Properties of probability measures on the set of quantum states and their applications.

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1 Introduction

The notion of ensemble as a set of states with corresponding set of probabilities is widely used in quantum information theory. In particular, important characteristics such as the Holevo capacity of a quantum channel and the entanglement of formation of a quantum state are defined by optimization of the particular functionals depending on ensemble of states [10].

An ensemble of quantum states can be considered as an atomic probability measure on the set of all states, which atoms correspond to the states of the ensemble. So, it is natural to consider an arbitrary Borel probability measure on the set of all states as a generalized ensemble. This point of view is especially useful in dealing with infinite dimensional quantum channels and systems since in this case there are no reasons for existence of ensembles called optimal, at which extrema of several important functionals are achieved, but under some conditions it is possible to show existence of optimal measures [7]. Moreover, by considering probability measures as generalized ensembles it is possible to prove results, which formally have no relations to probability measures [18]. The advantage of this approach is based on application of general results of the theory of probability measures on complete separable metric spaces [6],[14].

In this paper some observations concerning the Choquet ordering [15] on the set of all probability measures on the set of quantum states are considered (proposition 1, corollary 1 and lemma 3). They imply, in particular, that arbitrary measures supported by pure states can be weakly approximated by

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a sequence of atomic measures supported by pure states and having the same barycenter (corollary 2).

The important property of finite ensembles of quantum states proved in [17] (lemma 3) can be expressed figuratively speaking as follows: *an arbitrary continuous deformation of the average state of any finite ensemble can be realized by appropriate continuous deformation of the states of the ensemble and their weights*.[1] This property is the basic point of the proof of lower semicontinuity of the χ-function of an arbitrary quantum channel (proposition 3 in [17]). In this paper we show that it implies the openness properties of the mapping, which associates with a probability measure the barycenter of this measure (propositions 2 and 3). These properties and the compactness criterion for subsets of measures obtained in [7] result in interesting observations on properties of functions on the set of quantum states (theorems 1, 2, propositions 4, 5, corollaries 8, 10).

In particular, it is shown that every continuous bounded function on the set of quantum states has continuous bounded convex closure (proposition 4). It is also shown that every continuous bounded function on the set of pure quantum states has convex (concave) continuous bounded extension on the set of all states having the particular minimality (maximality) property (proposition 5).

The above general observations have several applications to quantum information theory (corollaries 6,7 and 9). In particular, they provide a necessary and sufficient condition of boundedness and continuity of the convex closure of the output entropy of a quantum channel (corollary 9). These results can also be used for construction of continuous bounded characteristics of quantum states as the above convex (concave) extensions of continuous bounded functions defined on the set of pure states. As an example, we consider the construction of quasimeasure of entanglement, which is a continuous bounded function on the whole state space of a infinite dimensional bipartite system closely related to the entanglement of formation (remark 3).

In [18] the open problem of coinciding of two definitions of the entanglement of formation of a state in infinite dimensional bipartite system is discussed. In this paper we construct an example showing that this problem can not be solved by using only simple analytical properties of the quantum entropy (remark 2 and the note below).

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[1]Even in $\mathbb{R}^3$ there exist convex sets for which the analogue of this assertion is not valid.

2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ - the set of all bounded operators in $\mathcal{H}$ with the cone $\mathfrak{B}_+(\mathcal{H})$ of all positive operators, $\mathfrak{T}(\mathcal{H})$ - the Banach space of all trace-class operators with the trace norm $\|\cdot\|_1$ and $\mathfrak{S}(\mathcal{H})$ - the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of all positive operators with the unit trace - density operators in $\mathcal{H}$, which is complete separable metric space with the metric defined by the trace norm. Each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$ [5], so, in what follows we will also for brevity use the term "state".

We denote by $\mathrm{co}\mathcal{A}$ ($\overline{\mathrm{co}}\mathcal{A}$) the convex hull (closure) of a set $\mathcal{A}$ and by $\mathrm{co}f$ ($\overline{\mathrm{co}}f$) the convex hull (closure) of a function f [13]. We denote by $\mathrm{ext}\mathcal{A}$ the set of all extreme points of a convex set $\mathcal{A}$.

Let $\mathcal{P}$ be the set of all Borel probability measures on $\mathfrak{S}(\mathcal{H})$ endowed with the topology of weak convergence [6],[14]. Since $\mathfrak{S}(\mathcal{H})$ is a complete separable metric space $\mathcal{P}$ is a complete separable metric space as well [14]. Let $\widehat{\mathcal{P}}$ be the closed subset of $\mathcal{P}$ consisting of all measures supported by the closed set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$ of all pure states.

The *barycenter* of the measure μ is the state defined by the Bochner integral

$$\bar{\rho}(\mu) = \int\limits_{\mathfrak{S}(\mathcal{H})} \sigma\mu(d\sigma).$$

For arbitrary subset $\mathcal{A}$ of $\mathfrak{S}(\mathcal{H})$ let $\mathcal{P}_{\mathcal{A}}$ (corresp. $\widehat{\mathcal{P}}_{\mathcal{A}}$) be the subset of $\mathcal{P}$ (corresp. $\widehat{\mathcal{P}}$) consisting of measures with the barycenter in $\mathcal{A}$. By using Prokhorov's theorem [16] the following compactness criterion for subsets of $\mathcal{P}$ is established in [7] (proposition 2): *The set $\mathcal{P}_{\mathcal{A}}$ is compact if and only if the set $\mathcal{A}$ is compact.* It follows, in particular, that the subset $\mathcal{P}_{\{\rho\}}$ of $\mathcal{P}$, consisting of all measures with the barycenter ρ, is compact as well as the corresponding subset $\widehat{\mathcal{P}}_{\{\rho\}}$ of $\widehat{\mathcal{P}}$.

An atomic probability measure consisting of atoms $\{\rho_i\}$ with corresponding weights $\{\pi_i\}$ is denoted by $\{\pi_i, \rho_i\}$. A probability measure consisting of *finite* number of atoms is denoted by $\{\pi_i, \rho_i\}^{\mathrm{f}}$ and is called a *measure with finite support*.

The following version of the Choquet decomposition [15] adapted to the case of closed convex subsets of $\mathfrak{S}(\mathcal{H})$ is obtained in [8] (lemma 1): *Let $\mathcal{A}$ be a closed subset of $\mathfrak{S}(\mathcal{H})$. Then every state in $\overline{\mathrm{co}}\mathcal{A}$ can be represented as the barycenter of some Borel probability measure supported by $\mathcal{A}$.*

3 Properties of probability measures

Let $\mathcal{S}$ be the set of all bounded continuous convex functions on $\mathfrak{S}(\mathcal{H})$ and $\widehat{\mathcal{S}}$ be the set of all functions on $\mathfrak{S}(\mathcal{H})$, which can be represented as pointwise limits of monotonous sequences of functions from the set $\mathcal{S}$. It is easy to see that the set $\widehat{\mathcal{S}}$ consists of convex functions, which are either lower semicontinuous or upper semicontinuous.

