcal{H})^* \mapsto \mathfrak{B}(\mathcal{H}')^*$ continuous with respect to the $*$ -weak topologies on the both spaces and such that $\Phi^{**}|_{\mathfrak{S}(\mathcal{H})} = \Phi$. By using the compactness argument and $*$ -weak density of the set $\mathfrak{S}(\mathcal{H})$ in $\widehat{\mathfrak{S}}(\mathcal{H})$ it is easy to show that $\Phi^{**}(\widehat{\mathfrak{S}}(\mathcal{H})) \subseteq \overline{\widehat{\Phi}(\mathfrak{S}(\mathcal{H}))}$. Since each one-to-one continuous mapping from compact topological space onto Hausdorff topological space is a homeomorphism [7] we can conclude that the identity mapping from $\overline{\widehat{\Phi}(\mathfrak{S}(\mathcal{H}))}$ with the trace norm topology onto itself with the $*$ -weak topology has a continuous converse. This and the previous observation imply that Φ^{**} is continuous with respect to the $*$ -weak topology on $\widehat{\mathfrak{S}}(\mathcal{H})$ and the trace norm topology on $\mathfrak{S}(\mathcal{H}')$. So, the mapping $\Phi^{**}|_{\widehat{\mathfrak{S}}(\mathcal{H})}$ has all the properties of the extension $\widehat{\Phi}$, stated in the proposition. \square

Definition 2. For an arbitrary channel Φ with relatively compact output the mapping $\widehat{\Phi}$ introduced in proposition 3 is called its channel extension.

Proposition 3 and theorem 1 imply the following observation.

Corollary 3. If Φ is a channel with finite χ -capacity then it has the channel extension $\widehat{\Phi}$.

By this corollary for given channel Φ with finite χ -capacity it is possible to consider the entropic characteristics of its channel extension $\widehat{\Phi}$ such as the minimal output entropy $H_{\min}(\widehat{\Phi}) = \inf_{\rho \in \widehat{\mathfrak{S}}(\mathcal{H})} H(\widehat{\Phi}(\rho))$ and the χ -capacity

$$\bar{C}(\widehat{\Phi}) = \sup_{\{\pi_i, \hat{\rho}_i\}} \sum_i \pi_i H(\widehat{\Phi}(\hat{\rho}_i) \| \widehat{\Phi}(\sum_j \pi_j \hat{\rho}_j)), \quad (12)$$

where the supremum is over all finite ensembles $\{\pi_i, \hat{\rho}_i\}$ of states in $\widehat{\mathfrak{S}}(\mathcal{H})$. By proposition 3 and theorem 2B in [12] we have

$$\bar{C}(\widehat{\Phi}) = \bar{C}(\overline{\widehat{\Phi}(\mathfrak{S}(\mathcal{H}))}) = \bar{C}(\Phi(\mathfrak{S}(\mathcal{H}))) = \bar{C}(\Phi). \quad (13)$$

This means that one can not increase the χ -capacity by using nonnormal states.

In the same way as for the initial channel Φ we may define the notions of an approximating sequence of ensembles and of an input optimal average state for the channel extension $\widehat{\Phi}$. In contrast to the case of the initial channel Φ compactness of the set $\widehat{\mathfrak{S}}(\mathcal{H})$ guarantees existence of at least one

input optimal average state for the channel extension $\widehat{\Phi}$. By using lower semicontinuity of the relative entropy and (13) it is possible to show that each *normal* input optimal average state for the channel extension $\widehat{\Phi}$ is an input optimal average state for the initial channel Φ and vice versa. For the channel extension $\widehat{\Phi}$ it is possible to prove the analog of theorem 1, in particular, to show that the image of any input optimal average state for the channel extension $\widehat{\Phi}$ coincides with the output optimal average state $\Omega(\Phi)$.

