

The Convex Closure of the Output Entropy of Infinite Dimensional Channels and the Additivity Problem

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1 Introduction

The central role in quantum information theory is played by the notion of a quantum channel, transforming states of one (input) quantum system into states of other (output) quantum system. The important characteristics of a quantum channel are the output entropy and its convex closure¹ (cf.[16]), which is widely used in quantum information theory, sometimes, implicitly. For example, the Holevo capacity of a quantum channel with finite output entropy can be defined as the maximal difference between the output entropy and its convex closure. Another example is the notion of entanglement of formation (EoF) of a state in bipartite system. Indeed, it is easy to see that the definition of the EoF in the finite dimensional case implies its coincidence with the convex closure of the output entropy of a partial trace [2],[1]. In the infinite dimensional case the EoF can be directly defined as the convex closure of the output entropy of a partial trace. The advantages of this definition and its relations with some others are discussed in [19].

The representation of the convex closure of the output entropy (in what follows, CCoE) of an arbitrary infinite dimensional channel and some of its properties are obtained in [19]. In this paper we develop these results and show their applications to the additivity problem.

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¹It is also used the term "lower envelope" [4].

In section 3 the continuity conditions for the CCoOE are considered (propositions 1 and 2, corollary 1). The nontrivial example of channel with continuous and bounded CCoOE is presented.

In section 4 the superadditivity property of the CCoOE is considered. For partial trace channels this property means strong superadditivity of the EoF [1]. In the finite dimensional case superadditivity of the CCoOE for given two channels is equivalent to the additivity of the Holevo capacity for these channels with arbitrary constraints [10]. Moreover, the global additivity conjecture can be expressed as the superadditivity of the CCoOE for partial trace channels (= superadditivity of the EoF) [23]. In the infinite dimensional case the relations between different additivity properties are more complex. The main problem in this case consists in existence of pure states with infinite entropy of partial traces, which can be called superentangled (see remark 4 in [18]). It is this problem that up to now prevented to prove the CCoOE-analog of theorem 3 in [18], stated that the additivity of the Holevo capacity for all finite dimensional channels implies the additivity of the Holevo capacity for all infinite dimensional channels with arbitrary constraints and to show the superadditivity of the CCoOE (and even additivity of the minimal output entropy) for two infinite dimensional channels with one of them a direct sum of noiseless and entanglement-breaking channels despite the fact that additivity of the Holevo capacity for these channels with arbitrary constraints is derived in [18] from the corresponding finite dimensional results [22],[10]. In this paper we overcome these difficulties by using special approximation result (lemma 1), based on the continuity property of the CCoOE and some other observations from [19].

The main result of this paper is the statement that the superadditivity of the CCoOE for all finite dimensional channels implies the superadditivity of the CCoOE for all infinite dimensional channels (theorem 1 and corollary 2), which implies the analogous statements for the strong superadditivity of the EoF (corollary 3) and for the additivity of the minimal output entropy (corollary 4). This result and theorem 3 in [18] provide infinite dimensional generalization of Shor's theorem [23], stated equivalence of different additivity conjectures (theorem 2). The approximation technique used in the proof of the above result also makes possible to derive the superadditivity of the CCoOE (and hence the additivity of the minimal output entropy) for two infinite dimensional channels with one of them a direct sum of noiseless and entanglement-breaking channels (proposition 3) from the additivity of the Holevo capacity for these channels with arbitrary constraints. Some conse-

quences of this result are considered (corollary 5 and remark 5).

The role of the superadditivity of the CCoOE is stressed by the observation in [9],[14] that validity of this property for some pair of channels means its validity for the pair of complementary channels. By theorem 1 in [10] in the finite dimensional case this result can be reformulated in terms of the additivity of the Holevo capacity. In the infinite dimensional case the situation is more difficult, but some conditional result in this direction can be proved (proposition 4). This and the above observations leads to extension of the class of infinite dimensional channels for which the superadditivity of the CCoOE and the additivity of the Holevo capacity with arbitrary constraints are proved (corollary 6).

In the Appendix 5.1 some general continuity condition for the quantum entropy applicable to noncompact and nonconvex sets of states is considered (proposition 5, corollaries 7 and 8).

2 Preliminaries

Let \mathcal{H} be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ be the set of all bounded operators on \mathcal{H} , $\mathfrak{T}(\mathcal{H})$ be the Banach space of all trace-class operators with the trace norm $\|\cdot\|_1$ and $\mathfrak{S}(\mathcal{H})$ be the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of all density operators on \mathcal{H} , which is complete separable metric space with the metric defined by the trace norm. Each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$ [4], so, in what follows we will also for brevity use the term "state".

We denote by $\text{co}\mathcal{A}$ ($\overline{\text{co}}\mathcal{A}$) the convex hull (closure) of a set \mathcal{A} and by $\text{co}f$ ($\overline{\text{co}}f$) the convex hull (closure) of a function f [16]. We denote by $\text{ext}\mathcal{A}$ the set of all extreme points of a convex set \mathcal{A} .

Let \mathcal{P} be the set of all Borel probability measures on $\mathfrak{S}(\mathcal{H})$ endowed with the topology of weak convergence [3],[17]. Since $\mathfrak{S}(\mathcal{H})$ is a complete separable metric space \mathcal{P} is a complete separable metric space as well [17]. Let $\hat{\mathcal{P}}$ be the closed subset of \mathcal{P} consisting of all measures supported by the closed set $\text{ext}\mathfrak{S}(\mathcal{H})$ of all pure states.

The *barycenter* of the measure μ is the state defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \sigma \mu(d\sigma).$$

For arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ let $\mathcal{P}_{\{\rho\}}$ (corresp. $\widehat{\mathcal{P}}_{\{\rho\}}$) be the subset of \mathcal{P} (corresp. $\widehat{\mathcal{P}}$) consisting of measures with the barycenter ρ .

A collection of states $\{\rho_i\}$ with corresponding probability distribution $\{\pi_i\}$ is conventionally called *ensemble* and is denoted by $\{\pi_i, \rho_i\}$. In this paper we will consider ensemble of states as a partial case of probability measure, so that notation $\{\pi_i, \rho_i\} \in \mathcal{P}_{\{\rho\}}$ means that $\rho = \sum_i \pi_i \rho_i$. An ensemble consisting of finite number of states is denoted by $\{\pi_i, \rho_i\}^f$ and is also called a measure with finite support.

We will use the following extension of the von Neumann entropy $S(\rho) = -\text{Tr} \rho \log \rho$ of a state ρ to the set of all positive trace class operators (cf.[15])

$$H(A) = -\text{Tr} A \log A + \text{Tr} A \log \text{Tr} A = (\text{Tr} A) S(A/\text{Tr} A), \quad \forall A \in \mathfrak{T}_+(\mathcal{H}).$$

Nonnegativity, concavity and lower semicontinuity of the von Neumann entropy S on the set $\mathfrak{S}(\mathcal{H})$ imply the same properties of the entropy H on the set $\mathfrak{T}_+(\mathcal{H})$.

Let $\mathcal{H}, \mathcal{H}'$ be a pair of separable Hilbert spaces which we shall call correspondingly input and output space. A channel Φ is a linear positive trace preserving map from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ such that the dual map $\Phi^* : \mathfrak{B}(\mathcal{H}') \mapsto \mathfrak{B}(\mathcal{H})$ is completely positive.

The important characteristic of a quantum channel Φ is the output entropy $H_\Phi(\rho) = H(\Phi(\rho))$ - concave lower semicontinuous function on the input state space $\mathfrak{S}(\mathcal{H})$. It is shown in [19] that the convex closure of the output entropy (CCoOE) of an arbitrary quantum channel Φ is defined by the expression

$$\overline{\text{co}}H_\Phi(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} H_\Phi(\sigma) \mu(d\sigma) = \inf_{\mu \in \widehat{\mathcal{P}}_{\{\rho\}}} \int_{\text{ext} \mathfrak{S}(\mathcal{H})} H_\Phi(\sigma) \mu(d\sigma) \quad (1)$$

for all ρ in $\mathfrak{S}(\mathcal{H})$ and that the last infimum in this expression is always achieved at some measure μ_ρ^Φ in $\widehat{\mathcal{P}}_{\{\rho\}}$.

