On approximation of quantum channels

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1 Introduction

Although a major attention in quantum information theory so far was paid to finite-dimensional systems and channels, there is an increasing interest in infinite-dimensional generalizations (see [4], [8], [9], [15]-[18] and references therein). In the present paper we develop an approximation approach to infinite dimensional quantum channels based on detailed investigation of the continuity properties of entropic characteristics of quantum channels, related to the classical capacity, as functions of a pair "channel, input state". It appears that often it is convenient to approximate a channel by trace-nonincreasing completely positive (CP) maps – *operations*, rather than by channels. Thus it is necessary to generalize the definitions of the channel characteristics to operations and to consider continuity properties of these characteristics on the extended domain.

The essential feature of infinite dimensional channels is discontinuity and unboundedness of the main entropic characteristics which prevents from straightforward generalization of the results obtained in finite dimensions. A natural way to study quantum channels with singular characteristics is to approximate them in appropriate topology by channels (or, more generally, by operations) with continuous characteristics, for example, by channels with finite dimensional output space. This approach was used (implicitly) in [15] to derive the strong additivity of the Holevo capacity (χ-*capacity* in what follows) for some classes of infinite dimensional channels from the corresponding

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finite dimensional results and to prove that validity of the additivity conjecture in finite dimensions implies strong additivity of the χ-capacity for all infinite-dimensional channels.

The content of this paper is as follows. Section 2 presents basic notions and some results of previous works used in this paper. In section 3 we consider the topology of strong convergence on the set of all quantum operations, which appears to be a proper topology for the purposes of approximation. It is shown that it is this topology in which the set of all quantum operations is isomorphic to a particular subset of states of composite system (the generalized Choi-Jamiolkowski isomorphism). This isomorphism implies simple compactness criterion for subsets of quantum operations. In section 4 the continuity properties of the convex closure of the output entropy and of the χ-function (the constrained χ-capacity) as functions of pair (quantum operation, input state) are explored. Several continuity conditions are obtained. In section 5 the obtained results are applied to the following problems:

1) continuity of the χ-capacity as function of a channel;

2) strong additivity of the χ-capacity for infinite dimensional channels;

3) the representation for the convex closure of the output entropy of arbitrary quantum channel.

Thus approximation of infinite dimensional quantum channels by operations in the topology of strong convergence appears as a useful tool in study the characteristics related to the classical capacity. In subsequent work we plan to apply it to other characteristics of quantum channels, such as entanglement-assisted capacity and quantum capacity.

2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ – the set of all bounded operators on $\mathcal{H}$, $\mathfrak{T}(\mathcal{H})$ – the Banach space of all trace-class operators with the trace norm $\|\cdot\|_1$. Let

$$\mathfrak{T}_1(\mathcal{H}) = \{A \in \mathfrak{T}(\mathcal{H}) \,|\, A \geq 0, \mathrm{Tr}A \leq 1\} \text{ and } \mathfrak{S}(\mathcal{H}) = \{A \in \mathfrak{T}_1(\mathcal{H}) \,|\, \mathrm{Tr}A = 1\}$$

be the closed convex subsets of $\mathfrak{T}(\mathcal{H})$, which are complete separable metric spaces with the metric defined by the trace norm. Operators in $\mathfrak{S}(\mathcal{H})$ are called density operators. Each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$ [2], so, in what follows we will also for brevity use the term "state".

We denote by $\mathrm{co}\mathcal{A}$ ($\overline{\mathrm{co}}\mathcal{A}$) the convex hull (closure) of a set $\mathcal{A}$ and by $\mathrm{co}f$ ($\overline{\mathrm{co}}f$) the convex hull (closure) of a function f [12]. We denote by $\mathrm{extr}\mathcal{A}$ the set of all extreme points of a convex set $\mathcal{A}$.

Let $\mathcal{P}(\mathcal{A})$ be the set of all Borel probability measures on complete separable metric space $\mathcal{A}$ endowed with the topology of weak convergence [13]. This set can be considered as a complete separable metric space as well [13]. The subset of $\mathcal{P}(\mathcal{A})$ consisting of measures with finite support will be denoted by $\mathcal{P}^{\mathrm{f}}(\mathcal{A})$. In what follows we will also use the abbreviations $\mathcal{P} = \mathcal{P}(\mathfrak{S}(\mathcal{H}))$ and $\widehat{\mathcal{P}} = \mathcal{P}(\mathrm{extr}\mathfrak{S}(\mathcal{H}))$.

The *barycenter* of the measure $\mu \in \mathcal{P}$ is the state defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \sigma\mu(d\sigma).$$

For arbitrary subset $\mathcal{A} \subset \mathfrak{S}(\mathcal{H})$ let $\mathcal{P}_{\mathcal{A}}$ (corresp. $\widehat{\mathcal{P}}_{\mathcal{A}}$) be the subset of $\mathcal{P}$ (corresp. $\widehat{\mathcal{P}}$) consisting of all measures with the barycenter in $\mathcal{A}$.

A collection of states $\{\rho_i\}$ with corresponding probability distribution $\{\pi_i\}$ is conventionally called *ensemble* and is denoted by $\{\pi_i, \rho_i\}$. In this paper we will consider ensemble of states as a partial case of probability measure, so that notation $\{\pi_i, \rho_i\} \in \mathcal{P}_{\{\rho\}}$ means that $\rho = \sum_i \pi_i \rho_i$.