In contrast to the χ -capacity the minimal output entropies for the channel Φ and for its extension $\widehat{\Phi}$ may be different as it follows from the below simple example. Let $\{\rho_n\}_{n=1}^{+\infty}$ be a sequence of states in $\mathfrak{S}(\mathcal{H})$ with infinite entropy, converging to pure state ρ_0 and such that $\bar{C}(\{\rho_n\}_{n=1}^{+\infty}) < +\infty$. Consider the channel $\Phi(\rho) = \sum_{n=1}^{+\infty} \langle n|\rho|n\rangle \rho_n$, where $\{|n\rangle\}$ is some orthonormal basis of \mathcal{H} . Since an arbitrary state in $\Phi(\mathfrak{S}(\mathcal{H}))$ majorizes at least one trace class operator with infinite entropy it has infinite entropy as well [15]. So we have $H_{\min}(\Phi) = +\infty$. But by proposition 3 we have $\widehat{\Phi}(\widehat{\mathfrak{S}}(\mathcal{H})) = \overline{\text{co}}(\{\rho_n\}_{n=1}^{+\infty})$ and hence there exists a state $\widehat{\rho}_0$ in $\widehat{\mathfrak{S}}(\mathcal{H})$ such that $\widehat{\Phi}(\widehat{\rho}_0) = \rho_0$. This implies $H_{\min}(\widehat{\Phi}) = 0$.

Let $\widehat{\mathcal{P}}$ be the set of all regular Borel measures on the set $\widehat{\mathfrak{S}}(\mathcal{H})$ endowed with the $*$ -weak topology. In contrast to the set \mathcal{P} the set $\widehat{\mathcal{P}}$ is not metrizable but it is compact in the weak topology [1].

In the same way ⁴ as in [4] it is possible to show that

$$\bar{C}(\widehat{\Phi}) = \sup_{\hat{\pi} \in \widehat{\mathcal{P}}} \int_{\widehat{\mathfrak{S}}(\mathcal{H})} H(\widehat{\Phi}(\hat{\rho}) \| \widehat{\Phi}(\bar{\rho}(\hat{\pi}))) \hat{\pi}(d\hat{\rho})$$

where the supremum is over all probability measures on $\widehat{\mathfrak{S}}(\mathcal{H})$ and $\bar{\rho}(\hat{\pi})$ is a barycenter of the measure $\hat{\pi}$. The measure $\hat{\pi}_*$ at which the above supremum is achieved (if it exists) is called *optimal* for the channel extension $\widehat{\Phi}$.

By using lower semicontinuity of the relative entropy, equality (13) and theorem 1 it is possible to prove that any optimal measure $\hat{\pi}$ for the channel extension $\widehat{\Phi}$ has the generalized maximal distance property:

$$H(\widehat{\Phi}(\hat{\rho}) \| \Omega(\Phi)) = \bar{C}(\widehat{\Phi}) \quad \text{for } \hat{\pi}_*\text{-almost all } \hat{\rho} \text{ in } \widehat{\mathfrak{S}}(\mathcal{H}).$$

⁴The only difference in the argumentation is a necessity to use proposition I.2.3 in [1] instead of lemma 1 in [4] since the set $\widehat{\mathfrak{S}}(\mathcal{H})$ is not complete separable metric space, but it is compact.

The following proposition shows, in particular, that the condition of existence of an optimal measure for the channel extension $\widehat{\Phi}$ is substantially weaker than the condition of existence of an optimal measure for the initial channel Φ .

Proposition 4. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be a channel with finite χ -capacity and $\widehat{\Phi} : \widehat{\mathfrak{S}}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be its channel extension.*

The following properties are equivalent:

- *there exists an optimal measure for the set $\overline{\Phi(\mathfrak{S}(\mathcal{H}))}$;*
- *there exists an optimal measure for the channel extension $\widehat{\Phi}$.*

The above equivalent properties hold if the set $\overline{\Phi(\mathfrak{S}(\mathcal{H}))}$ contains regular subset with the same χ -capacity.

Proof. If $\hat{\mu}_*$ is an optimal measure for the channel extension $\widehat{\Phi}$ then its image $\hat{\mu}_* \circ \widehat{\Phi}^{-1}$ corresponding to the mapping $\widehat{\Phi}$ is an optimal measure for the set $\Phi(\mathfrak{S}(\mathcal{H}))$.