Remark 1. There exist channels Φ and states ρ such that each optimal measure μ_ρ^Φ is purely nonatomic. Indeed, let Φ_0 be the partial trace channel and ρ_0 be the separable state constructed in [12], such that any measure with the barycenter ρ_0 has no atoms in the set of pure product states, then $\overline{\text{co}}H_{\Phi_0}(\rho_0) = 0$ and by the construction of the state ρ_0 any optimal measure $\mu_{\rho_0}^{\Phi_0}$ has no atoms. \square

It is also shown in [19] that

$$\overline{\text{co}}H_{\Phi}(\rho) = \text{co}H_{\Phi}(\rho) = \inf_{\{\pi_i, \rho_i\}^f \in \mathcal{P}_{\{\rho\}}} \sum_i \pi_i H_{\Phi}(\rho_i) \quad (2)$$

in the case $H_{\Phi}(\rho) < +\infty$. By decomposing each state of finite ensemble $\{\pi_i, \rho_i\}^f$ into convex countable combination of pure states and by using concavity of the (output) entropy it is easy to obtain from (1) and (2) that in the case $H_{\Phi}(\rho) < +\infty$ the following representation holds

$$\overline{\text{co}}H_{\Phi}(\rho) = \inf_{\{\pi_i, \rho_i\} \in \hat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_{\Phi}(\rho_i). \quad (3)$$

Remark 2. Validity of representations (2) and (3) for all states with finite output entropy implies that the definition of the entanglement of formation as the CCoOE of a partial trace, proposed in [19], is *reasonable* from the physical point of view. Indeed, since in a physical experiment we can prepare only states with finite mean energy, it is reasonable to consider in the definition of the entanglement of formation $E_F(\omega)$ of a given state ω with finite energy only convex decompositions of this state consisting of states with finite energy. Representations (2) and (3) imply that for any such state ω the infimum in definition (1) of the value $\overline{\text{co}}H_{\Phi}(\omega)$, where Φ is a partial trace, can be taken only over the set of atomic measures whose atoms are states with finite energy.²□

In the case $H_{\Phi}(\rho) = +\infty$ the general properties of the entropy [15],[25] implies $\text{co}H_{\Phi}(\rho) = +\infty$, but $\overline{\text{co}}H_{\Phi}(\rho)$ may be finite (for example, if Φ is the noiseless channel then $\overline{\text{co}}H_{\Phi}(\rho) = 0$ for all states ρ) and hence representation (2) does not hold in this case. Nevertheless this does not imply that representation (3) is not true. Moreover, for all studied examples of quantum channels Φ this representation holds for all states ρ and some general sufficient condition for its validity can be formulated (proposition 2 below). Note that " \leq " in (3) is obviously follows from (1), but the proof of " $=$ " remains an open problem. Validity of representation (3) for all channels Φ and states ρ is a very desirable property from the technical point of view since in proving general results it is more convenient to deal with a countable sum instead of an integral over an arbitrary probability measure. In [21] (remark 2 and the note below) it is shown that representation (3) can not be proved

²The author is grateful to J.I.Cirac for pointing the importance of this observation.

by using only such analytical properties of the output entropy as concavity and lower semicontinuity.

For brevity and according to the tradition the convex closure $\overline{\text{co}}H_\Phi$ of the output entropy H_Φ of a channel Φ will be denoted below by \hat{H}_Φ .

The χ -function of the channel Φ is defined by (cf.[10],[11])

$$\chi_\Phi(\rho) = \sup_{\{\pi_i, \rho_i\}^f \in \mathcal{P}_{\{\rho\}}} \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\rho)). \quad (4)$$

The χ -function of an arbitrary infinite dimensional channel Φ is a non-negative concave and lower semicontinuous function, such that

$$\chi_\Phi(\rho) = H_\Phi(\rho) - \hat{H}_\Phi(\rho) \quad (5)$$

for all states ρ with finite output entropy $H_\Phi(\rho)$ [18].

We will denote by $h_2(p)$ the binary entropy $-p \log p - (1-p) \log(1-p)$.

3 Continuity properties

For arbitrary channel Φ the function $\hat{H}_\Phi = \overline{\text{co}}H_\Phi$ is lower semicontinuous by definition but it is not continuous in general. Nevertheless the function \hat{H}_Φ can have continuous restrictions to some sets of states. Moreover for some channels Φ it can be continuous and bounded on the whole state space (for example, if Φ is the noiseless channel then $\hat{H}_\Phi \equiv 0$). In this section we consider two continuity conditions for this function.

In [19] the sufficient condition of continuity of the restriction of the function \hat{H}_Φ to a subset of $\mathfrak{S}(\mathcal{H})$ is obtained (proposition 7). This condition can be reformulated as follows.

Proposition 1. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be an arbitrary quantum channel. If $\{\rho_n\}$ is a sequence of states converging to the state ρ_0 such that $\lim_{n \rightarrow +\infty} H_\Phi(\rho_n) = H_\Phi(\rho_0) < +\infty$ then $\lim_{n \rightarrow +\infty} \hat{H}_\Phi(\rho_n) = \hat{H}_\Phi(\rho_0)$.*

Remark 3. The statement of proposition 1 seems surprising by the following reason. A value of the output entropy H_Φ at a particular state ρ is completely defined by the output state $\Phi(\rho)$ and it does not depend on the action of the channel Φ on other input states. Hence the condition $\lim_{n \rightarrow +\infty} H_\Phi(\rho_n) = H_\Phi(\rho_0)$ depends only on the action of the channel Φ on the states of the sequence $\{\rho_n\}$ and its limit state ρ_0 so that it is a *local* condition. In contrast to this a value of the function \hat{H}_Φ at a particular state

ρ depends on the action of the channel Φ on the whole state space.³ It follows as from the general definition of the convex closure as from representation (1). Hence the property $\lim_{n \rightarrow +\infty} \hat{H}_\Phi(\rho_n) = \hat{H}_\Phi(\rho_0)$ depends on the action of the channel Φ on the whole state space so that it is *nonlocal*. Nevertheless proposition 1 provides a sufficient condition of its validity in terms of the above local condition. \square

It is shown in [19] that proposition 1 implies continuity of the entanglement of formation on the set of all states of bipartite system with bounded mean energy. In this paper we will use this proposition in proving our basic approximation result (lemma 1 in section 4), concerning the additivity problem.

Note also that proposition 1 implies the following observation, which shows continuity of the CCoOE with respect to the convergence defined by the quantum relative entropy $H(\cdot\|\cdot)$ [15].

Corollary 1. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be a quantum channel and ρ_0 be a state in $\mathfrak{S}(\mathcal{H})$ such that $\text{Tr}(\Phi(\rho_0))^\lambda < +\infty$ for some $\lambda < 1$. Then $\lim_{n \rightarrow +\infty} \hat{H}_\Phi(\rho_n) = \hat{H}_\Phi(\rho_0)$ for arbitrary sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$ such that $\lim_{n \rightarrow +\infty} H(\rho_n\|\rho_0) = 0$.*

Proof. By proposition 1 it is sufficient to show that $\lim_{n \rightarrow +\infty} H_\Phi(\rho_n) = H_\Phi(\rho_0) < +\infty$. But this follows from proposition 2 in [20] since by monotonicity of the relative entropy the condition $\lim_{n \rightarrow +\infty} H(\rho_n\|\rho_0) = 0$ implies $\lim_{n \rightarrow +\infty} H(\Phi(\rho_n)\|\Phi(\rho_0)) = 0$. \square

Note that the conditions of corollary 1 are valid for arbitrary Gaussian channel Φ and arbitrary Gaussian state ρ_0 .