Lemma 1. *The set $\widehat{\mathcal{S}}$ contains the characteristic function*

$$\chi_{\mathcal{A}}(\rho) = \begin{cases} 1 & \rho \in \mathcal{A} \\ 0, & \rho \in \mathfrak{S}(\mathcal{H}) \setminus \mathcal{A} \end{cases}$$

of a subset $\mathcal{A}$ of $\mathfrak{S}(\mathcal{H})$ in the following cases:

1) *$\mathcal{A}$ is an arbitrary closed subset of $\mathrm{ext}\,\mathfrak{S}(\mathcal{H})$;*

2) *$\mathcal{A} = \mathfrak{S}(\mathcal{H}_0) = \{\rho \in \mathfrak{S}(\mathcal{H}) \,|\, \mathrm{supp}\rho \subseteq \mathcal{H}_0\}$ for arbitrary subspace $\mathcal{H}_0 \subseteq \mathcal{H}$;*

3) *$\mathcal{A} = \mathfrak{S}_n(\mathcal{H}) = \{\rho \in \mathfrak{S}(\mathcal{H}) \,|\, \mathrm{rank}\rho \leq n\}$ for arbitrary $n \in \mathbb{N}$.*

Proof. 1) It is sufficient to show that the function $g(\rho) = \sup_{\sigma \in \mathcal{A}} \mathrm{Tr}\rho\sigma$ lies in the class $\mathcal{S}$ since this implies that the functions $f_n(\rho) = 1 - \sqrt[n]{1 - g(\rho)}$ lie in the class $\mathcal{S}$ for all natural n (due to concavity of the increasing function $\sqrt[n]{x}$) and the decreasing sequence of functions $\{f_n(\rho)\}$ pointwise converges to the function $\chi_{\mathcal{A}}(\rho)$.

Boundedness, convexity and lower semicontinuity of the function $g(\rho)$ follows from its representation as the least upper bound of the family $\{\mathrm{Tr}\rho\sigma\}_{\sigma \in \mathcal{A}}$ of bounded continuous affine functions on $\mathfrak{S}(\mathcal{H})$.

Suppose that the function $g(\rho)$ is not upper semicontinuous. This implies existence of a sequence $\{\rho_n\}$ of states converging to some state ρ_0 such that

$$\lim_{n \to +\infty} g(\rho_n) > g(\rho_0). \tag{1}$$

Let $\mathfrak{A} = \{|\varphi\rangle \in \mathcal{H} \,|\, |\varphi\rangle\langle\varphi| \in \mathcal{A}\}$ be subset of $\mathcal{H}$ and $\overline{\mathfrak{A}}$ be its closure in the weak topology in $\mathcal{H}$. Lemma 2 on p.284 in [9] [2] implies

$$g(\rho_0) = \sup_{\sigma \in \mathcal{A}} \mathrm{Tr}\rho_0\sigma = \sup_{\varphi \in \mathfrak{A}} \langle\varphi|\rho_0|\varphi\rangle = \sup_{\varphi \in \overline{\mathfrak{A}}} \langle\varphi|\rho_0|\varphi\rangle. \tag{2}$$

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[2]Let $\{x_n\}$ be an arbitrary sequence of vectors in a Hilbert space weakly converging to the vector x and A be an arbitrary compact operator. Then $\lim_{n \to +\infty} \langle x_n|A|x_n\rangle = \langle x|A|x\rangle$.

For arbitrary $\varepsilon > 0$ and arbitrary n there exists a vector $\varphi_n^\varepsilon \in \mathfrak{A}$ such that $\langle \varphi_n^\varepsilon | \rho_n | \varphi_n^\varepsilon \rangle > g(\rho_n) - \varepsilon$. Since the unit ball of the space $\mathcal{H}$ is compact in the weak topology we can find a subsequence $\{\varphi_{n_k}^\varepsilon\}_k$ of the sequence $\{\varphi_n^\varepsilon\}_n$ weakly converging to some vector $\varphi^\varepsilon \in \overline{\mathfrak{A}}$. By lemma 2 on p.284 in [9] the sequence $\{\langle \varphi_{n_k}^\varepsilon | \rho_0 | \varphi_{n_k}^\varepsilon \rangle\}_k$ converges to $\langle \varphi^\varepsilon | \rho_0 | \varphi^\varepsilon \rangle$ as k tends to the infinity. This and the estimation $|\langle \varphi_{n_k}^\varepsilon | \rho_{n_k} - \rho_0 | \varphi_{n_k}^\varepsilon \rangle| \leq \|\rho_{n_k} - \rho_0\|_1$ imply

$$\lim_{k \to +\infty} g(\rho_{n_k}) \leq \lim_{k \to +\infty} \langle \varphi_{n_k}^\varepsilon | \rho_{n_k} | \varphi_{n_k}^\varepsilon \rangle - \varepsilon = \langle \varphi^\varepsilon | \rho_0 | \varphi^\varepsilon \rangle - \varepsilon \leq g(\rho_0) - \varepsilon,$$

where the last inequality follows from (2). But this contradicts to the assumption (1) due to the freedom of the choice of ε.

2) Let P_0 be the projector on the subspace $\mathcal{H}_0$. Since $g(\rho) = \mathrm{Tr} P_0 \rho$ is a continuous affine function on $\mathfrak{S}(\mathcal{H})$ the functions $f_n(\rho) = 1 - \sqrt[n]{1 - g(\rho)}$ lie in the class $\mathcal{S}$ for all natural n (due to concavity of the increasing function $\sqrt[n]{x}$) and the decreasing sequence of functions $\{f_n(\rho)\}$ pointwise converges to the function $\chi_{\mathfrak{S}(\mathcal{H}_0)}(\rho)$.

3) For given n consider the continuous function $g(\rho) = \sum_{i=1}^{n} \lambda_i$, where $\{\lambda_i\}_{i=1}^n$ is the set of n maximal eigen values of the state ρ. The function $g(\rho)$ is convex since it can be represented as the least upper bound of the family $\{\mathrm{Tr} P \rho\}_{P \in \mathfrak{P}_n(\mathcal{H})}$ of affine functions, where $\mathfrak{P}_n(\mathcal{H})$ is the set of all projectors in $\mathcal{H}$ of rank n. Hence the functions $f_n(\rho) = 1 - \sqrt[n]{1 - g(\rho)}$ lie in the class $\mathcal{S}$ for all natural n (due to concavity of the increasing function $\sqrt[n]{x}$) and the decreasing sequence of functions $\{f_n(\rho)\}$ pointwise converges to the function $\chi_{\mathfrak{S}_n(\mathcal{H})}(\rho)$. $\square$

Consider the following partial order on the set $\mathcal{P}$ of all probability measures on $\mathfrak{S}(\mathcal{H})$. We say that $\mu \succ \nu$ if and only if

$$\int_{\mathfrak{S}(\mathcal{H})} f(\sigma) \mu(d\sigma) \geq \int_{\mathfrak{S}(\mathcal{H})} f(\sigma) \nu(d\sigma) \quad \text{for all } f \text{ in } \mathcal{S}.$$

The partial order of this type is widely used in the theory of integral representation and is often called the Choquet ordering [15],[4].