Let ν_* be an optimal measure for the set $\overline{\Phi(\mathfrak{S}(\mathcal{H}))}$ and let $\{\nu_n = \{\pi_i^n, \rho_i^n\}\}_n$ be a sequence of measures in $\mathcal{P}(\overline{\Phi(\mathfrak{S}(\mathcal{H}))})$ with finite support weakly converging to the measure ν_* . Since $\overline{\Phi(\mathfrak{S}(\mathcal{H}))} = \widehat{\Phi}(\widehat{\mathfrak{S}}(\mathcal{H}))$ for each n and i there exists a state $\hat{\rho}_i^n$ in $\widehat{\mathfrak{S}}(\mathcal{H})$ such that $\widehat{\Phi}(\hat{\rho}_i^n) = \rho_i^n$. The sequence $\{\hat{\mu}_n = \{\pi_i^n, \hat{\rho}_i^n\}\}_n$ of measures in the weakly compact set $\mathcal{P}(\widehat{\mathfrak{S}}(\mathcal{H}))$ has a weak limit point $\hat{\mu}_*$. Since the mapping $\widehat{\Phi}$ is continuous with respect to the $*$ -weak topology on $\widehat{\mathfrak{S}}(\mathcal{H})$ and the trace norm topology on $\mathfrak{S}(\mathcal{H}')$ the image $\hat{\mu}_* \circ \widehat{\Phi}^{-1}$ of the weak limit point $\hat{\mu}_*$ of the set $\{\hat{\mu}_n\}_n$ is a weak limit point of the set $\{\nu_n = \hat{\mu}_n \circ \widehat{\Phi}^{-1}\}_n$, so that $\hat{\mu}_* \circ \widehat{\Phi}^{-1} = \nu_*$. Thus $\hat{\mu}_*$ is an optimal measure for the channel extension $\widehat{\Phi}$.

The last assertion of the proposition follows from the previous one and theorem 3 in [12]. \square

Remark 2. It is clear that each optimal measure for the channel Φ is an optimal measure for the channel extension $\widehat{\Phi}$. Proposition 4 shows the meaning of the conditions of existence of an optimal measure for the channel Φ in theorem 2. Namely, the regularity condition implies that the set of optimal measures for the channel extension $\widehat{\Phi}$ is not empty while the condition of existence of an input optimal average state for the channel Φ implies that this set contains at least one optimal measure for the channel Φ . This observation is illustrated by the example in the next section.

5 On a class of channels with finite χ -capacity

In this section we consider nontrivial class of entanglement-breaking channels generalizing the example considered in [5]. For channels of this class the χ -capacity and the minimal output entropy can be explicitly calculated. There also exists simple necessary and sufficient of continuity of the output entropy for these channels.

Let G be a compact group, $\{V_g\}$ be unitary representation of G on \mathcal{H}' , $M(dg)$ be a positive operator-valued measure (POVM) on G , such that the set of probability measures $\{\text{Tr}\rho M(\cdot)\}_{\rho \in \mathfrak{S}(\mathcal{H})}$ is weakly dense in the set of all probability measures on G .

For arbitrary state σ in $\mathfrak{S}(\mathcal{H}')$ consider the channel

$$\Phi_\sigma(\rho) = \int_G V_g \sigma V_g^* \text{Tr}\rho M(dg).$$

Let $\omega(G, V_g, \sigma) = \int_G V_g \sigma V_g^* \mu_H(dg)$, where μ_H is the Haar measure on G . It follows from the assumption of weak density of the set $\{\text{Tr}\rho M(\cdot)\}_{\rho \in \mathfrak{S}(\mathcal{H})}$ in the set of all Borel probability measures on G that $\overline{\Phi_\sigma(\mathfrak{S}(\mathcal{H}))} = \overline{\text{co}\{V_g \sigma V_g^*\}_{g \in G}}$. Thus proposition 12 in [12] implies the following observation.

Proposition 5. *The χ -capacity of the channel Φ_σ is equal to*

$$\bar{C}(\Phi_\sigma) = H(\sigma \| \omega(G, V_g, \sigma)).$$

If this capacity is finite then $\Omega(\Phi_\sigma) = \omega(G, V_g, \sigma)$ and $H_{\min}(\Phi_\sigma) = H(\sigma)$.

The channel Φ_σ is a CE-channel if and only if $H(\omega(G, V_g, \sigma)) < +\infty$. In this case $\bar{C}(\Phi_\sigma) = H(\omega(G, V_g, \sigma)) - H(\sigma) = H(\Omega(\Phi_\sigma)) - H_{\min}(\Phi_\sigma)$.