The assertion of proposition 1 can not be converted as it follows from the example of the noiseless channel Φ .

In contrast to proposition 1 the following proposition provides a necessary and sufficient condition of continuity of the CCoOE on the whole state space.

Proposition 2. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be an arbitrary quantum channel.*

The convex closure \hat{H}_Φ of the output entropy H_Φ is bounded and continuous on the set $\mathfrak{S}(\mathcal{H})$ if and only if the output entropy H_Φ is bounded and continuous on the set $\text{ext}\mathfrak{S}(\mathcal{H})$. In this case representation (3) holds for arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$.

The above equivalent properties hold if there exist separable Hilbert space \mathcal{K} and set $\{\alpha_\psi\}_{\psi \in \mathcal{H}, \|\psi\|=1}$ of isomorphism from $\mathfrak{S}(\mathcal{H}')$ onto $\mathfrak{S}(\mathcal{K})$ such that $\alpha_\psi(\Phi(|\psi\rangle\langle\psi|))$ lies for each ψ in some compact subset \mathcal{A} of $\mathfrak{S}(\mathcal{K})$, on which

³More precisely, on the union of supports of all measures with the barycenter ρ .

the entropy is continuous.

Proof The first part of this proposition follows from proposition 5 and corollary 9 in [21]. The second one follows from corollary 7 in the Appendix 5.1, where general condition of continuity of the entropy is considered. \square

Note that continuity of the output entropy on the set of pure states is not very restrictive requirement. Indeed, this continuity trivially holds for noiseless channel, for which the output entropy coincides with the entropy of a state and is far from being continuous on the whole state space. Note that in this case the sufficient condition in proposition 2 is also trivially verified with $\mathcal{K} = \mathcal{H}'$, $\mathcal{A} = \{|\psi_0\rangle\langle\psi_0|\}$ and $\alpha_\psi = U_\psi(\cdot)U_\psi^*$, where U_ψ is any unitary such that $U_\psi|\psi\rangle = |\psi_0\rangle$ and $|\psi_0\rangle$ is some fixed unit vector in \mathcal{H}' .

The nontrivial application of the sufficient condition in proposition 2 is the proof of continuity of the CCoOE for the class of channels considered in the following example.

Example. Let \mathcal{H}_a be the Hilbert space $\mathcal{L}_2([-a, +a])$, where $a \leq +\infty$ and $\{U_t\}_{t \in \mathbb{R}}$ be the group of unitary operators in \mathcal{H}_a defined by

$$(U_t\varphi)(x) = \exp(-itx)\varphi(x), \quad \forall \varphi \in \mathcal{H}_a.$$

For given probability density function $p(t)$ consider the channel

$$\Phi_p^a : \mathfrak{S}(\mathcal{H}_a) \ni \rho \mapsto \int_{-\infty}^{+\infty} U_t \rho U_t^* p(t) dt \in \mathfrak{S}(\mathcal{H}_a).$$

Assume that the differential entropy $\int_{-\infty}^{+\infty} p(t)(-\log p(t))dt$ of the distribution $p(t)$ is finite and that the function $p(t)$ is bounded and monotonous on $(-\infty, -b]$ and on $[+b, +\infty)$ for sufficiently large b^4 . By using the sufficient condition in proposition 2 it is possible to show (see Appendix 5.2 for details) that:

- if $a < +\infty$ then the equivalent properties in proposition 2 hold for the channel Φ_p^a ;
- if $a = +\infty$ then for the channel Φ_p^a there exists pure state in $\mathfrak{S}(\mathcal{H}_a)$ with infinite output entropy.

Since for each $a > 0$ the space \mathcal{H}_a can be considered as a subspace of $\mathcal{H}_{+\infty}$ the set $\mathfrak{S}(\mathcal{H}_a)$ can be considered as a subset of $\mathfrak{S}(\mathcal{H}_{+\infty})$. The restriction

⁴The last assumption is for technical convenience.

of the channel $\Phi_p^{+\infty}$ to the subset $\mathfrak{S}(\mathcal{H}_a)$ coincides with the channel Φ_p^a and hence the restriction of the function $\hat{H}_{\Phi_p^{+\infty}}$ to the subset $\mathfrak{S}(\mathcal{H}_a)$ coincides with the function $\hat{H}_{\Phi_p^a}$. By the above observation the unbounded function $\hat{H}_{\Phi_p^{+\infty}}$ has bounded and continuous restriction to each subset from the increasing family $\{\mathfrak{S}(\mathcal{H}_a)\}_{a>0}$ such that $\bigcup_{a>0} \mathfrak{S}(\mathcal{H}_a) = \mathfrak{S}(\mathcal{H}_{+\infty})$.

4 The additivity problem

Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi : \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ be two channels. In this section we consider the superadditivity property of the CCoOE, which means validity of the inequality

$$\hat{H}_{\Phi \otimes \Psi}(\omega) \geq \hat{H}_{\Phi}(\omega^{\mathcal{H}}) + \hat{H}_{\Psi}(\omega^{\mathcal{K}}) \quad (6)$$

for all states ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, where $\omega^{\mathcal{H}} = \text{Tr}_{\mathcal{K}}\omega$ and $\omega^{\mathcal{K}} = \text{Tr}_{\mathcal{H}}\omega$. If Φ and Ψ are partial trace channels then this property means the strong superadditivity of the entanglement of formation [1]. Hence, as it is proved by P.Shor [23], the global additivity conjecture in the finite dimensional case can be expressed as the superadditivity of the CCoOE for partial trace channels.

The superadditivity of the CCoOE for given two finite dimensional channels is equivalent to the additivity of the Holevo capacity for these channels with arbitrary constraints, which can be expressed as the subadditivity property of the χ -function, consisting in validity of the inequality

$$\chi_{\Phi \otimes \Psi}(\omega) \leq \chi_{\Phi}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) \quad (7)$$

for all states ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ [10].

The continuity condition for the CCoOE, considered in section 3, implies the following approximation result, which plays a basic role in this section.

Lemma 1. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi : \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ be quantum channels. If for arbitrary finite rank state ω_0 in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that $H_{\Phi \otimes \Psi}(\omega_0) < +\infty$ there exists sequence $\{\omega_n\}$ of states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that*

- $\lim_{n \rightarrow +\infty} \omega_n = \omega_0$ and $\lim_{n \rightarrow +\infty} H_{\Phi \otimes \Psi}(\omega_n) = H_{\Phi \otimes \Psi}(\omega_0)$,
- inequality (6) holds with $\omega = \omega_n$ for each $n \in \mathbb{N}$,

then inequality (6) holds for all states ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$.

Proof. Note first that the condition of the lemma implies inequality (6) for arbitrary finite rank state ω such that $H_{\Phi \otimes \Psi}(\omega) < +\infty$. Let ω_0 be such a state. By proposition 1 the first property of the corresponding sequence $\{\omega_n\}$ implies $\lim_{n \rightarrow +\infty} \hat{H}_{\Phi \otimes \Psi}(\omega_n) = \hat{H}_{\Phi \otimes \Psi}(\omega_0)$. By this and due to lower semicontinuity of the functions \hat{H}_Φ and \hat{H}_Ψ the second property of the sequence $\{\omega_n\}$ implies inequality (6) with $\omega = \omega_0$.