We will use the following two extensions of the von Neumann entropy $S(\rho) = -\mathrm{Tr}\rho\log\rho$ of a state ρ to the set $\mathfrak{T}_1(\mathcal{H})$ (cf.[11])

$$S(A) = -\mathrm{Tr}A\log A \quad \text{and} \quad H(A) = S(A) - \eta(\mathrm{Tr}A), \quad \forall A \in \mathfrak{T}_1(\mathcal{H}),$$

where $\eta(x) = -x\log x$.

Nonnegativity, concavity and lower semicontinuity of the von Neumann entropy S on the set $\mathfrak{S}(\mathcal{H})$ imply the same properties of the functions S and H on the set $\mathfrak{T}_1(\mathcal{H})$. We will use the following properties

$$H(\lambda A) = \lambda H(A), \quad A \in \mathfrak{T}_1(\mathcal{H}),\ \lambda \geq 0, \tag{1}$$

$$H(A) + H(B-A) \leq H(B) \leq H(A) + H(B-A) + \mathrm{Tr}Bh_2\left(\frac{\mathrm{Tr}A}{\mathrm{Tr}B}\right), \tag{2}$$

where $A, B \in \mathfrak{T}_1(\mathcal{H}),\ A \leq B$, and $h_2(x) = \eta(x) + \eta(1-x)$.

Subadditivity property of the quantum entropy implies the following inequality

$$S(C) \leq S(\mathrm{Tr}_{\mathcal{H}}C) + S(\mathrm{Tr}_{\mathcal{K}}C) - \eta(\mathrm{Tr}C), \quad \forall C \in \mathfrak{T}_1(\mathcal{H} \otimes \mathcal{K}). \tag{3}$$

The relative entropy for two operators A and B in $\mathfrak{T}_1(\mathcal{H})$ is defined by (cf.[11])

$$H(A \,\|B) = \sum_i \langle i| \left(A \log A - A \log B + B - A\right) |i\rangle$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of A.

Let $\mathcal{H}, \mathcal{H}'$ be a pair of separable Hilbert spaces which we call correspondingly input and output space. A quantum operation Φ is a linear positive trace-nonicreasing map from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ such that the dual map $\Phi^* : \mathfrak{B}(\mathcal{H}') \mapsto \mathfrak{B}(\mathcal{H})$ is completely positive. The convex set of all quantum operations from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ will be denoted by $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$. If Φ is trace preserving then it is called quantum channel. The convex set of all channels from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ will be denoted by $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$.

Since the functions $\rho \mapsto H_\Phi(\rho) = H(\Phi(\rho))$, $\rho \mapsto S_\Phi(\rho) = S(\Phi(\rho))$ and $\rho \mapsto H(\Phi(\rho)\|A)$, where Φ is a given quantum operation in $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ and A is a given operator in $\mathfrak{T}_1(\mathcal{H})$, are nonnegative and lower semicontinuous on the set $\mathfrak{S}(\mathcal{H})$, the functionals

$$\hat{H}_\Phi(\mu) = \int_{\mathfrak{S}(\mathcal{H})} H_\Phi(\rho)\mu(d\rho), \quad \hat{S}_\Phi(\mu) = \int_{\mathfrak{S}(\mathcal{H})} S_\Phi(\rho)\mu(d\rho)$$

and

$$\chi_\Phi(\mu) = \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\mu)))\mu(d\rho)$$

are well defined on the set $\mathcal{P}$.

Proposition 1. *The functionals $\hat{H}_\Phi(\mu)$, $\hat{S}_\Phi(\mu)$ and $\chi_\Phi(\mu)$ are lower semicontinuous on the set $\mathcal{P}$. If $S_\Phi(\bar{\rho}(\mu)) < +\infty$ then*

$$\chi_\Phi(\mu) = S_\Phi(\bar{\rho}(\mu)) - \hat{S}_\Phi(\mu). \tag{4}$$

This proposition can be proved by obvious modification of the arguments used in the proof of proposition 1 in [8].

Corollary 1. *Let $\mathcal{P}_0$ be such subset of $\mathcal{P}$ that the function S_Φ is continuous on the set $\{\bar{\rho}(\mu)\}_{\mu \in \mathcal{P}_0}$. Then the functionals $\hat{H}_\Phi(\mu)$, $\hat{S}_\Phi(\mu)$ and $\chi_\Phi(\mu)$ are continuous on the set $\mathcal{P}_0$.*

Corollary 1 implies in particular continuity of the functionals $\hat{H}_{\Phi}(\mu)$, $\hat{S}_{\Phi}(\mu)$ and $\chi_{\Phi}(\mu)$ on the set $\mathcal{P}_{\{\rho\}}$ if $S_{\Phi}(\rho)<+\infty$.

The important characteristics of the quantum channel Φ is the convex closure $\overline{\mathrm{co}}H_{\Phi}$ of the output entropy $H_{\Phi}(=S_{\Phi})$ [18]. In this paper we consider the convex closures $\overline{\mathrm{co}}H_{\Phi}$ and $\overline{\mathrm{co}}S_{\Phi}$ of the functions H_{Φ} and S_{Φ} correspondingly for arbitrary quantum operation Φ in $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$.