Since the set $\overline{\Phi_\sigma(\mathfrak{S}(\mathcal{H}))} = \overline{\text{co}\{V_g \sigma V_g^*\}_{g \in G}}$ is compact for arbitrary σ proposition 3 implies existence of the extension $\widehat{\Phi}_\sigma$ of the channel Φ_σ to the set $\widehat{\mathfrak{S}}(\mathcal{H})$ such that $\widehat{\Phi}_\sigma(\widehat{\mathfrak{S}}(\mathcal{H})) = \overline{\Phi_\sigma(\mathfrak{S}(\mathcal{H}))} = \overline{\text{co}\{V_g \sigma V_g^*\}_{g \in G}}$.

Proposition 4 and proposition 12 in [12] show existence of an optimal measure for the channel extension $\widehat{\Phi}_\sigma$ provided $\bar{C}(\widehat{\Phi}_\sigma) = \bar{C}(\Phi_\sigma) < +\infty$. The support of any optimal measure for the channel extension $\widehat{\Phi}_\sigma$ consists of states $\hat{\rho}$ in $\widehat{\mathfrak{S}}(\mathcal{H})$ such that $\widehat{\Phi}_\sigma(\hat{\rho}) = V_g \sigma V_g^*$ for some $g \in G$. As it is shown by the following example we can not assert existence of an optimal measure for the channel Φ_σ even in the case when Φ_σ is a CE-channel.

Example 2. Consider the case $G = \mathbb{T}$, which can be identified with $[0, 2\pi)$. In this case the Haar measure μ_H is the normalized Lebesgue measure $\frac{dx}{2\pi}$. Let V_g is the group of shifts in $\mathcal{H}' = \mathcal{L}_2(\mathbb{T})$ and $M(dg)$ is the

spectral measure of the operator of multiplication by an independent variable in $\mathcal{H} = \mathcal{L}_2(\mathbb{T})$. It is easy to see that the above density assumption is valid in this case and hence all the above results hold for the corresponding channel Φ_σ with arbitrary $\sigma \in \mathfrak{S}(\mathcal{L}_2(\mathbb{T}))$. Suppose $\sigma = |\varphi\rangle\langle\varphi|$, where φ be an arbitrary function in $\mathcal{L}_2(\mathbb{T})$ with unit norm. This implies $\omega(\mathbb{T}, V_g, |\varphi\rangle\langle\varphi|) = (2\pi)^{-1} \int_0^{2\pi} |\varphi_x\rangle\langle\varphi_x| dx$, where $\varphi_x(t) = \varphi(t-x)$. This channel was originally used in [5] as an example of entanglement-breaking channel which has no canonical representation with purely atomic POVM and hence has no Kraus representation with operators of rank 1. Here we will use this channel as an example of CE-channel, which demonstrates the essential features of infinite-dimensional channels.

By using proposition 5 and direct calculation it is easy to obtain (cf.[5]) that

$$\bar{C}(\Phi_{|\varphi\rangle\langle\varphi|}) = H(\omega(\mathbb{T}, V_g, |\varphi\rangle\langle\varphi|)) = - \sum_{k=-\infty}^{+\infty} |\varphi_k|^2 \log |\varphi_k|^2, \quad (14)$$

where $\{\varphi_k = (2\pi)^{-1} \int_0^{2\pi} e^{-ixk} \varphi(x) dx\}_{k \in \mathbb{Z}}$ is the sequence of the Fourier coefficients of the function φ , and that finiteness of $\bar{C}(\Phi_{|\varphi\rangle\langle\varphi|})$ implies CE-property of the channel $\Phi_{|\varphi\rangle\langle\varphi|}$ and

$$\Omega(\Phi_{|\varphi\rangle\langle\varphi|}) = \omega(\mathbb{T}, V_g, |\varphi\rangle\langle\varphi|) = \sum_{k=-\infty}^{+\infty} |\varphi_k|^2 |\tau_k\rangle\langle\tau_k|,$$

where $\{\tau_k(x) = \exp(ikx)\}_{k \in \mathbb{Z}}$ is the trigonometric basis in $\mathcal{L}_2(\mathbb{T})$. It is interesting to note that the properties of the channel $\Phi_{|\varphi\rangle\langle\varphi|}$ are determined by the rate of vanishing of the Fourier coefficients of the function φ .