It is sufficient to prove inequality (6) for an arbitrary state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that $\hat{H}_{\Phi \otimes \Psi}(\omega) < +\infty$. Let ω_0 be such a state. By lemma 3 in [19] there exists a sequence $\{\omega_n\}$ of finite rank states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ converging to the state ω_0 such that

$$H_{\Phi \otimes \Psi}(\omega_n) < +\infty, \forall n \quad \text{and} \quad \lim_{n \rightarrow +\infty} \hat{H}_{\Phi \otimes \Psi}(\omega_n) = \hat{H}_{\Phi \otimes \Psi}(\omega_0). \quad (8)$$

By the previous observation inequality (6) holds with $\omega = \omega_n$ for all $n = 1, 2, \dots$. This, lower semicontinuity of the functions \hat{H}_Φ and \hat{H}_Ψ and (8) imply inequality (6) with $\omega = \omega_0$. \square

Remark 4. By lemma 1 to prove the superadditivity of the CCoOE for given two channels Φ and Ψ it is sufficient to prove inequality (6) for arbitrary finite rank state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that $H_{\Phi \otimes \Psi}(\omega) < +\infty$. \square

In [18] it is proved that additivity of the Holevo capacity for all finite dimensional channels implies additivity of the Holevo capacity for all infinite dimensional channels with arbitrary constraints. Lemma 1 makes possible to prove the analogous result, concerning the superadditivity of the CCoOE.

Theorem 1. *If the superadditivity of the CCoOE holds for all finite dimensional channels then the superadditivity of the CCoOE holds for all infinite dimensional channels.*

Proof. By the observation in [13] any channel is unitarily equivalent to a particular subchannel⁵ of a partial trace. Since the superadditivity of the CCoOE for arbitrary two channels implies the same property for any their subchannels it is sufficient to consider the case of partial trace channels Φ and Ψ . Let $\mathcal{H}, \mathcal{L}, \mathcal{K}, \mathcal{N}$ be separable Hilbert spaces and

$$\Phi(\rho) = \text{Tr}_{\mathcal{L}} \rho, \quad \rho \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{L}) \quad \text{and} \quad \Psi(\sigma) = \text{Tr}_{\mathcal{N}} \sigma, \quad \sigma \in \mathfrak{S}(\mathcal{K} \otimes \mathcal{N}).$$

Let ω_0 be a state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{K} \otimes \mathcal{N})$ such that $H_{\Phi \otimes \Psi}(\omega_0) < +\infty$. Let $\{P_n\}, \{R_n\}, \{Q_n\}, \{S_n\}$ be increasing sequences of n -dimensional projectors in the spaces $\mathcal{H}, \mathcal{L}, \mathcal{K}, \mathcal{N}$, strongly converging to the identity operators

⁵Subchannel of a channel is a restriction of this channel to the set of all states supported by a particular subspace.

$I_{\mathcal{H}}, I_{\mathcal{L}}, I_{\mathcal{K}}, I_{\mathcal{N}}$ correspondingly. Consider the sequence of states

$$\omega_n = \frac{W_n \omega_0 W_n}{\text{Tr} W_n \omega_0}, \quad \text{where} \quad W_n = P_n \otimes R_n \otimes Q_n \otimes S_n,$$

converging to the state ω_0 .

For each n the following operator inequality holds

$$\begin{aligned} \Phi \otimes \Psi(\omega_n) &= \text{Tr}_{\mathcal{L} \otimes \mathcal{N}} \omega_n \\ &\leq (\text{Tr} W_n \omega_0)^{-1} P_n \otimes Q_n \cdot (\text{Tr}_{\mathcal{L} \otimes \mathcal{N}} \omega_0) \cdot P_n \otimes Q_n \\ &= (\text{Tr} W_n \omega_0)^{-1} P_n \otimes Q_n \cdot \Phi \otimes \Psi(\omega_0) \cdot P_n \otimes Q_n. \end{aligned} \tag{9}$$

Indeed, let $\{|i\rangle\}$ and $\{|j\rangle\}$ be ONB in the spaces \mathcal{L} and \mathcal{N} correspondingly such that

$$R_n = \sum_{i=1}^n |i\rangle\langle i| \quad \text{and} \quad S_n = \sum_{j=1}^n |j\rangle\langle j|.$$

For arbitrary $|\varphi\rangle$ in $P_n \otimes Q_n(\mathcal{H} \otimes \mathcal{K})$ we have

$$\langle \varphi | \text{Tr}_{\mathcal{L} \otimes \mathcal{N}} \omega_n | \varphi \rangle = \sum_{i,j=1}^{+\infty} \langle \varphi \otimes i \otimes j | \omega_n | \varphi \otimes i \otimes j \rangle =$$

$$(\text{Tr} W_n \omega_0)^{-1} \sum_{i,j=1}^n \langle \varphi \otimes i \otimes j | \omega_0 | \varphi \otimes i \otimes j \rangle \leq (\text{Tr} W_n \omega_0)^{-1} \langle \varphi | \text{Tr}_{\mathcal{L} \otimes \mathcal{N}} \omega_0 | \varphi \rangle,$$

which implies (9).

Since inequality (9) means decomposition

$$P_n \otimes Q_n \cdot \Phi \otimes \Psi(\omega_0) \cdot P_n \otimes Q_n = \lambda \Phi \otimes \Psi(\omega_n) + (1 - \lambda)(\text{positive operator}),$$

where $\lambda = \text{Tr} W_n \omega_0$, concavity of the entropy and proposition 4 in [15] imply

$$\begin{aligned} H(\Phi \otimes \Psi(\omega_n)) &\leq (\text{Tr} W_n \omega_0)^{-1} H(P_n \otimes Q_n \cdot \Phi \otimes \Psi(\omega_0) \cdot P_n \otimes Q_n) \\ &\leq (\text{Tr} W_n \omega_0)^{-1} H(\Phi \otimes \Psi(\omega_0)) < +\infty. \end{aligned}$$

It follows from this and lower semicontinuity of the entropy that

$$\lim_{n \rightarrow +\infty} H(\Phi \otimes \Psi(\omega_n)) = H(\Phi \otimes \Psi(\omega_0)) < +\infty.$$

By the assumption inequality (6) holds with $\omega = \omega_n$ for each n .

Lemma 1 implies the superadditivity of the CCoOE for the channels Φ and Ψ . \square

By using corollary 3 in [6] theorem 1 can be strengthened as follows.

Corollary 2. *If the superadditivity of the CCoOE holds for all finite dimensional unital channels then the superadditivity of the CCoOE holds for all infinite dimensional channels.*

By the observation in [13] theorem 1 can be reformulated in terms of the strong superadditivity of the entanglement of formation.

Corollary 3. *The strong superadditivity of the EoF in the finite dimensional case implies the strong superadditivity of the EoF in the infinite dimensional case.*⁶

Since the superadditivity of the CCoOE for two channels obviously implies the additivity of the minimal output entropy for these channels theorem 1, Shor's theorem [23] and corollary 3 in [6] imply the following result.

Corollary 4. *The additivity of the minimal output entropy for all finite dimensional unital channels implies the additivity of the minimal output entropy for all infinite dimensional channels.*

Theorem 1 and theorem 3 in [18] provide infinite dimensional generalization of Shor's theorem [23].

Theorem 2. *The following properties are equivalent:*

- *the additivity of the Holevo capacity holds for all infinite dimensional channels with arbitrary constraints;*
- *the additivity of the minimal output entropy holds for all infinite dimensional channels;*
- *the strong superadditivity of the EoF holds for arbitrary state in infinite dimensional bipartite system.*

Lemma 1, proposition 7 and theorem 2 in [18] make possible to derive the superadditivity of the CCoOE for nontrivial classes of infinite dimensional channels from the corresponding finite dimensional results [22],[10].

Proposition 3. *Let Ψ be an arbitrary channel. The superadditivity the CCoOE holds in each of the following cases:*

- (i) Φ is the noiseless channel;
- (ii) Φ is an entanglement breaking channel;

⁶In the last case we use the definition of the EoF as the CCoOE of a partial trace [19].