Proposition 2. *Let Φ be an arbitrary quantum operation in $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ and ρ be an arbitrary state in $\mathfrak{S}(\mathcal{H})$.*

A) *The following expressions hold*

$$\overline{\mathrm{co}}H_{\Phi}(\rho)=\inf_{\mu\in\mathcal{P}_{\{\rho\}}}\hat{H}_{\Phi}(\mu)=\inf_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}}\hat{H}_{\Phi}(\mu) \tag{5}$$

and

$$\overline{\mathrm{co}}S_{\Phi}(\rho)=\inf_{\mu\in\mathcal{P}_{\{\rho\}}}\hat{S}_{\Phi}(\mu)=\inf_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}}\hat{S}_{\Phi}(\mu) \tag{6}$$

The infima in these expressions are achieved at some measures in $\widehat{\mathcal{P}}_{\{\rho\}}$.

B) *The following inequalities hold*

$$\overline{\mathrm{co}}H_{\Phi}(\rho)\leq\overline{\mathrm{co}}S_{\Phi}(\rho)\leq\overline{\mathrm{co}}H_{\Phi}(\rho)+\eta(\mathrm{Tr}\Phi(\rho)).$$

C) *If $\overline{\mathrm{co}}S_{\Phi}(\rho)<+\infty$ then*

$$\{S_{\Phi}(\rho)<+\infty\}\Leftrightarrow\{\overline{\mathrm{co}}S_{\Phi}(\rho)=\mathrm{co}S_{\Phi}(\rho)\},$$

where $\mathrm{co}S_{\Phi}$ is the convex hull of the function S_{Φ} defined by the expression

$$\mathrm{co}S_{\Phi}(\rho)=\inf_{\{\pi_i,\rho_i\}\in\mathcal{P}^{\mathrm{f}}_{\{\rho\}}}\sum_i\pi_iS_{\Phi}(\rho_i).$$

Proof. All assertions in A follow from theorem 1 in [17].

The inequalities in B are easily deduced from the representations in A and concavity of the function η.

The implication $\Rightarrow$ in C follows from lemma 1 in [8] and corollary 1. Since the set of all states ρ with finite $S_{\Phi}(\rho)$ is convex, $S_{\Phi}(\rho)=+\infty$ implies $\mathrm{co}S_{\Phi}(\rho)=+\infty$. This observation proves the implication $\Leftarrow$ in C. $\square$

The χ-function of the channel Φ is defined by the expression (cf.[7],[8])

$$\chi_{\Phi}(\rho)=\sup_{\{\pi_i,\rho_i\}\in\mathcal{P}^{\mathrm{f}}_{\{\rho\}}}\chi_{\Phi}(\{\pi_i,\rho_i\})=\sup_{\mu\in\mathcal{P}_{\{\rho\}}}\chi_{\Phi}(\mu), \tag{7}$$

where the last equality follows from lower semicontinuity of the functional χ_Φ and lemma 1 in [8].

In this paper we will consider the χ-function of arbitrary quantum operation Φ in $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$. By using propositions 1 and 2 it is easy to deduce from (7) that

$$\chi_\Phi(\rho) = S_\Phi(\rho) - \mathrm{co}S_\Phi(\rho) = S_\Phi(\rho) - \overline{\mathrm{co}}S_\Phi(\rho) \tag{8}$$

for arbitrary state $\rho \in \mathfrak{S}(\mathcal{H})$ such that $S_\Phi(\rho) < +\infty$.

3 The topology of strong convergence

The set $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ of all quantum operations from $\mathfrak{T}(\mathcal{H})$ into $\mathfrak{T}(\mathcal{H}')$ can be endowed with different topologies, in particular, with the topology of *uniform convergence*, defined by the metric

$$d(\Phi,\Psi) = \sup_{\rho\in\mathfrak{S}(\mathcal{H})} \|\Phi(\rho) - \Psi(\rho)\|_1,$$

or with the topology defined by the norm of complete boundedness [14].

But for realization of the idea of approximation of an arbitrary quantum channel by a sequence of quantum operations with "smooth characteristics" described in the Introduction it is convenient to use the weaker topology of *strong convergence* on the set $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$, generated by the strong operator topology on the set of all linear bounded operators from the Banach space $\mathfrak{T}(\mathcal{H})$ into the Banach space $\mathfrak{T}(\mathcal{H}')$. Strong convergence of the sequence $\{\Phi_n\} \subset \mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ to the quantum operation $\Phi_0 \in \mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ means that

$$\lim_{n\to+\infty} \Phi_n(\rho) = \Phi_0(\rho), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

In what follows we will consider the set $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ as a topological space with the topology of strong convergence. Separability of the set $\mathfrak{S}(\mathcal{H})$ implies that the topology of strong convergence on the set $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ is metrisable (can be defined by some metric).

Remark 1. Since the operator norm of any quantum operation in $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ is ≤ 1, it is easy to see that the topology of strong convergence on the set $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ coincides with the topology of uniform convergence on compact subsets of $\mathfrak{S}(\mathcal{H})$.$\square$

The advantage of the topology of strong convergence consists in possibility to approximate arbitrary channel Φ in $\mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ by sequence of quantum operations with finite dimensional output space, for example, by the sequence $\{\Phi_n(\cdot) = P_n\Phi(\cdot)P_n\}$, where $\{P_n\}$ is an arbitrary sequence of finite rank projectors in $\mathfrak{B}(\mathcal{H}')$ increasing to the unit operator $I_{\mathcal{H}'}$.

The following proposition shows that it is the topology of strong convergence that makes the set of all operations to be topologically isomorphic to the special subset of states of composite system (generalized Choi-Jamiolkowski isomorphism [3]).