It is easy to see (by considering affine continuous functions on $\mathfrak{S}(\mathcal{H})$) that a relation $\mu \succ \nu$ implies $\bar{\rho}(\mu) = \bar{\rho}(\nu)$.

Intuitively speaking, a relation $\mu \succ \nu$ means that "the mass of μ is removed farther away from the common barycenter of μ and ν, and comes close to the extreme boundary" [1]. Note that the extreme boundary (=the set of extreme points) of the set $\mathfrak{S}(\mathcal{H})$ is the set of all pure states (=states of

rank 1) and that for arbitrary subspace $\mathcal{H}_0$ of the space $\mathcal{H}$ the subset $\mathfrak{S}(\mathcal{H}_0)$ is a *face* of the set $\mathfrak{S}(\mathcal{H})$. Thus the above characterization of the partial order " $\succ$ " is confirmed by the following observations.

Proposition 1. *Let μ and ν be arbitrary measures in $\mathcal{P}$ such that $\mu \succ \nu$. Then*

- $\mu(\mathcal{A}) \geq \nu(\mathcal{A})$ *for arbitrary Borel subset $\mathcal{A}$ of the set* ext$\mathfrak{S}(\mathcal{H})$;

- $\mu(\mathfrak{S}(\mathcal{H}_0)) \geq \nu(\mathfrak{S}(\mathcal{H}_0))$ *for arbitrary subspace $\mathcal{H}_0$ of the space $\mathcal{H}$;*

- $\mu(\mathfrak{S}_n(\mathcal{H})) \geq \nu(\mathfrak{S}_n(\mathcal{H}))$ *for arbitrary $n = 1, 2, ...$*

Proof. Since every measure in $\mathcal{P}$ is a Radon measure the assertions of the proposition directly follow from lemma 1 and the part A of lemma 2 below. $\square$

The monotonous convergence theorem [9] and the definition of weak convergence [6] imply the following observation.

Lemma 2. A) *Let μ and ν be measures in $\mathcal{P}$ such that $\mu \succ \nu$. Then*

$$\int_{\mathfrak{S}(\mathcal{H})} g(\sigma)\mu(d\sigma) \geq \int_{\mathfrak{S}(\mathcal{H})} g(\sigma)\nu(d\sigma) \qquad (3)$$

for arbitrary function g in $\widehat{\mathcal{S}}$.

B) *Let $\{\mu_n\}$ and $\{\nu_n\}$ be two sequences of measures in $\mathcal{P}$ converging to measures μ and ν correspondingly such that $\mu_n \succ \nu_n$ for all n. Then $\mu \succ \nu$.*

According to [15] a measure μ in $\mathcal{P}$ is called *maximal* if $\nu \succ \mu$ implies $\nu = \mu$ for any measure ν in $\mathcal{P}$.

Corollary 1. *The set of all maximal measures in $\mathcal{P}$ coincides with $\widehat{\mathcal{P}}$.*

For every measure μ in $\mathcal{P}$ there exists a maximal measure $\hat{\mu}$ in $\mathcal{P}$ such that $\hat{\mu} \succ \mu$.

Proof. The assertions of this corollary can be deduced from the general results of the theory of measures on convex set [1],[4],[15], but we want to show that proposition 1 and the compactness criterion for subsets of $\mathcal{P}$ provide simple and constructive way of their proof.

Let μ be an arbitrary measure in $\widehat{\mathcal{P}}$. By proposition 1 the assumption $\nu \succ \mu$ for some measure ν in $\mathcal{P}$ implies $\nu(\mathcal{A}) \geq \mu(\mathcal{A})$ for arbitrary Borel set $\mathcal{A}$ of pure states. Since μ and ν are *probability* measures and μ is supported by pure states equality necessarily holds in the above inequality, which means that $\mu = \nu$. Thus μ is a maximal measure in $\mathcal{P}$.

If μ is a maximal measure in $\mathcal{P}$ then by the below observation there exists a measure $\hat{\mu}$ in $\widehat{\mathcal{P}}$ such that $\hat{\mu} \succ \mu$ and hence $\mu = \hat{\mu}$.

Let μ be an arbitrary measure in $\mathcal{P}$. By lemma 1 in [7] there exists sequence $\{\mu_n\}$ of measures in $\mathcal{P}$ with finite support, converging to the measure μ. By using lemma 3 below we obtain the measure $\hat{\mu}$ such that $\hat{\mu} \succ \mu$. $\square$

The following observation based on the compactness criterion for subsets of $\mathcal{P}$ plays an important role in this paper.

Lemma 3. *Let μ_0 be an arbitrary measure in $\mathcal{P}$ and $\{\mu_n\}$ be a sequence of measures in $\mathcal{P}$ with finite support converging to the measure μ_0. There exist a subsequence $\{\mu_{n_k}\}$ of the sequence $\{\mu_n\}$ and a sequence $\{\hat{\mu}_k\}$ of atomic measures in $\widehat{\mathcal{P}}$ converging to some measure $\hat{\mu}_0$ in $\widehat{\mathcal{P}}$ such that*

$$\hat{\mu}_k \succ \mu_{n_k}, \quad \bar{\rho}(\hat{\mu}_k) = \bar{\rho}(\mu_{n_k}), \ \forall k, \quad and \quad \hat{\mu}_0 \succ \mu_0.$$

Proof. Decomposing each atom of the measure μ_n into convex combination of pure states we obtain (as in the proof of the theorem in [7]) the measure $\hat{\mu}_n$ with the same barycenter supported by pure states. It is easy to see by definition that $\hat{\mu}_n \succ \mu_n$. Continuity of the mapping $\mu \mapsto \bar{\rho}(\mu)$ implies compactness of the set $\{\bar{\rho}(\mu_n)\}_{n \geq 0} \supseteq \bar{\rho}(\{\hat{\mu}_n\}_{n>0})$. By the compactness criterion for subsets of $\mathcal{P}$ the set $\{\hat{\mu}_n\}_{n>0}$ is a relatively compact subset of $\mathcal{P}$. This implies existence of subsequence $\{\hat{\mu}_{n_k}\}$ converging to a measure $\{\hat{\mu}_0\}$ supported by pure states due to theorem 6.1 in [14]. Since $\hat{\mu}_{n_k} \succ \mu_{n_k}$ for all k, the part B of lemma 2 implies $\hat{\mu}_0 \succ \mu_0$. Denoting $\hat{\mu}_k = \hat{\mu}_{n_k}$ we complete the proof of the lemma. $\square$

Corollary 1 and lemma 3 make possible to obtain the following analog of lemma 1 in [7], which means weak density of atomic measures in the set of all maximal measures with given barycenter.