(iii) Φ is a direct sum mixture (cf. [10]) of a noiseless channel and a channel Φ_0 such that the superadditivity the CCoOE holds for Φ_0 and Ψ (in particular, an entanglement breaking channel).

Proof. (i) ⁷ Let $\Phi = \text{Id}$ be the noiseless channel. Let ω_0 be a state such that $H_{\Phi \otimes \Psi}(\omega_0) < +\infty$ and $\{P_n\}$ be the increasing sequence of spectral projectors of the state $\omega_0^{\mathcal{H}} = \text{Tr}_{\mathcal{K}} \omega_0$. Consider the sequence of states

$$\omega_n = \lambda_n^{-1} P_n \otimes I \omega_0 P_n \otimes I,$$

converging to the state ω_0 , where $\lambda_n = \text{Tr}(P_n \otimes I \omega_0)$.

Since $\Phi \otimes \Psi(\omega_n) = \lambda_n^{-1} P_n \otimes I \Phi \otimes \Psi(\omega_0) P_n \otimes I$ for all n these states have finite entropy and Simon's convergence theorem for entropy [24] implies

$$\lim_{n \rightarrow +\infty} H_{\Phi \otimes \Psi}(\omega_n) = H_{\Phi \otimes \Psi}(\omega_0) < +\infty.$$

Since $\omega_n^{\mathcal{H}} = \Phi(\omega_n^{\mathcal{H}})$ is a finite rank state $H_{\Phi}(\omega_n^{\mathcal{H}}) < +\infty$. This and well known inequality

$$|H_{\Phi}(\omega_n^{\mathcal{H}}) - H_{\Psi}(\omega_n^{\mathcal{K}})| \leq H_{\Phi \otimes \Psi}(\omega_n) < +\infty$$

implies $H_{\Psi}(\omega_n^{\mathcal{K}}) < +\infty$. By proposition 7 and theorem 2 in [18] inequality (6) holds with $\omega = \omega_n$ for each n .

Thus the sequence $\{\omega_n\}$ satisfies the both conditions in lemma 1 for given state ω_0 . By this lemma inequality (6) holds for all states ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$.

(ii) Let Φ be an entanglement-breaking channel. By lemma 1 it is sufficient to prove inequality (6) for all states ω such that $H_{\Phi \otimes \Psi}(\omega) < +\infty$. Let ω_0 be such a state. Since the channel Φ is entanglement-breaking the state $\Phi \otimes \Psi(\omega_0)$ is separable and hence

$$\max\{H_{\Phi}(\omega_0^{\mathcal{H}}), H_{\Psi}(\omega_0^{\mathcal{K}})\} \leq H_{\Phi \otimes \Psi}(\omega_0) < +\infty.$$

By proposition 7 and theorem 2 in [18] inequality (6) holds with $\omega = \omega_0$.

(iii) Let $\Phi_q = q\text{Id} \oplus (1-q)\Phi_0$. Note that $H_{\Phi_q} = qH_{\text{Id}} + (1-q)H_{\Phi_0} + h_2(q)$ and $H_{\Phi_q \otimes \Psi} = qH_{\text{Id} \otimes \Psi} + (1-q)H_{\Phi_0 \otimes \Psi} + h_2(q)$. By using this, (i) and the condition on the channel Φ_0 we obtain

$$\begin{aligned} \hat{H}_{\Phi_q \otimes \Psi}(\omega) &\geq q\hat{H}_{\text{Id} \otimes \Psi}(\omega) + (1-q)\hat{H}_{\Phi_0 \otimes \Psi}(\omega) + h_2(q) \\ &\geq q\hat{H}_{\Psi}(\omega^{\mathcal{K}}) + (1-q)(\hat{H}_{\Phi_0}(\omega^{\mathcal{H}}) + \hat{H}_{\Psi}(\omega^{\mathcal{K}})) + h_2(q) = \hat{H}_{\Phi_q}(\omega^{\mathcal{H}}) + \hat{H}_{\Psi}(\omega^{\mathcal{K}}), \end{aligned}$$

⁷This part of the proposition can be also proved by using the notion of complementary channel described below.

for any state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, where the last equality follows from the observation that the infimum in expression (1) for the value $\hat{H}_{\text{Id}}(\omega^{\mathcal{H}}) = 0$ is achieved at any measure supported by pure states and having the barycenter $\omega^{\mathcal{H}}$. \square

Proposition 3 implies in particular the following result, which seems to be nontrivial in the infinite dimensional case.

Corollary 5. *For arbitrary channels Φ and Ψ the following inequality holds*

$$\hat{H}_{\Phi \otimes \Psi}(\omega) \geq \max \left\{ \hat{H}_{\Phi}(\omega^{\mathcal{H}}), \hat{H}_{\Psi}(\omega^{\mathcal{K}}) \right\}, \quad \forall \omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}),$$

which implies in particular that

$$H_{\min}(\Phi \otimes \Psi) \geq \max \{H_{\min}(\Phi), H_{\min}(\Psi)\}.$$

Proof. By proposition 3 $\hat{H}_{\Phi \otimes \text{Id}}(\omega) \geq \hat{H}_{\Phi}(\omega^{\mathcal{H}})$ for any state ω . By using this and noting that $\hat{H}_{\Phi \otimes \text{Id}}(\text{Id} \otimes \Psi(\omega)) \leq \hat{H}_{\Phi \otimes \Psi}(\omega)$ for any state ω (this follows from expression (1) for the CCoE) we obtain the first assertion of the corollary. The second one obviously follows from the first. \square

Remark 5. By corollary 5 if the output entropy of one of the channels Φ and Ψ is everywhere infinite then the output entropy of the channel $\Phi \otimes \Psi$ is everywhere infinite as well despite the properties of the another channel. \square

In [9],[14] it is noted that the notion of complementary channel introduced in [5] is very useful in connection with the additivity problem.

If $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ is an arbitrary channel then by the Stinespring dilation theorem there exist a Hilbert space \mathcal{H}'' and an isometry $V : \mathcal{H} \mapsto \mathcal{H}' \otimes \mathcal{H}''$ such that

$$\Phi(\rho) = \text{Tr}_{\mathcal{H}''} V \rho V^*, \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

The channel $\hat{\Phi} : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}'')$ defined by

$$\hat{\Phi}(\rho) = \text{Tr}_{\mathcal{H}'} V \rho V^*, \quad \forall \rho \in \mathfrak{S}(\mathcal{H}),$$

is called complementary to the channel Φ . The complementary channel is defined uniquely up to unitary equivalence.

The basic properties of the complementary relation are the following:

- $H_\Phi(\rho) = H_{\widehat{\Phi}}(\rho)$ for arbitrary channel Φ and any pure state ρ ;
- $\widehat{\Phi \otimes \Psi} = \widehat{\Phi} \otimes \widehat{\Psi}$ for arbitrary channels Φ and Ψ .

The first property and the definition of the CCoOE imply $\widehat{H}_\Phi \equiv \widehat{H}_{\widehat{\Phi}}$ despite the fact that H_Φ and $H_{\widehat{\Phi}}$ may be substantially different functions. This and the second property show that the superadditivity of the CCoOE for channels Φ and Ψ is equivalent to the same property for the complementary channels $\widehat{\Phi}$ and $\widehat{\Psi}$ (theorem 1 in [9]).

In the finite dimensional case the superadditivity of the CCoOE for particular channels Φ and Ψ is equivalent to additivity of the Holevo capacity for these channels with arbitrary constraints. Hence in this case the above observation can be formulated in terms of this property: additivity of the Holevo capacity for channels Φ and Ψ with arbitrary constraints is equivalent to the same property for the complementary channels $\widehat{\Phi}$ and $\widehat{\Psi}$. Note that in contrast to the CCoOE the Holevo capacities for channels related by the complementary relation may be substantially different, but the common CCoOE plays, loosely speaking, the role of bridge between the additivity properties of the Holevo capacity for pairs of complementary channels.