For given full rank state $\sigma = \sum_i \lambda_i|i\rangle\langle i|$ in $\mathfrak{S}(\mathcal{K})$ let $\mathfrak{T}(\sigma)$ be the subset of $\mathfrak{T}_1(\mathcal{K})$ consisting of all operators A such that $\left\|\frac{\langle i|A|j\rangle}{\sqrt{\lambda_i\lambda_j}}\right\| \leq E$, where E is the unit matrix (this means that $\sum_{i,j}\frac{\langle i|A|j\rangle}{\sqrt{\lambda_i\lambda_j}}|i\rangle\langle j| \leq I_{\mathcal{K}}$).

Proposition 3. *Let $\mathcal{H}$, $\mathcal{H}'$ and $\mathcal{K}$ be separable Hilbert spaces and $|\Omega\rangle$ be an unit vector in $\mathcal{H}\otimes\mathcal{K}$ such that $\sigma = \mathrm{Tr}_{\mathcal{H}}|\Omega\rangle\langle\Omega|$ is a full rank state in $\mathcal{K}$. Then the map*

$$\mathfrak{Y} : \Phi \mapsto A_{\Phi} = \Phi\otimes\mathrm{Id}(|\Omega\rangle\langle\Omega|)$$

is a topological isomorphism from $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ onto the subset

$$\mathfrak{T}_1(\mathcal{H}')\otimes\mathfrak{T}(\sigma) = \{A \in \mathfrak{T}_1(\mathcal{H}'\otimes\mathcal{K}) \,|\, \mathrm{Tr}_{\mathcal{H}'}A \in \mathfrak{T}(\sigma)\}.$$

The restriction of the map $\mathfrak{Y}$ to the set $\mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ of channels is a topological isomorphism from $\mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ onto the subset

$$\mathfrak{S}(\mathcal{H}')\otimes\{\sigma\} = \{\omega \in \mathfrak{S}(\mathcal{H}'\otimes\mathcal{K}) \,|\, \mathrm{Tr}_{\mathcal{H}'}\omega = \sigma\}.$$

Proof. The second assertion of the proposition obviously follows from the first.

Let $\sigma = \sum_i \lambda_i|i\rangle\langle i|$ and $|\Omega\rangle = \sum_i \sqrt{\lambda_i}|i\rangle\otimes|i\rangle$, where $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H}\cong\mathcal{H}'\cong\mathcal{K}$.

Let $\Phi(\cdot) = \sum_k V_k(\cdot)V_k^*$ be a quantum operation in $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ so that $\sum_k V_k^*V_k \leq I_{\mathcal{H}}$. We have

$$\langle i|\mathrm{Tr}_{\mathcal{H}'}\Phi\otimes\mathrm{Id}(|\Omega\rangle\langle\Omega|)|j\rangle = \sqrt{\lambda_i\lambda_j}\mathrm{Tr}\Phi(|i\rangle\langle j|) = \sqrt{\lambda_i\lambda_j}\langle i|\sum_k V_k^*V_k|j\rangle.$$

This implies $\mathrm{Tr}_{\mathcal{H}'}\Phi\otimes\mathrm{Id}(|\Omega\rangle\langle\Omega|) \in \mathfrak{T}(\sigma)$.

It is clear that the map $\mathfrak{Y}$ is continuous. It is injective since

$$\Phi \otimes \mathrm{Id}(|\Omega\rangle\langle\Omega|) = \sum_{i,j} \sqrt{\lambda_i \lambda_j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|, \tag{9}$$

and hence the operator $\Phi \otimes \mathrm{Id}(|\Omega\rangle\langle\Omega|)$ determines action of the quantum operation Φ on the operators $|i\rangle\langle j|$ for all i and j. By generalizing the arguments in [10] to the infinite dimensional case we will show that for each operator A in $\mathfrak{T}_1(\mathcal{H}')\otimes\mathfrak{T}(\sigma)$ there exists quantum operation Φ_A in $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ such that $A = \mathfrak{Y}(\Phi_A)$.

Let $A = \sum_k \pi_k |\psi_k\rangle\langle\psi_k|$, where $|\psi_k\rangle = \sum_{i,j} c^k_{ij} |i\rangle \otimes |j\rangle$ is a unit vector in $\mathcal{H}' \otimes \mathcal{K}$ for each k. Let $\mathrm{Tr}_{\mathcal{H}'} A = \sum_{i,j} a_{ij} |i\rangle\langle j|$. The equality

$$\sum_{i,j} a_{ij} |i\rangle\langle j| = \mathrm{Tr}_{\mathcal{H}'} A = \mathrm{Tr}_{\mathcal{H}'} \sum_{k,i,j,p,t} \pi_k c^k_{ij} \overline{c^k_{pt}} |i\rangle\langle p| \otimes |j\rangle\langle t| = \sum_{k,i,j,t} \pi_k c^k_{ij} \overline{c^k_{it}} |j\rangle\langle t|$$

implies that

$$\sum_{k,i} \pi_k c^k_{ij} \overline{c^k_{it}} = a_{jt}, \quad \forall j, t, \tag{10}$$