Suppose series (14) is finite for a function φ and hence $\Phi_{|\varphi\rangle\langle\varphi|}$ is a CE-channel. By proposition 5 $H_{\min}(\Phi_{|\varphi\rangle\langle\varphi|}) = 0$ in spite of the fact that the set $\Phi_{|\varphi\rangle\langle\varphi|}(\mathfrak{S}(\mathcal{H}))$ does not contain pure states. The set $\{|\varphi_x\rangle\langle\varphi_x|\}_{x \in \mathbb{T}}$ of pure states in $\overline{\Phi_{|\varphi\rangle\langle\varphi|}(\mathfrak{S}(\mathcal{H}))}$ is contained in the output set of the channel extension $\widehat{\Phi}_{|\varphi\rangle\langle\varphi|}$ and corresponds to a particular subset of nonnormal states in $\widehat{\mathfrak{S}}(\mathcal{H})$. By proposition 1 the functions $\chi_{\Phi_{|\varphi\rangle\langle\varphi|}}$ and $\widehat{H}_{\Phi_{|\varphi\rangle\langle\varphi|}}$ are bounded and continuous on $\mathfrak{S}(\mathcal{H})$. Nevertheless there exists no optimal measure and hence there exists no input optimal average state for the channel $\Phi_{|\varphi\rangle\langle\varphi|}$ (since otherwise corollary 2 implies existence of an optimal measure). The continuous functions $\chi_{\Phi_{|\varphi\rangle\langle\varphi|}}$ and $\widehat{H}_{\Phi_{|\varphi\rangle\langle\varphi|}}$ do not achieve their finite supremum

$\bar{C}(\Phi_{|\varphi\rangle\langle\varphi|})$ and infimum $H_{\min}(\Phi_{|\varphi\rangle\langle\varphi|}) = 0$ correspondingly on noncompact set $\mathfrak{S}(\mathcal{H})$.

The family of channels $\{\Phi_{|\varphi\rangle\langle\varphi|}\}_{\varphi \in \mathcal{L}_2(\mathbb{T})}$ provides an another example showing that in the infinite dimensional case the χ -capacity is not continuous but only lower semicontinuous function of a channel [10].

It is easy to see that for arbitrary unit vector $|\varphi_*\rangle$ in $\mathcal{L}_2(\mathbb{T})$ and for arbitrary sequence $\{|\varphi_n\rangle\}$ of unit vectors in $\mathcal{L}_2(\mathbb{T})$ converging to the vector $|\varphi_*\rangle$ the sequence $\sup_{\rho \in \mathfrak{S}(\mathcal{H})} \|\Phi_{|\varphi_n\rangle\langle\varphi_n|}(\rho) - \Phi_{|\varphi_*\rangle\langle\varphi_*|}(\rho)\|_1$ tends to zero. It means that $\varphi \mapsto \Phi_{|\varphi\rangle\langle\varphi|}$ is a continuous mapping from $\mathcal{L}_2(\mathbb{T})$ into the set of all channels endowed with the topology of uniform convergence.

Let $\{\varphi_n\}$ be the sequence of function in $\mathcal{L}_2(\mathbb{T})$ with the following Fourier coefficients

$$(\varphi_n)_k = \begin{cases} \sqrt{1 - q_n}, & k = 0 \\ \sqrt{q_n/n}, & 0 < k \leq n \\ 0, & \text{for others } k, \end{cases}$$

where $\{q_n\}$ is a sequence of numbers in $(0, 1)$ such that $\lim_{n \rightarrow +\infty} q_n \log n = C > 0$. It is easy to see that the sequence $\{\varphi_n\}$ converges in $\mathcal{L}_2(\mathbb{T})$ to the function $\varphi_*(x) \equiv 1$. The above observation implies that the sequence $\{\Phi_{|\varphi_n\rangle\langle\varphi_n|}\}$ of channels uniformly converges to the channel $\Phi_{|\varphi_*\rangle\langle\varphi_*|}$, for which $\Phi_{|\varphi_*\rangle\langle\varphi_*|}(\rho) = |\varphi_*\rangle\langle\varphi_*|$ for all ρ in $\mathfrak{S}(\mathcal{H})$ and hence $\bar{C}(\Phi_{|\varphi_*\rangle\langle\varphi_*|}) = 0$. But by (14) we have

$$\bar{C}(\Phi_{|\varphi_n\rangle\langle\varphi_n|}) = -q_n \log q_n - (1 - q_n) \log(1 - q_n) + q_n \log n, \quad \forall n \in \mathbb{N},$$

and hence $\lim_{n \rightarrow +\infty} \bar{C}(\Phi_{|\varphi_n\rangle\langle\varphi_n|}) = C > 0$.

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