In the infinite dimensional case there exist only conditional relations between the above additivity properties (theorem 2 in [18]) and hence we can not prove the infinite dimensional version of the above observation, concerning the additivity of the Holevo capacity for pairs of complementary channels. Nevertheless the following conditional version can be proved.

Proposition 4. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi : \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ be such channels that*

$$H_\Phi(\rho) < +\infty, \forall \rho \in \text{ext}\mathfrak{S}(\mathcal{H}) \quad \text{and} \quad H_\Psi(\sigma) < +\infty, \forall \sigma \in \text{ext}\mathfrak{S}(\mathcal{K}).$$

Then the additivity of the Holevo capacity for the channels Φ and Ψ with arbitrary constraints is equivalent to the same property for the complementary channels $\widehat{\Phi}$ and $\widehat{\Psi}$.

Proof. By the first property of the complementary relation the condition of the proposition is symmetrical with respect to the pairs (Φ, Ψ) and $(\widehat{\Phi}, \widehat{\Psi})$. Hence it is sufficient to show that the additivity of the Holevo capacity for channels Φ and Ψ with arbitrary constraints implies the same property for the complementary channels $\widehat{\Phi}$ and $\widehat{\Psi}$.

Let $\mathfrak{S}_f(\mathcal{H} \otimes \mathcal{K})$ be the subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ consisting of all states ω such that $\text{rank } \omega^{\mathcal{H}} < +\infty$ and $\text{rank } \omega^{\mathcal{K}} < +\infty$. Note that the condition of

the proposition implies $H_\Phi(\omega^{\mathcal{H}}) < +\infty$ and $H_\Psi(\omega^{\mathcal{K}}) < +\infty$ for arbitrary state $\omega \in \mathfrak{S}_f(\mathcal{H} \otimes \mathcal{K})$. By the part (i) \Rightarrow (iii) of theorem 2 in [18] the additivity of the Holevo capacity for the channels Φ and Ψ with arbitrary constraints implies validity of inequality (6) for these channels and arbitrary state $\omega \in \mathfrak{S}_f(\mathcal{H} \otimes \mathcal{K})$. By the properties of the complementary relation this means validity of inequality (6) for the channels $\widehat{\Phi}$ and $\widehat{\Psi}$ and arbitrary state $\omega \in \mathfrak{S}_f(\mathcal{H} \otimes \mathcal{K})$. By the part (iii) \Rightarrow (ii) of theorem 2 in [18] this implies validity of inequality (7) for the channels $\widehat{\Phi}$ and $\widehat{\Psi}$ and arbitrary state $\omega \in \mathfrak{S}_f(\mathcal{H} \otimes \mathcal{K})$. By proposition 6 in [18] the last property implies validity of inequality (7) for the channels $\widehat{\Phi}$ and $\widehat{\Psi}$ and arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, which means the additivity of the Holevo capacity for the channels $\widehat{\Phi}$ and $\widehat{\Psi}$ with arbitrary constraints. \square

Propositions 3 and 4 imply the following result.

Corollary 6. *Let Φ be a channel such that its complementary channel $\widehat{\Phi}$ belongs to one of the classes considered in proposition 3 and Ψ be an arbitrary channel. Then the superadditivity of the CCoOE holds for the channels Φ and Ψ .*

If $H_\Phi(\rho) < +\infty, \forall \rho \in \text{ext}\mathfrak{S}(\mathcal{H})$ and $H_\Psi(\sigma) < +\infty, \forall \sigma \in \text{ext}\mathfrak{S}(\mathcal{K})$ then the additivity of the Holevo capacity holds for the channels Φ and Ψ with arbitrary constraints.

By the first part of corollary 6 the superadditivity of the CCoOE holds for arbitrary channel Ψ and the channel Φ_p^a considered in the example in section 3 with arbitrary probability density function $p(t)$ and $a \leq +\infty$. Indeed, by using lemma 2 in Appendix 5.2 it is easy to see that

$$\widehat{\Phi}_p^{+\infty}(\rho) = \int_{-\infty}^{+\infty} V_t |\varphi_p\rangle \langle \varphi_p| (V_t)^* \text{Tr}(\rho M(dt)), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}_{+\infty}),$$

where $\varphi_p(x) = \sqrt{p(x)}$, V_t - the shift operator and $M(\cdot)$ - the projector valued measure of the operator of multiplication on x in $\mathcal{H}_{+\infty} = L_2^x(\mathbb{R})$. By theorem 2 in [12] the channel $\widehat{\Phi}_p^{+\infty}$ is entanglement-breaking. Since the channel Φ_p^a with $a < +\infty$ can be considered as a subchannel of the channel $\Phi_p^{+\infty}$ corresponding to the subspace $L_2^x([-a, a]) \subset L_2^x(\mathbb{R})$ the channel $\widehat{\Phi}_p^a$ is entanglement-breaking as well.

By the second part of corollary 6 if $a < +\infty$ and the function $p(t)$ satisfies the conditions assumed in proving continuity and boundedness of the function $\widehat{H}_{\Phi_p^a}$ in the example in section 3 then the additivity of the Holevo capacity holds for the channels Φ_p^a and Ψ with arbitrary constraints, where Ψ is an

arbitrary channel such that $H_{\Psi}(\sigma) < +\infty$ for all σ in $\text{ext}\mathfrak{S}(\mathcal{K})$, in particular, $\Psi = \Phi_p^a$.

5 Appendices

5.1 Continuity condition for entropy

For many application it is necessary to have sufficient condition of continuity of the entropy on some subset of states. There exist several results of this type, in particular, Simon's convergence theorems for the entropy [24]. The important and widely used continuity condition is the following [25]: *if H is a positive unbounded operator in separable Hilbert space \mathcal{H} such that $\text{Tr} \exp(-\lambda H) < +\infty$ for all $\lambda > 0$ then the entropy is continuous on the set $\mathcal{K}_{H,h} = \{\rho \in \mathfrak{S}(\mathcal{H}) \mid \text{Tr} H \rho \leq h\}$ for any $h > 0$.*

The feature of this (and some other conditions) consists in compactness and convexity of the set, on which continuity of the entropy is established⁸. Thus by using such conditions we can not prove boundedness and continuity of the entropy on noncompact or nonconvex sets such that the entropy is not continuous on their convex closure (note that by corollary 5 in [20] boundedness of the entropy on a convex set implies relative compactness of this set). A necessity to deal with such sets arises when we want to prove boundedness and continuity of the output entropy of a quantum channel on the set of all pure states if the output entropy of this channel is not bounded on the whole state space. This problem can be solved by using the following simple observation.

Let $\text{aut}\mathfrak{S}(\mathcal{H})$ be the set of all automorphism of the state space $\mathfrak{S}(\mathcal{H})$.

Proposition 5. *Let \mathcal{K} be a compact subset of $\mathfrak{S}(\mathcal{H})$, on which the entropy is continuous. Then the entropy is bounded and continuous on the set*

$$\widehat{\mathcal{K}} = \bigcup_{\alpha \in \text{aut}\mathfrak{S}(\mathcal{H})} \alpha(\mathcal{K}).$$

It is clear that the entropy is continuous on the "translation" $\alpha(\mathcal{K})$ of the set \mathcal{K} by given $\alpha \in \text{aut}\mathfrak{S}(\mathcal{H})$. Compactness of the set \mathcal{K} implies, loosely speaking, that the entropy continuously changes under transition between different translations.

⁸Compactness of the set $\mathcal{K}_{H,h}$ is proved in [8].