in particular

$$\sum_{k,i} \pi_k |c^k_{ij}|^2 = a_{jj}, \quad \forall j. \tag{11}$$

By using the condition $\mathrm{Tr}_{\mathcal{H}'} A \in \mathfrak{T}(\sigma)$ and equality (11) it is easy to show that $\pi_k \sum_t |c^k_{ti}|^2 \leq \lambda_i$ for each i and k. Hence for each k we can define bounded operator V_k from $\mathcal{H}$ into $\mathcal{H}'$ by its action on the vectors $\{|i\rangle\}$ as follows

$$V_k |i\rangle = \sqrt{\frac{\pi_k}{\lambda_i}} \sum_t c^k_{ti} |t\rangle.$$

Direct calculation shows that

$$A = \sum_k V_k \otimes I_{\mathcal{K}} \, |\Omega\rangle\langle\Omega| \, V_k^* \otimes I_{\mathcal{K}} = \Phi_A \otimes \mathrm{Id}(|\Omega\rangle\langle\Omega|),$$

where $\Phi_A(\cdot) = \sum_k V_k(\cdot)V_k^*$ is a CP map from $\mathfrak{T}(\mathcal{H})$ into $\mathfrak{T}(\mathcal{H}')$.

It follows from equality (10) that $\langle i| \sum_k V_k^* V_k |j\rangle = \frac{a_{ij}}{\sqrt{\lambda_i \lambda_j}}$. Hence the condition $\mathrm{Tr}_{\mathcal{H}'} A \in \mathfrak{T}(\sigma)$ means $\sum_k V_k^* V_k \leq I_{\mathcal{H}}$ so that $\Phi_A \in \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$.

To complete the proof it is necessary to prove openness of the map $\mathfrak{Y}$. By using expression (9) it is easy to see that for any sequence $\{A_n\}$ of operators in $\mathfrak{T}_1(\mathcal{H}')\otimes\mathfrak{T}(\sigma)$, converging to the operator A_0, the sequence $\{\Phi_{A_n}(|i\rangle\langle j|)\}$ of trace class operators converges to the operator $\Phi_{A_0}(|i\rangle\langle j|)$ (in the trace norm topology) for each i and j. Since the operator norm of quantum operation in $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ is ≤ 1, this implies strong convergence of the sequence $\{\Phi_{A_n}\}$ to the quantum operation Φ_{A_0}. $\square$

Remark 2. It follows from the proof of proposition 3 that in infinite dimensions the set of *all* CP maps is not isomorphic to the set of states of composite quantum system in contrast to the finite dimensional case (cf.[10]).$\square$

Proposition 3 makes possible to study properties of subsets of quantum operations (resp. channels) by identifying these subsets with subsets of trace class operators (resp. states). For example, it implies that the set $\mathfrak{F}_{\sigma\mapsto\rho}$ of all channels transforming a given full rank state σ into a given arbitrary state ρ is topologically isomorphic to the set $\mathcal{C}(\rho,\sigma)$ of all states ω in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{H}')$ such that $\mathrm{Tr}_{\mathcal{H}'}\omega=\sigma$ and $\mathrm{Tr}_{\mathcal{H}}\omega=\rho$.

Proposition 3 provides the simple proof of the following compactness criterion for subsets of quantum operations in the topology of strong convergence.

Corollary 2. 1) *The subset $\mathfrak{F}_0\subseteq\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ is compact if there exists full rank state σ in $\mathfrak{S}(\mathcal{H})$ such that $\{\Phi(\sigma)\}_{\Phi\in\mathfrak{F}_0}$ is a compact subset of $\mathfrak{T}_1(\mathcal{H}')$.*

2) *If the subset $\mathfrak{F}_0\subseteq\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ is compact then the set $\{\Phi(\sigma)\}_{\Phi\in\mathfrak{F}_0}$ is a compact subset of $\mathfrak{T}_1(\mathcal{H}')$ for arbitrary state σ in $\mathfrak{S}(\mathcal{H})$.*

Proof. 1) For arbitrary state $\sigma=\sum_i\lambda_i|i\rangle\langle i|$ in $\mathfrak{S}(\mathcal{K})$ the set $\mathfrak{T}(\sigma)$ is a compact subset of $\mathfrak{T}_1(\mathcal{K})$. It follows from the compactness criterion for subsets of $\mathfrak{T}_1(\mathcal{K})$ (the proposition in the Appendix). Indeed, if $P_n=\sum_{i=1}^n|i\rangle\langle i|$ then

$$\mathrm{Tr}A(I_{\mathcal{K}}-P_n)=\sum_{i>n}\langle i|A|j\rangle\leq\sum_{i>n}\lambda_i,\quad\forall A\in\mathfrak{T}(\sigma).$$

Hence compactness of the set $\mathfrak{F}_0$ in the topology of strong convergence follows from proposition 3 and the corollary in the Appendix.

2) This assertion obviously follows from the definition of the topology of strong convergence. $\square$

Example 1. Let σ be a full rank state in $\mathfrak{S}(\mathcal{H})$ and A be an arbitrary operator in $\mathfrak{T}_1(\mathcal{H}')$. By corollary 2 the set

$$\mathfrak{F}_{\sigma\mapsto A}=\{\Phi\in\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')\,|\,\Phi(\sigma)=A\}$$

is compact in the topology of strong convergence. Note that this set is not compact in the topology of uniform convergence. Note also that the set of *all* CP maps transforming the state σ into the operator A is not compact in the topology of strong convergence.