Corollary 2. *An arbitrary measure $\hat{\mu}_0$ in $\widehat{\mathcal{P}}$ can be weakly approximated by a sequence $\{\hat{\mu}_n\}$ of atomic measures in $\widehat{\mathcal{P}}$ such that $\bar{\rho}(\hat{\mu}_n) = \bar{\rho}(\hat{\mu}_0), \ \forall n$.*

Proof. By lemma 1 in [7] for given measure $\hat{\mu}_0$ there exists a sequence $\{\mu_n\}$ of measures in $\mathcal{P}$ with finite support converging to the measure $\hat{\mu}_0$ such that $\bar{\rho}(\mu_n) = \bar{\rho}(\hat{\mu}_0), \ \forall n$. Applying lemma 3 and corollary 1 it is easy to construct from the above sequence a sequence $\{\hat{\mu}_n\}$ with the required properties. $\square$

The mapping $\mu \mapsto \bar{\rho}(\mu)$ is a continuous mapping from $\mathcal{P}$ onto $\mathfrak{S}(\mathcal{H})$ [7]. Lemma 3 in [17] makes possible to prove the another important topological property of this mapping.

Proposition 2. *The mapping $\mathcal{P} \ni \mu \mapsto \bar{\rho}(\mu) \in \mathfrak{S}(\mathcal{H})$ is open.*

Proof. Let U be an arbitrary open subset of $\mathcal{P}$. Suppose $\bar{\rho}(U)$ is not open. Then there exist a state $\rho_0 \in \bar{\rho}(U)$ and a sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H}) \setminus \bar{\rho}(U)$ converging to the state ρ_0.

Let μ_0 be a measure in U such that $\bar{\rho}(\mu_0) = \rho_0$. Since U is an open set we may consider due to lemma 1 in [7] that μ_0 is a measure with finite support, so that $\mu_0 = \{\pi_i^0, \rho_i^0\}_{i=1}^m$. By lemma 3 in [17] there exists a sequence of measures $\mu_n = \{\pi_i^n, \rho_i^n\}_{i=1}^m$ converging to the measure $\mu_0 = \{\pi_i^0, \rho_i^0\}_{i=1}^m$ such that $\bar{\rho}(\mu_n) = \rho_n$, $\forall n$. Openness of the set U implies $\mu_n \in U$ for all sufficiently large n contradicting to the choice of the sequence $\{\rho_n\}$.$\square$

Note that openness of the mapping $\mu \mapsto \bar{\rho}(\mu)$ does not follow from continuity of this mapping and compactness criterion for subsets of $\mathcal{P}$. There exist compact convex subsets in $\mathbb{R}^n, n \geq 3$, for which the analogous mapping is not open.

Practically it is convenient to use the following reformulation of proposition 2.

Corollary 3. *Let μ_0 be an arbitrary measure in $\mathcal{P}$ and $\{\rho_n\}$ be an arbitrary sequence of states converging to the state $\bar{\rho}(\mu_0) = \rho_0$. There exist a subsequence $\{\rho_{n_k}\}$ of the sequence $\{\rho_n\}$ and a sequence $\{\mu_k\}$ of measures with finite support converging to the measure μ_0 such that $\bar{\rho}(\mu_k) = \rho_{n_k}$, $\forall k$.*

Proof. Note that $\mathcal{P}$ is a complete separable metric space [14].

For given $k \in \mathbb{N}$ let U_k be the open ball in $\mathcal{P}$ with the center μ_0 and radius $1/k$. By proposition 2 the set $\bar{\rho}(U_k)$ is an open vicinity of the state ρ_0 and hence it contains at least one state ρ_{n_k} of the sequence $\{\rho_n\}$ so that $\rho_{n_k} = \bar{\rho}(\mu_k)$ for some measure $\mu_k \in U_k$. Since the set U_k is open it follows from lemma 1 in [7] that we may consider that the measure μ_k has finite support. By the construction the sequence $\{\mu_k\}$ converges to the measure μ_0. $\square$

The above observations concerning properties of the partial order ” $\succ$ ” makes possible to strengthen proposition 2 as follows.

Proposition 3. *The restriction of the mapping $\mathcal{P} \ni \mu \rightarrow \bar{\rho}(\mu) \in \mathfrak{S}(\mathcal{H})$ to the set $\widehat{\mathcal{P}}$ is open.*

Proof. It is sufficient to show that for arbitrary measure $\hat{\mu}_0$ in $\widehat{\mathcal{P}}$ and arbitrary sequence of states $\{\rho_n\}$ converging to the state $\bar{\rho}(\hat{\mu}_0) = \rho_0$ there exist a subsequence $\{\rho_{n_k}\}$ of the sequence $\{\rho_n\}$ and a sequence $\{\hat{\mu}_k\}$ of measures in $\widehat{\mathcal{P}}$ converging to the measure $\hat{\mu}_0$ such that $\bar{\rho}(\hat{\mu}_k) = \rho_{n_k}$ for all k. By corollary 3 for given measure $\hat{\mu}_0$ and sequence $\{\rho_n\}$ there exist subsequence $\{\rho_{n_k}\}$ of the sequence $\{\rho_n\}$ and a sequence $\{\nu_k\}$ of measures with finite support

converging to the measure $\hat{\mu}_0$ such that $\bar{\rho}(\nu_k) = \rho_{n_k}$ for all k. By lemma 3 there exist a subsequence $\{\rho_{n_{k_m}}\}$ of the sequence $\{\rho_{n_k}\}$ and a sequence $\{\hat{\nu}_m\}$ of atomic measures in $\widehat{\mathcal{P}}$ converging to some measure $\hat{\nu}_0$ in $\widehat{\mathcal{P}}$ such that

$$\bar{\rho}(\hat{\nu}_m) = \bar{\rho}(\nu_{k_m}) = \rho_{n_{k_m}}, \; \forall m \quad \text{and} \quad \hat{\nu}_0 \succ \hat{\mu}_0.$$

Since $\hat{\mu}_0 \in \widehat{\mathcal{P}}$ corollary 1 implies $\hat{\nu}_0 = \hat{\mu}_0$. Thus the subsequence $\{\rho_{n_{k_m}}\}$ and the sequence of measures $\{\hat{\nu}_m\}$ have the required properties.$\square$

In the proof of proposition 3 the following observation was established.