Proof. Boundedness of the entropy is obvious. Suppose the entropy is not continuous on the set $\widehat{\mathcal{K}}$. Then there exists a sequence $\{\rho_n\}$ of states in $\widehat{\mathcal{K}}$ converging to a state ρ_0 in $\widehat{\mathcal{K}}$ such that $\lim_{n \rightarrow +\infty} H(\rho_n) > H(\rho_0)$. By definition of the set $\widehat{\mathcal{K}}$ there exists a sequence $\{\alpha_n\} \subset \text{aut}\mathfrak{S}(\mathcal{H})$ such that $\sigma_n = \alpha_n(\rho_n) \in \mathcal{K}$ for all n . Since the set \mathcal{K} is compact there exists a subsequence $\{\sigma_{n_k}\}$ of the sequence $\{\sigma_n\}$ converging to some state σ_0 . Since the states ρ_{n_k} and $\sigma_{n_k} = \alpha_{n_k}(\rho_{n_k})$ are isomorphic for all k they have the same entropy. Moreover it follows that the limit states σ_0 and ρ_0 of the sequences $\{\rho_{n_k}\}$ and $\{\sigma_{n_k}\}$ are isomorphic and hence have the same entropy. By the assumption of continuity of the entropy on the set \mathcal{K} we have

$$\lim_{k \rightarrow +\infty} H(\rho_{n_k}) = \lim_{k \rightarrow +\infty} H(\sigma_{n_k}) = H(\sigma_0) = H(\rho_0)$$

contradicting to the definition of the sequence $\{\rho_n\}$. \square

For applications it is convenient to reformulate proposition 5 as follows.

Corollary 7. *The entropy is continuous and bounded on a subset \mathcal{A} of $\mathfrak{S}(\mathcal{H})$ if there exist separable Hilbert space \mathcal{K} and set $\{\alpha_\rho\}_{\rho \in \mathcal{A}}$ of isomorphisms from $\mathfrak{S}(\mathcal{H})$ onto $\mathfrak{S}(\mathcal{K})$ such that $\alpha_\rho(\rho)$ lies for each ρ in some compact subset \mathcal{B} of $\mathfrak{S}(\mathcal{K})$, on which the entropy is continuous.*

Proposition 5 and the continuity condition for the entropy stated before imply the following observation. Let $\mathfrak{U}(\mathcal{H})$ be the group of all unitaries in \mathcal{H} .

Corollary 8. *Let H be a positive unbounded operator in \mathcal{H} such that $\text{Tr} \exp(-\lambda H) < +\infty$ for all $\lambda > 0$. Then the entropy is continuous on the set*

$$\widehat{\mathcal{K}}_{H,h} = \{\rho \in \mathfrak{S}(\mathcal{H}) \mid \inf_{U \in \mathfrak{U}(\mathcal{H})} \text{Tr}(HU\rho U^*) \leq h\}, \quad \forall h > 0.$$

By using corollary 8 it is easy to obtain all Simon's convergence theorems for entropy [24].

5.2 The proof of the properties of the channel Φ_p^a

In this subsection the properties of the channel Φ_p^a described in the example in section 3 are proved. We will show that the sufficient condition in proposition 2 of boundedness and continuity of the function $\rho \mapsto H(\Phi_p^a(\rho))$ on the set $\text{ext}\mathfrak{S}(\mathcal{H})$ holds for the channel Φ_p^a under the assumed conditions.

Let $\{U_t\}_{t \in \mathbb{R}}$ and $\{V_t\}_{t \in \mathbb{R}}$ are the unitary groups in the Hilbert space $\mathcal{L}_2(\mathbb{R})$ defined by the expressions

$$(U_t \varphi)(x) = \exp(-itx) \varphi(x) \quad \text{and} \quad (V_t \varphi)(x) = \varphi(x - t), \quad \forall \varphi \in \mathcal{L}_2(\mathbb{R}).$$

Lemma 2. *The states*

$$\int_{-\infty}^{+\infty} U_t|\psi\rangle\langle\psi|U_t^*|\widehat{\varphi}(t)|^2 dt \quad \text{and} \quad \int_{-\infty}^{+\infty} V_t|\varphi\rangle\langle\varphi|V_t^*|\psi(t)|^2 dt$$

are isomorphic for arbitrary φ and ψ in $\mathcal{L}_2(\mathbb{R})$ with $\|\varphi\| = 1$ and $\|\psi\| = 1$, where $\widehat{\varphi}$ is the Fourier transformation of φ .

Proof. By direct calculation it is possible to show that the above states coincide with the partial traces $\text{Tr}_{\mathcal{L}_2^x}|\theta\rangle\langle\theta|$ and $\text{Tr}_{\mathcal{L}_2^y}|\theta\rangle\langle\theta|$ of the pure state $|\theta\rangle\langle\theta|$ in $\mathfrak{S}(\mathcal{L}_2^x \otimes \mathcal{L}_2^y)$, where $\theta(x, y) = \varphi(x - y)\psi(y)$. \square

Suppose first that the function $p(t)$ has finite support. Then there exist $\lambda > 0$ and $b > 0$ such that $\lambda p(t) \leq \frac{\sin^2 bt}{\pi bt^2}$ for all $t \in \mathbb{R}$ and hence $\lambda\rho_\psi \leq \sigma_\psi$, where

$$\rho_\psi = \int_{-\infty}^{+\infty} U_t|\psi\rangle\langle\psi|U_t^*p(t)dt \quad \text{and} \quad \sigma_\psi = \int_{-\infty}^{+\infty} U_t|\psi\rangle\langle\psi|U_t^*\frac{\sin^2 bt}{\pi bt^2}dt.$$

for arbitrary ψ in $\mathcal{L}_2([-a, a]) = \mathcal{H}_a$ with $\|\psi\| = 1$.

By lemma 2 for each $\psi \in \mathcal{H}_a$ with $\|\psi\| = 1$ there exists unitary transformation $W_\psi : \mathcal{H}_a \mapsto \mathcal{K} = \mathcal{L}_2(\mathbb{R})$ such that

$$W_\psi\sigma_\psi W_\psi^* = \tilde{\sigma}_\psi = \int_{-\infty}^{+\infty} V_t|\varphi_b\rangle\langle\varphi_b|V_t^*|\psi(t)|^2 dt,$$

where $\varphi_b(x) = \begin{cases} 1, & x \in [-b, b] \\ 0, & x \in \mathbb{R} \setminus [-b, b]. \end{cases}$

Let $c = a + b$ and \tilde{V}_t^c be the operator of cyclic shift on t in $\mathcal{L}_2([-c, c])$ considered as a subspace of $\mathcal{L}_2(\mathbb{R})$ consisting of functions supported by $[-c, c]$.

Note that $\tilde{V}_t^c|\varphi_b\rangle = V_t|\varphi_b\rangle$ for all t in $[-a, a]$. Since $\text{supp}\psi \subseteq [-a, a]$ we have

$$\tilde{\sigma}_\psi = \int_{-a}^a V_t|\varphi_b\rangle\langle\varphi_b|V_t^*|\psi(t)|^2 dt = \int_{-a}^a \tilde{V}_t^c|\varphi_b\rangle\langle\varphi_b|(\tilde{V}_t^c)^*|\psi(t)|^2 dt. \quad (10)$$

It is clear that $\lambda\tilde{\rho}_\psi \leq \tilde{\sigma}_\psi$, where $\tilde{\rho}_\psi = W_\psi\rho_\psi W_\psi^*$.