Example 2. Let σ be a full rank state in $\mathfrak{S}(\mathcal{H})$ and H' be a $\mathfrak{H}$-operator (positive operator with eigenvalues of finite multiplicity tending to the infinity, which can be interpreted as a Hamiltonian of a quantum system [8]) in the space $\mathcal{H}'$. Corollary 2 and the lemma in [5] imply that the set of channels

$$\{\Phi \in \mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}') \,|\, \mathrm{Tr}H'\Phi(\sigma) \leq h\}$$

is compact in the topology of strong convergence for each $h > 0$.

Let H be an arbitrary $\mathfrak{H}$-operator in the space $\mathcal{H}$. For given $k > 0$ consider the set of channels

$$\mathfrak{F}_{H,H',k} = \left\{ \Phi \in \mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}') \;\middle|\; \sup_{\rho\in\mathfrak{S}(\mathcal{H}),\mathrm{Tr}H\rho<+\infty} \frac{\mathrm{Tr}H'\Phi(\rho)}{\mathrm{Tr}H\rho} \leq k \right\} \tag{12}$$

Considering the $\mathfrak{H}$-operators H and H' as the Hamiltonians of the input and output systems correspondingly, the set $\mathfrak{F}_{H,H',k}$ can be treated as the set of channels with the energy amplification factor not increasing k. By the above observation the set $\mathfrak{F}_{H,H',k}$ is compact in the topology of strong convergence for each k.

4 Continuity properties of the entropic characteristics related to the classical capacity

For realization of the approximation procedures described in the Introduction it is necessary to obtain sufficient conditions for convergence of the characteristics to be explored. In this section we consider analytical properties of the functions $(\Phi,\rho) \mapsto \chi_\Phi(\rho)$ and $(\Phi,\rho) \mapsto \overline{\mathrm{co}}H_\Phi(\rho)$ defined on the Cartesian product of the set $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ of quantum operations (with the topology of strong convergence) and the set $\mathfrak{S}(\mathcal{H})$ (with the topology of the trace norm).

Proposition 4. *The functions* $(\Phi,\rho) \mapsto \chi_\Phi(\rho)$ *and* $(\Phi,\rho) \mapsto \overline{\mathrm{co}}H_\Phi(\rho)$ *are lower semicontinuous on the set* $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}') \times \mathfrak{S}(\mathcal{H})$.

Proof. Lower semicontinuity of the function $(\Phi,\rho) \mapsto \chi_\Phi(\rho)$ can be proved by the simple modification of the proof of lower semicontinuity of the function $\rho \mapsto \chi_\Phi(\rho)$ (proposition 3 in [15]).

The proof of lower semicontinuity of the function $(\Phi,\rho) \mapsto \overline{\mathrm{co}}H_\Phi(\rho)$ is based on lemma 1 below and the compactness criterion for subsets of $\mathcal{P}$.

Suppose that the function $(\Phi,\rho)\mapsto\overline{\mathrm{co}}H_{\Phi}(\rho)$ is not lower semicontinuous. This means existence of sequences $\{\Phi_n\}\subset\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ and $\{\rho_n\}\subset\mathfrak{S}(\mathcal{H})$, converging to the operation Φ_0 and to the state ρ_0, such that

$$\lim_{n\to+\infty}\overline{\mathrm{co}}H_{\Phi_n}(\rho_n)<\overline{\mathrm{co}}H_{\Phi_0}(\rho_0). \tag{13}$$

For each $n>0$ proposition 2 guarantees existence of measure $\mu_n\in\mathcal{P}_{\{\rho_n\}}$ such that

$$\overline{\mathrm{co}}H_{\Phi_n}(\rho_n)=\hat{H}_{\Phi_n}(\mu_n).$$

By the compactness criterion for subsets of $\mathcal{P}$ (proposition 2 in [8]) the sequence $\{\mu_n\}_{n>0}$ is relatively compact and hence there exists subsequence $\{\mu_{n_k}\}_k$ converging to some measure μ_0. Continuity of the map $\mu\mapsto\bar{\rho}(\mu)$ implies $\mu_0\in\mathcal{P}_{\{\rho_0\}}$. By using lemma 1 we obtain

$$\liminf_{k\to+\infty}\overline{\mathrm{co}}H_{\Phi_{n_k}}(\rho_{n_k})=\liminf_{k\to+\infty}\hat{H}_{\Phi_{n_k}}(\mu_{n_k})\geq\hat{H}_{\Phi_0}(\mu_0)\geq\overline{\mathrm{co}}H_{\Phi_0}(\rho_0),$$

which contradicts to (13).$\square$

Lemma 1. *The functional* $(\Phi,\mu)\mapsto\hat{H}_{\Phi}(\mu)$ *is lower semicontinuous on the set* $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')\times\mathcal{P}$.

Proof. Suppose that there exist such sequences $\{\Phi_n\}\subset\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ and $\{\mu_n\}\subset\mathcal{P}$, converging to the operation Φ_0 and to the measure μ_0, that

$$\lim_{n\to+\infty}\hat{H}_{\Phi_n}(\mu_n)<\hat{H}_{\Phi_0}(\mu_0). \tag{14}$$

Let $\nu_n=\mu_n\circ\Phi_n^{-1}$ be the image of the measure μ_n under the map Φ_n for each n. By proposition 6.1 in [13] to prove that the sequence $\{\nu_n\}$ of measures in $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}'))$ weakly converges to the measure $\nu_0=\mu_0\circ\Phi_0^{-1}$ it is sufficient to show that

$$\lim_{n\to+\infty}\int_{\mathfrak{T}_1(\mathcal{H}')}f(A)\nu_n(dA)=\int_{\mathfrak{T}_1(\mathcal{H}')}f(A)\nu_0(dA) \tag{15}$$

for any bounded uniformly continuous function f on the set $\mathfrak{T}_1(\mathcal{H}')$. By the construction of the sequence $\{\nu_n\}$ relation (15) is equivalent to the following one

$$\lim_{n\to+\infty}\int_{\mathfrak{S}(\mathcal{H})}f(\Phi_n(\rho))\mu_n(d\rho)=\int_{\mathfrak{S}(\mathcal{H})}f(\Phi_0(\rho))\mu_0(d\rho). \tag{16}$$