Corollary 4. *Let $\hat{\mu}_0$ be an arbitrary measure in $\widehat{\mathcal{P}}$ and $\{\rho_n\}$ be an arbitrary sequence of states converging to the state $\bar{\rho}(\hat{\mu}_0) = \rho_0$. There exist a subsequence $\{\rho_{n_k}\}$ of the sequence $\{\rho_n\}$ and a sequence $\{\hat{\mu}_k\}$ of atomic measures in $\widehat{\mathcal{P}}$ converging to the measure $\hat{\mu}_0$ such that $\bar{\rho}(\hat{\mu}_k) = \rho_{n_k}$ for all k.*

4 Applications

The results in the previous section concerning properties of probability measures on the set of quantum states imply some nontrivial properties of functions defined on this set.

As it was noted in [2] the general results of the convex analysis can be successfully applied for study some characteristics of quantum channels and systems. In particular, the entanglement of formation of a state in finite dimensional bipartite system [3] can be considered as the convex closure of the output entropy of partial trace channel and this observation can be used to define the entanglement of formation of a state in infinite dimensional bipartite system [18].

The convex closure[3] $\overline{\text{co}}f$ of a function f on a convex topological space X is defined as the maximal closed (lower semicontinuous) convex function on X majorized by f and generally does not coincide with the convex hull $\text{co}f$ of the function f defined as the maximal convex function on X majorized by f [1],[13].[4] Coincidence of $\overline{\text{co}}f$ with $\text{co}f$ always holds if the function f is continuous and the space X is compact [1], but even in $\mathbb{R}^3$ there exist convex compact set X and continuous function f on X such that $\overline{\text{co}}f = \text{co}f$ is not continuous on X.

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[3]The lower envelope in terms of [1].
[4]See also Appendix in [18] for brief description.

In this section it is shown that for an arbitrary continuous and bounded function f on the set $X = \mathfrak{S}(\mathcal{H})$ (which is noncompact if $\dim \mathcal{H} = +\infty$) the convex closure $\overline{\mathrm{co}}f$ of the function f is a continuous and bounded function on the set $\mathfrak{S}(\mathcal{H})$, coinciding with the convex hull $\mathrm{co}f$ of the function f. This result (proposition 4 below) is a corollary of the following more general observation.

Theorem 1. *Let f be a lower bounded lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$.*

A) *The functions*

$$\check{f}(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma) \quad and \quad \hat{f}(\rho) = \sup_{\mu \in \mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma)$$

are lower semicontinuous functions on the set $\mathfrak{S}(\mathcal{H})$.

B) *The function $\check{f}$ is the convex closure $\overline{\mathrm{co}}f$ of the function f;*

C) *For an arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ there exists a measure μ_ρ^f in $\mathcal{P}_{\{\rho\}}$ such that*

$$\check{f}(\rho) = \int_{\mathfrak{S}(\mathcal{H})} f(\sigma)\mu_\rho^f(d\sigma).$$

If in addition f is a concave function such that $-f \in \widehat{\mathcal{S}}$ then the measure μ_ρ^f can be chosen in $\widehat{\mathcal{P}}_{\{\rho\}}$.[5]

D) *The function $\hat{f}$ can be also defined as follows*

$$\hat{f}(\rho) = \sup_{\{\pi_i \rho_i\}^f \in \mathcal{P}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}),$$

which means that $\hat{f}$ is the minimal concave function majorizing the function f (the concave hull of f).

The expression for the function $\hat{f}$ in the part D means that for any state ρ the supremum in the definition of the value $\hat{f}(\rho)$ can be taken over the subset of $\mathcal{P}_{\{\rho\}}$ consisting of measures with finite support.

Remark 1. The analogies of the assertions C and D in theorem 1 are not true for the functions $\hat{f}$ and $\check{f}$ correspondingly.

It is easy to construct bounded lower semicontinuous function f and a state ρ_0 such that the supremum in the definition of the value $\hat{f}(\rho_0)$ is not

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[5]The set $\widehat{\mathcal{S}}$ is defined at the begin of section 3. It is sufficient to require that the assertion A of lemma 2 holds for the function $-f$.

achieved. Indeed, let

$$f(\rho) = \begin{cases} \operatorname{Tr}\rho^2, & \rho \text{ is a mixed state} \\ 0, & \rho \text{ is a pure state} \end{cases}$$

be a lower semicontinuous bounded function and $\rho_0 = \frac{1}{2\pi}\int_0^{2\pi} V_t|\psi\rangle\langle\psi|V_t^* dt$, where V_t is the unitary representation in the Hilbert space $\mathcal{H}$ of the torus $\mathbb{T}$, identified with $[0, 2\pi)$, and $|\psi\rangle$ is an arbitrary vector in $\mathcal{H}$. Then $\hat{f}(\rho_0) = 1$ and hence the supremum in the definition of the value $\hat{f}(\rho_0)$ is not achieved since $f(\rho) < 1$ for all ρ. To show that $\hat{f}(\rho_0) = 1$ it is sufficient to note that the mixed state $\rho_\delta = \delta^{-1}\int_0^\delta V_t|\psi\rangle\langle\psi|V_t^* dt$ tends to the pure state $|\psi\rangle\langle\psi|$ and hence $f(\rho_\delta)$ tends to 1 as $\delta \to +0$.

To show that the infimum in the definition of the value $\check{f}(\rho)$ can not be taken over the subset of $\mathcal{P}_{\{\rho\}}$ consisting of measures with finite support it is sufficient to consider the quantum entropy in the role of function f (cf. lemma 2 in [18]). $\square$

Proof of theorem 1. Convexity and concavity of the functions $\check{f}$ and $\hat{f}$ correspondingly easily follows from the definitions and convexity of the set $\mathcal{P}$.

By using the arguments from the proof of theorem 2.1 in [6] it is possible to show that the assumption of the theorem implies that the functional

$$\mu \mapsto \int f(\sigma)\mu(d\sigma) \tag{4}$$

is lower semicontinuous on $\mathcal{P}$. It follows from this and compactness of the set $\mathcal{P}_{\{\rho\}}$ that for arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ the infimum in the definition of the value $\check{f}(\rho)$ is achieved at some measure μ_ρ^f in $\mathcal{P}_{\{\rho\}}$. If f is a concave function such that $-f \in \widehat{\mathcal{S}}$ and $\hat{\mu}_\rho^f$ is any maximal measure such that $\hat{\mu}_\rho^f \succ \mu_\rho^f$ existing by corollary 1 then by the part A of lemma 2 optimality of the measure μ_ρ^f implies optimality of the measure $\hat{\mu}_\rho^f$.

Lower semicontinuity of the functional (4) and lemma 1 in [7] also imply that for any state ρ the supremum in the definition of the value $\hat{f}(\rho)$ can be taken over the subset of $\mathcal{P}_{\{\rho\}}$ consisting of measures with finite support.