Let $H = \sum_k \log^2(|k|+1)|k\rangle\langle k|$ be a positive operator in $\mathcal{L}_2([-c, c])$, where $\{|k\rangle \sim (2c)^{1/2} \exp(i\pi c^{-1}kx)\}$ - trigonometric basis in $\mathcal{L}_2([-c, c])$. Note that $\text{Tr} \exp(-\lambda H) < +\infty$ for all $\lambda > 0$. Since $\tilde{V}_t^c = \sum_{k=-\infty}^{k=+\infty} \exp(-i\pi c^{-1}kt)|k\rangle\langle k|$

we obtain from (10) that

$$\begin{aligned}\mathrm{Tr}\tilde{\sigma}_\psi H &= \sum_{k=-\infty}^{k=+\infty} \log^2(|k|+1) \langle k|\tilde{\sigma}_\psi|k\rangle = \sum_{k=-\infty}^{k=+\infty} \log^2(|k|+1) |\langle \varphi_b|k\rangle|^2 \\ &= \sum_{k=-\infty}^{k=+\infty} \log^2(|k|+1) \left| \frac{c \sin^2(\pi c^{-1}kb)}{b\pi^2 k^2} \right| = h < +\infty, \quad \forall \psi,\end{aligned}$$

and hence $\mathrm{Tr}\tilde{\rho}_\psi H \leq \lambda^{-1} \mathrm{Tr}\tilde{\sigma}_\psi H = \lambda^{-1} h < +\infty$, $\forall \psi$.

Thus the sufficient condition in proposition 2 is fulfilled with $\mathcal{K} = \mathcal{L}_2(\mathbb{R})$, $\alpha_\psi = W_\psi(\cdot)W_\psi^*$ and $\mathcal{A} = \{\rho \in \mathfrak{S}(\mathcal{L}_2([-c, c])) \subset \mathfrak{S}(\mathcal{K}) \mid \mathrm{Tr}\rho H \leq \lambda^{-1}h\}$ (the set \mathcal{A} has the required properties by the observation in [11]). By this condition the function $\psi \mapsto H_{\mathbb{F}_p^a}(|\psi\rangle\langle\psi|) = H(\rho_\psi)$ is bounded and continuous.

Note that this implies in particular that

$$H(\omega(\psi)) \leq C < +\infty \text{ for all } \psi, \text{ where } \omega(\psi) = \int_0^1 U_t|\psi\rangle\langle\psi|U_t^* dt. \quad (11)$$

Let $p(t)$ be a bounded density function such that $|\int_{-\infty}^{+\infty} p(t) \log p(t) dt| < +\infty$ and for sufficiently large d the function $p(t)$ is monotonous on $(-\infty, -d)$ and on $(d, +\infty)$. In what follows we will assume that d is so large that the both functions $p(t)$ and $p(t) \log p(t)$ are monotonous on $(-\infty, -d+1)$ and on $(d-1, +\infty)$.

Let $\rho_\psi^d = \left(\int_{-d}^d p(t) dt \right)^{-1} \int_{-d}^d U_t|\psi\rangle\langle\psi|U_t^* p(t) dt$. Below we will prove that for all sufficiently large d the following inequality holds

$$\begin{aligned}|H(\rho_\psi) - H(\rho_\psi^d)| &\leq \alpha(d)H(\rho_\psi^d) + h_2(\alpha(d)) + \alpha(d-1)C + \beta(d-1) \\ &\leq \frac{\alpha(d)}{1-\alpha(d)}H(\rho_\psi) + h_2(\alpha(d)) + \alpha(d-1)C + \beta(d-1),\end{aligned} \quad (12)$$

where C is defined in (11), $\alpha(d) = \int_{|t|>d} p(t) dt$ and $\beta(d) = |\int_{|t|>d} p(t) \log p(t) dt|$. The first inequality in (12) with fixed d and the above observation (concerning the function $p(t)$ with finite support) imply uniform boundedness of the function $\psi \mapsto H_{\mathbb{F}_p^a}(|\psi\rangle\langle\psi|) = H(\rho_\psi)$. The second inequality in (12) with d increasing to $+\infty$ and the above observation imply continuity of the function $\psi \mapsto H_{\mathbb{F}_p^a}(|\psi\rangle\langle\psi|) = H(\rho_\psi)$.

To prove (12) consider convex decomposition $\rho_\psi = (1 - \alpha(d))\rho_\psi^d + \alpha(d)\sigma_\psi^d$, where $\sigma_\psi^d = \alpha^{-1}(d) \int_{|t|>d} U_t |\psi\rangle\langle\psi| U_t^* p(t) dt$. By the property of the entropy the above decomposition implies

$$\begin{aligned} & (1 - \alpha(d))H(\rho_\psi^d) + \alpha(d)H(\sigma_\psi^d) \\ & \leq H(\rho_\psi) \leq \end{aligned} \quad (13)$$

$$(1 - \alpha(d))H(\rho_\psi^d) + \alpha(d)H(\sigma_\psi^d) + h_2(\alpha(d))$$

and hence

$$|H(\rho_\psi) - H(\rho_\psi^d)| \leq \alpha(d)H(\rho_\psi^d) + \alpha(d)H(\sigma_\psi^d) + h_2(\alpha(d)). \quad (14)$$

By using monotonicity of the function $p(t)$ on $(-\infty, -d + 1)$ and on $(d - 1, +\infty)$ we obtain the following operator inequality

$$\begin{aligned} \alpha(d)\sigma_\psi^d &= \int_{-\infty}^{-d} U_t |\psi\rangle\langle\psi| U_t^* p(t) dt + \int_d^{+\infty} U_t |\psi\rangle\langle\psi| U_t^* p(t) dt \\ &\leq \sum_{k=-\infty}^{-1} p(-d + k + 1) \int_{-d+k}^{-d+k+1} U_t |\psi\rangle\langle\psi| U_t^* dt \\ &+ \sum_{k=0}^{+\infty} p(d + k) \int_{d+k}^{d+k+1} U_t |\psi\rangle\langle\psi| U_t^* dt = \gamma(d) \sum_{k=-\infty}^{k=+\infty} \lambda_k \omega_k(\psi), \end{aligned} \quad (15)$$

where $\gamma(d) = \sum_{k=-\infty}^{-1} p(-d + k + 1) + \sum_{k=0}^{+\infty} p(d + k)$,

$$\lambda_k = \begin{cases} \gamma^{-1}(d)p(-d + k + 1) & k < 0 \\ \gamma^{-1}(d)p(d + k), & k \geq 0, \end{cases} \quad \omega_k(\psi) = \int_{l_k}^{l_k+1} U_t |\psi\rangle\langle\psi| U_t^* dt,$$

$l_k = -d + k$ if $k < 0$ and $l_k = d + k$ if $k \geq 0$.

Since for each k the state $\omega_k(\psi)$ is isomorphic to the state $\omega(\psi)$ defined in (11) we have $H(\omega_k(\psi)) \leq C$. Inequality (15) and the property of the entropy imply

$$\alpha(d)H(\sigma_\psi^d) \leq \gamma(d)C - \gamma(d) \sum_{k=-\infty}^{k=+\infty} \lambda_k \log \lambda_k. \quad (16)$$

By using monotonicity of the function $p(t) \log p(t)$ on $(-\infty, -d + 1)$ and on $(d - 1, +\infty)$ we obtain

$$\begin{aligned}
& -\gamma(d) \sum_{k=-\infty}^{k=+\infty} \lambda_k \log \lambda_k = \gamma(d) \log \gamma(d) \\
& - \sum_{k=-\infty}^{-1} p(-d + k + 1) \log p(-d + k + 1) - \sum_{k=0}^{+\infty} p(d + k) \log p(d + k) \quad (17) \\
& \leq \gamma(d) \log \gamma(d) - \int_{|t|>d-1} p(t) \log p(t) dt \leq \gamma(d) \log \gamma(d) + \beta(d - 1).
\end{aligned}$$

By noting that $\gamma(d) \leq \alpha(d - 1) < 1$ for all sufficiently large d (due to monotonicity of the function $p(t)$ on $(-\infty, -d + 1)$ and on $(d - 1, +\infty)$ and since $\alpha(d) \rightarrow 0$ as $d \rightarrow +\infty$) we obtain from (14), (16) and (17) the first inequality in (12). The second inequality in (12) follows from the inequality $(1 - \alpha(d))H(\rho_\psi^d) \leq H(\rho_\psi)$ easily deduced from (13).

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