By Prohorov's theorem (cf. [13]) compactness of the sequence $\{\mu_n\}_{n\geq 0}$ (coming with separability and completeness of the space $\mathfrak{S}(\mathcal{H})$) implies that this sequence is *tight*, which means existence of such compact set $\mathcal{C}_\varepsilon \subset \mathfrak{S}(\mathcal{H})$ for each $\varepsilon > 0$ that $\mu_n(\mathcal{C}_\varepsilon) > 1-\varepsilon$ for all $n \geq 0$. For each n we have

$$|\int_{\mathfrak{S}(\mathcal{H})} f(\Phi_n(\rho))\mu_n(d\rho) - \int_{\mathfrak{S}(\mathcal{H})} f(\Phi_0(\rho))\mu_0(d\rho)|$$

$$\leq |\int_{\mathcal{C}_\varepsilon} f(\Phi_n(\rho))\mu_n(d\rho) - \int_{\mathcal{C}_\varepsilon} f(\Phi_0(\rho))\mu_0(d\rho)| + 2\varepsilon \sup_{A\in\mathfrak{T}_1(\mathcal{H})} |f(A)|$$

$$\leq \sup_{\rho\in\mathcal{C}_\varepsilon} |f(\Phi_n(\rho)) - f(\Phi_0(\rho))|$$

$$+|\int_{\mathcal{C}_\varepsilon} f(\Phi_0(\rho))\mu_n(d\rho) - \int_{\mathcal{C}_\varepsilon} f(\Phi_0(\rho))\mu_0(d\rho)| + 2\varepsilon \sup_{A\in\mathfrak{T}_1(\mathcal{H})} |f(A)|.$$

The first term in the right side of this inequality tends to zero as $n \to +\infty$ due to uniform continuity of the function f and uniform convergence of the sequence $\{\Phi_n\}$ to the quantum operation Φ_0 on the compact set $\mathcal{C}_\varepsilon$, provided by strong convergence (see remark 1). The second term tends to zero as $n \to +\infty$ due to weak convergence of the sequence $\{\mu_n\}$ to the measure μ_0. Since ε is arbitrary this observation proves (16) and hence (15). Weak convergence of the sequence $\{\nu_n = \mu_n \circ \Phi_n^{-1}\}$ to the measure $\nu_0 = \mu_0 \circ \Phi_0^{-1}$ and lower semicontinuity of the functional $\hat{H}(\nu) = \int_{\mathfrak{T}_1(\mathcal{H}')} H(A)\nu(dA)$ on the set $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}'))$ (which follows from nonnegativity and lower semicontinuity of the function $H(A)$ on the set $\mathfrak{T}_1(\mathcal{H}')$) imply

$$\liminf_{n\to+\infty} \hat{H}_{\Phi_n}(\mu_n) = \liminf_{n\to+\infty} \hat{H}(\nu_n) \geq \hat{H}(\nu_0) = \hat{H}_{\Phi_0}(\mu_0),$$

which contradicts to (14). $\square$

By concavity of the entropy and convexity of the relative entropy proposition 4 implies the following observation.

Corollary 3. *For arbitrary state σ in $\mathfrak{S}(\mathcal{H})$ the functions*

$$\Phi \mapsto \chi_\Phi(\sigma) \quad \textit{and} \quad \Phi \mapsto \overline{\mathrm{co}} H_\Phi(\sigma)$$

are lower semicontinuous convex and concave functions on the set $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ correspondingly.

By corollary 3 the function $\Phi \mapsto \overline{\mathrm{co}} H_\Phi(\sigma)$ achieves its infimum on any convex compact subset of $\mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ at some extreme point of this subset.

Hence the set $\mathfrak{F}_{\sigma\mapsto\rho}$ of all channels mapping given full rank state σ in $\mathfrak{S}(\mathcal{H})$ to given state ρ in $\mathfrak{S}(\mathcal{H}')$ (see example 1 in the previous section) contains such channel $\Phi_{\sigma,\rho}$ that

$$\overline{\mathrm{co}}H_{\Phi_{\sigma,\rho}}(\sigma) \leq \overline{\mathrm{co}}H_{\Phi}(\sigma), \quad \forall\Phi \in \mathfrak{F}_{\sigma\mapsto\rho}.$$

If $\rho \cong \sigma$ then $\Phi_{\sigma,\rho}(\cdot) = U(\cdot)U^*$ and $\overline{\mathrm{co}}H_{\Phi_{\sigma,\rho}}(\sigma) = 0$, where U is any unitary map from $\mathcal{H}$ onto $\mathcal{H}'$ such that $U\sigma U^* = \rho$. In general case the channel $\Phi_{\sigma,\rho}$ is the image of some extreme point of the compact convex set $\mathcal{C}(\sigma,\rho)$ (defined before corollary 2) under the map $\mathfrak{Y}^{-1}$ and in some sense can be considered as a channel with minimal noise transforming the state σ into the state ρ.