Suppose the function $\check{f}(\rho)$ is not lower semicontinuous. This implies existence of a sequence of states $\{\rho_n\}$ converging to some state ρ_0 such that

$$\lim_{n\to+\infty} \check{f}(\rho_n) < \check{f}(\rho_0). \tag{5}$$

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By the previous observation for each $n = 1, 2, ...$ there exists a measure μ_n in $\mathcal{P}_{\{\rho_n\}}$ such that

$$\check{f}(\rho_n) = \int f(\sigma)\mu_n(d\sigma).$$

Let $\mathcal{A} = \{\rho_n\}_{n=0}^{+\infty}$ be a compact subset of $\mathfrak{S}(\mathcal{H})$. By proposition 2 in [7] the set $\mathcal{P}_\mathcal{A}$ is compact. Since $\{\mu_n\} \subset \mathcal{P}_\mathcal{A}$ there exists subsequence $\{\mu_{n_k}\}$ of the sequence $\{\mu_n\}$ converging to some measure μ_0. Continuity of the mapping $\mu \mapsto \bar{\rho}(\mu)$ implies $\mu_0 \in \mathcal{P}_{\{\rho_0\}}$. By lower semicontinuity of the functional (4) we obtain

$$\check{f}(\rho_0) \leq \int f(\sigma)\mu_0(d\sigma) \leq \liminf_{k \to +\infty} \int f(\sigma)\mu_{n_k}(d\sigma) = \lim_{k \to +\infty} \check{f}(\rho_{n_k}),$$

which contradicts to (5).

Thus the convex function $\check{f}$ is lower semicontinuous and hence the definition of the convex closure implies $\check{f}(\rho) \leq \overline{\mathrm{co}}f(\rho)$ for all ρ in $\mathfrak{S}(\mathcal{H})$. Since $\overline{\mathrm{co}}f$ is a convex and lower semicontinuous function majorized by f Yensen's inequality implies

$$\overline{\mathrm{co}}f(\rho) \leq \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \int \overline{\mathrm{co}}f(\sigma)\mu(d\sigma) \leq \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \int f(\sigma)\mu(d\sigma) = \check{f}(\rho)$$

for all ρ in $\mathfrak{S}(\mathcal{H})$. It follows that $\check{f} = \overline{\mathrm{co}}f$.

Suppose the function $\hat{f}(\rho)$ is not lower semicontinuous. This implies existence of a sequence of states $\{\rho_n\}$ converging to some state ρ_0 such that

$$\lim_{n \to +\infty} \hat{f}(\rho_n) < \hat{f}(\rho_0). \tag{6}$$

Let ε be an arbitrary and μ_0^ε be a measure in $\mathcal{P}_{\{\rho_0\}}$ such that

$$\hat{f}(\rho_0) < \int f(\sigma)\mu_0^\varepsilon(d\sigma) + \varepsilon.$$

By corollary 3 there exist a subsequence $\{\rho_{n_k}\}$ of the sequence $\{\rho_n\}$ and a sequence $\{\mu_k\}$ of measures in $\mathcal{P}$ converging to the measure μ_0^ε such that $\bar{\rho}(\mu_k) = \rho_{n_k}$ for all k. By lower semicontinuity of the functional (4) we obtain

$$\hat{f}(\rho_0) \leq \int f(\sigma)\mu_0^\varepsilon(d\sigma) + \varepsilon \leq \liminf_{k \to +\infty} \int f(\sigma)\mu_k(d\sigma) + \varepsilon \leq \lim_{k \to +\infty} \hat{f}(\rho_{n_k}) + \varepsilon,$$

which contradicts to (6) since ε can be arbitrarily small. $\square$

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Since by proposition 4 in [11] the quantum entropy is a function represented as a pointwise limit of increasing sequence of concave continuous bounded functions theorem 1 implies, in particular, the following result [18].

Corollary 5. *The convex closure $\overline{\mathrm{co}}H_\Phi$ of the output entropy $H_\Phi = H(\Phi(\cdot))$ of an arbitrary channel Φ is defined by the expression*

$$\overline{\mathrm{co}}H_\Phi(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} H_\Phi(\sigma)\mu(d\sigma), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}),$$

in which the infimum is achieved at a particular measure in $\mathcal{P}_{\{\rho\}}$ supported by pure states.

If f is a continuous bounded function on the set $\mathfrak{S}(\mathcal{H})$ then theorem 1 is applicable for the functions f and $-f$ simultaneously resulting in the following observation.

Proposition 4. *Let f be a continuous bounded function on the set $\mathfrak{S}(\mathcal{H})$. The convex closure $\overline{\mathrm{co}}f$ of the function f is a continuous bounded function on the set $\mathfrak{S}(\mathcal{H})$ defined by the expression*

$$\overline{\mathrm{co}}f(\rho) = \inf_{\{\pi_i\rho_i\}^f \in \mathcal{P}_{\{\rho\}}} \sum_i \pi_i f(\rho_i) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}),$$

(7)

in which the last infimum is achieved at a particular measure in $\mathcal{P}_{\{\rho\}}$.

The above expression implies that the convex closure $\overline{\mathrm{co}}f$ of the function f coincides with its convex hull $\mathrm{co}f$.

As a simple application of this proposition consider the following sufficient condition of boundedness and continuity of the convex closure $\overline{\mathrm{co}}H_\Phi$ of the output entropy H_Φ of a quantum channel Φ, which implies, in particular, continuity of the entanglement of formation of states in tensor product of two systems with one of them finite dimensional.[6]

Corollary 6. *If the output entropy H_Φ of a particular channel Φ is bounded and continuous on the set $\mathfrak{S}(\mathcal{H})$ then the convex closure $\overline{\mathrm{co}}H_\Phi$ of the output entropy is bounded and continuous on the set $\mathfrak{S}(\mathcal{H})$ and coincides with its convex hull $\mathrm{co}H_\Phi$.*

Note that boundedness and continuity of the output entropy H_Φ is not a necessary condition of boundedness and continuity of its convex closure $\overline{\mathrm{co}}H_\Phi$. For example, if Φ is the noiseless channel then $\overline{\mathrm{co}}H_\Phi \equiv 0$ while H_Φ

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[6]Note that the proof of continuity of the entanglement of formation is not trivial even in the finite dimensional case [12].

as well as $\mathrm{co}H_\Phi$ take infinite values on a dense subset of states. Below by using proposition 3 we will obtain a necessary and sufficient condition of boundedness and continuity the function $\overline{\mathrm{co}}H_\Phi$ (corollary 9.)

One of the most common problem of the convex analysis is the problem of extension of a particular function defined on the set $\mathrm{ext}X$ of all extreme points of a particular convex set X to a convex (concave) function defined on the whole set X. In what follows we consider this problem for the set $X = \mathfrak{S}(\mathcal{H})$.

Below it is shown that every continuous and bounded function f defined on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$ of all pure states can be extended to continuous and bounded convex (concave) function on the set $\mathfrak{S}(\mathcal{H})$ with the particular maximality (minimality) property. This result (proposition 5 below) is a corollary of the following more general observation.

Theorem 2. *Let f be a lower bounded lower semicontinuous function on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$.*

A) *The functions*

$$f_*(\rho) = \inf_{\mu \in \widehat{\mathcal{P}}_{\{\rho\}}} \int_{\mathrm{ext}\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma) \quad and \quad f^*(\rho) = \sup_{\mu \in \widehat{\mathcal{P}}_{\{\rho\}}} \int_{\mathrm{ext}\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma)$$

are lower semicontinuous functions on the set $\mathfrak{S}(\mathcal{H})$;

B) *The function f_* is the maximal lower semicontinuous convex extension of the function f to the set $\mathfrak{S}(\mathcal{H})$;*

C) *For an arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ there exists a measure $\hat{\mu}_\rho^f$ in $\widehat{\mathcal{P}}_{\{\rho\}}$ such that*

$$f_*(\rho) = \int_{\mathrm{ext}\mathfrak{S}(\mathcal{H})} f(\sigma)\hat{\mu}_\rho^f(d\sigma);$$

D) *The function f^* can be defined as follows*

$$f^*(\rho) = \sup_{\{\pi_i \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}),$$

which means that f^* is the minimal lower bounded [7] concave extension of the function f to the set $\mathfrak{S}(\mathcal{H})$.

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[7] The requirement of lower boundedness is essential. Indeed, the function

$$g(\rho) = \begin{cases} f^*(\rho), & \mathrm{rank}\rho < +\infty \\ -\infty, & \mathrm{rank}\rho = +\infty \end{cases}$$

is a concave extension of the function f majorized by f^*.

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The expression for the function f^* in the part D means that for any state ρ the supremum in the definition of the value $f^*(\rho)$ can be taken over the subset of $\widehat{\mathcal{P}}_{\{\rho\}}$ consisting of atomic measures.

Remark 2. The analogies of the assertions C and D in theorem 2 are not true for the functions f^* and f_* correspondingly.

Below we construct bounded lower semicontinuous function f_s on the set of pure states and a state ρ_0 such that the supremum in the definition of the value $f_s^*(\rho_0)$ is not achieved.

Let $\mathcal{A}_s$ be the closed set of all pure product states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$ and ρ_0 be the separable state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$ constructed in [8] such that any measure with the barycenter ρ_0 has no atoms in the set $\mathcal{A}_s$. Let

$$
f_s(\rho) = \begin{cases} \sup_{\sigma \in \mathcal{A}_s} \mathrm{Tr}\rho\sigma, & \rho \in \mathrm{ext}\mathfrak{S}(\mathcal{H} \otimes \mathcal{H}) \setminus \mathcal{A}_s \\ 0, & \rho \in \mathcal{A}_s. \end{cases}
$$

By the observation in the proof of lemma 1 this function is bounded and lower semicontinuous on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$. Since $f_s(\rho) < 1$ for all ρ in $\mathrm{ext}\mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$ to show that the supremum in the definition of the value $f_s^*(\rho_0)$ is not achieved it is sufficient to show that $f_s^*(\rho_0) = 1$. Let $\widehat{\mu}_0$ be a measure (purely nonatomic) supported by $\mathcal{A}_s$ such that $\bar{\rho}(\hat{\mu}_0) = \rho_0$. By corollary 2 there exists a sequence $\{\hat{\mu}_n\}$ of atomic measures in $\widehat{\mathcal{P}}_{\{\rho_0\}}$ weakly converging to the measure $\hat{\mu}_0$. Since by the observation in the proof of lemma 1 the function $g_s(\rho) = \sup_{\sigma \in \mathcal{A}_s} \mathrm{Tr}\rho\sigma$ is bounded and continuous on the set $\mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$ the definition of weak convergence implies

$$
\lim_{n \to +\infty} \int g_s(\sigma)\hat{\mu}_n(d\sigma) = \int g_s(\sigma)\hat{\mu}_0(d\sigma) = 1. \tag{8}
$$

By the construction of the state ρ_0 for each n all atoms of the measure $\widehat{\mu}_n$ lie in $\mathrm{ext}\mathfrak{S}(\mathcal{H} \otimes \mathcal{H}) \setminus \mathcal{A}_s$ and hence

$$
\int g_s(\sigma)\hat{\mu}_n(d\sigma) = \int f_s(\sigma)\hat{\mu}_n(d\sigma).
$$

This and (8) imply $f^*(\rho_0) = 1$.

To show that the infimum in the definition of the value $f_*(\rho)$ can not be taken over the subset of $\widehat{\mathcal{P}}_{\{\rho\}}$ consisting of atomic measures consider the function $\chi_{\bar{\mathcal{A}}_s} = 1 - \chi_{\mathcal{A}_s}$, where $\mathcal{A}_s$ is the closed set of all pure product states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$ and ρ_0 be the separable state described in the above example. Since by the observation in the proof of lemma 1 the function $\chi_{\bar{\mathcal{A}}_s}$ can be

represented as a pointwise limit of increasing family of concave continuous and bounded functions it is concave and lower semicontinuous. We have

$$\inf_{\mu\in\widehat{\mathcal{P}}_{\{\rho_0\}}} \int \chi_{\bar{\mathcal{A}}_s}(\sigma)\mu(d\sigma) = 0, \quad \text{while} \quad \inf_{\{\pi_i\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho_0\}}} \sum_i \pi_i\chi_{\bar{\mathcal{A}}_s}(\rho_i) = 1, \quad (9)$$

since by the construction of the state ρ_0 each countable convex decomposition of this state does not contain states from $\mathcal{A}_s$. $\square$

The last example in remark 2 shows that the coincidence of two definitions of EoF for an arbitrary state in infinite dimensional bipartite system conjectured in [18] can not be proved by using only such analytical properties of the entropy as concavity and lower semicontinuity. Even more deeper property of the entropy, consisting in its representation as a pointwise limit of increasing sequence of continuous concave functions [11], can not help since by the construction in the proof of lemma 1 the above function $\chi_{\bar{\mathcal{A}}_s}$ has this property but relations (9) hold for this function.

Proof of theorem 2. Convexity and concavity of the functions f_* and f^* correspondingly easily follows from the definitions and convexity of the set $\widehat{\mathcal{P}}$.

The proof of almost all assertions of this theorem is a repetition of the proof of the corresponding assertions of theorem 1 with using corollary 2 and corollary 4 instead of lemma 1 in [7] and corollary 3 correspondingly. It is necessary only to show the maximality and minimality properties of the functions f_* and f^* correspondingly.

Suppose g is a convex lower bounded lower semicontinuous extension of the function f to the set $\mathfrak{S}(\mathcal{H})$. Yensen's inequality implies

$$g(\rho) \leq \inf_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}} \int g(\sigma)\mu(d\sigma) = \inf_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}} \int f(\sigma)\mu(d\sigma) = f_*(\rho)$$

for arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$. It follows that f_* is the maximal convex lower semicontinuous extension of the function f to the set $\mathfrak{S}(\mathcal{H})$.

Suppose g is a lower bounded concave extension of the function f to the set $\mathfrak{S}(\mathcal{H})$. Since for this function the discrete version of Yensen's inequality is valid we have

$$g(\rho) \geq \sup_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i g(\rho_i) = \sup_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i) = f^*(\rho)$$

for arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$. It follows that f^* is the minimal lower bounded concave extension of the function f to the set $\mathfrak{S}(\mathcal{H})$.$\square$

Note that any measure of entanglement (see [10] for the definition) coincides with the entropy of a partial trace on the set of pure states. Theorems 1C and 2B imply the following characterization of the entanglement of formation.

Corollary 7. *The entanglement of formation (defined as the convex closure of the output entropy of a partial trace channel) is the maximal lower semicontinuous function coinciding with the entropy of a partial trace on the set of pure states.*

If f is a continuous bounded function on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$ then theorem 2 is applicable for the functions f and $-f$ simultaneously resulting in the following observation.

Proposition 5. *Let f be a continuous bounded function on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$.*

The functions f_* and f^* introduced in theorem 2 are continuous functions on the set $\mathfrak{S}(\mathcal{H})$, which can be also defined as follows

$$f_*(\rho) = \inf_{\{\pi_i \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i) \ and \ f^*(\rho) = \sup_{\{\pi_i \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \ \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

The functions f_* and f^* are the maximal upper bounded convex and the minimal lower bounded concave extensions of the function f to the set $\mathfrak{S}(\mathcal{H})$ correspondingly.

Proposition 5 implies, in particular, the following observation, which seems to be nontrivial since the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$ is not compact if $\dim\mathcal{H} = +\infty$.

Corollary 8. *Arbitrary bounded continuous function on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$ can be extended to convex (concave) continuous bounded function on the set $\mathfrak{S}(\mathcal{H})$.*

Theorem 1 and proposition 5 also imply a necessary and sufficient condition of continuity and boundedness of the convex closure of the output entropy of a quantum channel.

Corollary 9. *The convex closure $\overline{\mathrm{co}}H_\Phi$ of the output entropy H_Φ of a particular channel Φ is bounded and continuous on the set $\mathfrak{S}(\mathcal{H})$ if and only if the output entropy H_Φ is bounded and continuous on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$. In this case*

$$\overline{\mathrm{co}}H_\Phi(\rho) = \inf_{\{\pi_i \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi(\rho_i), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

Due to proposition 4 in [11] this corollary is a partial case of the following one.

Corollary 10. *Let f be a concave lower bounded lower semicontinuous*

function such that $-f \in \widehat{\mathcal{S}}$.[8] *The convex closure* $\overline{\mathrm{co}}f$ *of the function* f *is bounded and continuous on the set* $\mathfrak{S}(\mathcal{H})$ *if and only if the function* f *is bounded and continuous on the set* $\mathrm{ext}\mathfrak{S}(\mathcal{H})$. *In this case*

$$\overline{\mathrm{co}}f(\rho) = \inf_{\{\pi_i\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

Proof. The assumption of the corollary and the parts B and C of theorem 1 imply

$$f_*(\rho) = \inf_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}} \int f(\sigma)\mu(d\sigma) = \overline{\mathrm{co}}f(\rho), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

If the function f is bounded and continuous on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$ then by proposition 5 the function $f_* = \overline{\mathrm{co}}f$ is bounded and continuous on the set $\mathfrak{S}(\mathcal{H})$.

The converse assertion is trivial since by theorem 1B the functions f and $\overline{\mathrm{co}}f$ coincide on the set $\mathrm{ext}\mathfrak{S}(\mathcal{H})$.

The expression for the function $\overline{\mathrm{co}}f = f_*$ follows from proposition 5. $\square$

Remark 3. Proposition 5 can be used for construction of continuous convex (concave) characteristics of quantum states as the above maximal convex (minimal concave) extensions of continuous bounded functions defined on the set of pure states. For example, it is possible to construct a quasimeasure[9] of entanglement, which is a continuous bounded function on the whole state space of infinite dimensional bipartite system closely related to the entanglement of formation E_F[10]. Let $n > 1$ be fixed natural number. For arbitrary pure state ω in the tensor product state space $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ let

$$H_n(\omega) = -\sum_{i=1}^n \lambda_i \log \lambda_i + \left(\sum_{i=1}^n \lambda_i\right) \log \left(\sum_{i=1}^n \lambda_i\right)$$

where $\{\lambda_i\}_{i=1}^n$ is the set of n maximal eigen values of the state $\mathrm{Tr}_{\mathcal{K}}\omega \cong \mathrm{Tr}_{\mathcal{H}}\omega$. By proposition 4 in [11] the sequence $\{H_n\}$ of continuous bounded functions

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[8]The set $\widehat{\mathcal{S}}$ is defined at the begin of section 3. It is sufficient to require that the assertion A of lemma 2 holds for the function $-f$.

[9]A measure of entanglement is a function on the set of states of a bipartite system satisfying the set of axioms [10], which imply, in particular, that any measure of entanglement coincides on the set of pure states with the entropy of partial trace and hence it can not be continuous in the infinite dimensional case.

[10]More precisely, to the function E_F^2, which is defined in [18] as the convex closure of the output entropy of partial trace channel

on ext$\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ is nondecreasing and pointwise converges to the quantum entropy of a partial trace.

Let E_F^n be the extension $(H_n)_*$ of the function H_n to the set $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ considered in proposition 5. By this proposition the function E_F^n is continuous convex bounded function on $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. By the construction the function E_F^n has the following properties:

1) $E_F^n(\omega) = 0$ if and only if ω is a separable state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$;

2) $E_F^n(\omega) \leq E_F(\omega)$ for arbitrary state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$;

3) $E_F^n(\omega) = E_F(\omega)$ for arbitrary state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that either $\mathrm{rank}\mathrm{Tr}_{\mathcal{K}}\omega \leq n$ or $\mathrm{rank}\mathrm{Tr}_{\mathcal{H}}\omega \leq n$;

4) $0 \leq E_F^n(\omega) \leq \log n$ for arbitrary state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$.

The sequence $\{E_F^n\}_n$ of continuous bounded functions on $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ is increasing. Hence its pointwise limit $E_F^{+\infty}$ is a convex lower semicontinuous function on $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, coinciding on the set of pure states with the entropy of a partial trace. An interesting open question is a relation between $E_F^{+\infty}$ and E_F. By the construction $E_F^{+\infty}(\omega) \leq E_F(\omega)$ for arbitrary state ω and equality here takes place if ω is a pure state or such a state that either $\mathrm{rank}\mathrm{Tr}_{\mathcal{K}}\omega < +\infty$ or $\mathrm{rank}\mathrm{Tr}_{\mathcal{H}}\omega < +\infty$. Coincidence of $E_F^{+\infty}$ and E_F would imply that E_F is a function of the class $\widehat{\mathcal{S}}$.

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