Proposition 4 and relation (8) imply the following sufficient condition of continuity of the functions $(\Phi,\rho) \mapsto \chi_{\Phi}(\rho)$ and $(\Phi,\rho) \mapsto \overline{\mathrm{co}}H_{\Phi}(\rho)$.

Proposition 5. [1] *Let $\{\Phi_n\}$ be a sequence of operations in $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$ strongly converging to the channel Φ_0 and $\{\rho_n\}$ be a sequence of states in $\mathfrak{S}(\mathcal{H})$ converging to the state ρ_0. If*

$$\lim_{n\to+\infty} H_{\Phi_n}(\rho_n) = H_{\Phi_0}(\rho_0) < +\infty$$

then

$$\lim_{n\to+\infty} \overline{\mathrm{co}}H_{\Phi_n}(\rho_n) = \lim_{n\to+\infty} \overline{\mathrm{co}}S_{\Phi_n}(\rho_n) = \overline{\mathrm{co}}H_{\Phi_0}(\rho_0) \ \text{ and } \lim_{n\to+\infty} \chi_{\Phi_n}(\rho_n) = \chi_{\Phi_0}(\rho_0).$$

As an application of this condition consider the compact set $\mathfrak{F}_{H,H',k}\times\mathcal{K}_{H,h}$, where $\mathfrak{F}_{H,H',k}$ is the compact subset of $\mathfrak{F}(\mathcal{H},\mathcal{H}')$ consisting of channels with bounded energy amplification factor (defined in example 2) and $\mathcal{K}_{H,h}$ is the compact subset of $\mathfrak{S}(\mathcal{H})$ consisting of states with bounded mean energy (defined by the inequality $\mathrm{Tr}H\rho \leq h$). Assume that $\mathrm{Tr}\exp(-\lambda H') < +\infty$ for all $\lambda > 0$. By using the observation in [20] it is easy to see that the function $(\Phi,\rho) \mapsto H_{\Phi}(\rho)$ is continuous on the set $\mathfrak{F}_{H,H',k} \times \mathcal{K}_{H,h}$ for any k and h. Proposition 5 implies that the functions $(\Phi,\rho) \mapsto \overline{\mathrm{co}}H_{\Phi}(\rho)$ and $(\Phi,\rho) \mapsto \chi_{\Phi}(\rho)$ are continuous on the set $\mathfrak{F}_{H,H',k} \times \mathcal{K}_{H,h}$.

The special choice of approximating sequence makes possible to ensure convergence of the functions $\overline{\mathrm{co}}H_{\Phi}$, $\overline{\mathrm{co}}S_{\Phi}$ and χ_{Φ} without reference to the output entropy.

Proposition 6. *Let $\{\Phi_n\}$ be a sequence of operations, strongly converging to the channel Φ_0. The relations*

$$\lim_{n\to+\infty} \overline{\mathrm{co}}H_{\Phi_n}(\rho) = \lim_{n\to+\infty} \overline{\mathrm{co}}S_{\Phi_n}(\rho) = \overline{\mathrm{co}}H_{\Phi_0}(\rho) \quad \text{and} \quad \lim_{n\to+\infty} \chi_{\Phi_n}(\rho) = \chi_{\Phi_0}(\rho)$$

[1] This proposition is a generalization of proposition 7 in [16].

hold for any state ρ in $\mathfrak{S}(\mathcal{H})$ in the following cases:

A) $\Phi_n(\cdot) = P_n\Phi_0(\cdot)P_n$ *for some sequence $\{P_n\}$ of projectors in $\mathfrak{B}(\mathcal{H}')$, increasing to the unit operator $I_{\mathcal{H}'}$;*

B) $\Phi_n(\rho) \leq \Phi_0(\rho)$ *for all ρ in $\mathfrak{S}(\mathcal{H})$ (in the operator order).*

Proof. A) For arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ lemma 3 in [11] and monotonicity of the relative entropy imply $\overline{\mathrm{co}}H_{\Phi_n}(\rho) \leq \overline{\mathrm{co}}H_{\Phi_0}(\rho)$ and $\chi_{\Phi_n}(\rho) \leq \chi_{\Phi_0}(\rho)$ correspondingly. Hence the limit relations in the proposition follow from proposition 4.

B) For arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ inequality (2) and lemma 2 below imply $\overline{\mathrm{co}}H_{\Phi_n}(\rho) \leq \overline{\mathrm{co}}H_{\Phi_0}(\rho)$ and $\chi_{\Phi_n}(\rho) \leq \chi_{\Phi_0}(\rho) + \eta(\mathrm{Tr}\Phi_n(\rho)) + h_2(\mathrm{Tr}\Phi_n(\rho))$ correspondingly. Hence the limit relations in the proposition follow from proposition 4.$\square$

Lemma 2. *Let $\{\pi_i, A_i\}$ and $\{\pi_i, B_i\}$ be two (finite) ensembles of operators in $\mathfrak{T}_1(\mathcal{H})$ such that $A_i \leq B_i$, $\forall i$. Then*

$$\sum_i \pi_i H(A_i \,\|\, A) \leq \sum_i \pi_i H(B_i \,\|\, B) + \eta(\mathrm{Tr}A) + \mathrm{Tr}B h_2\left(\frac{\mathrm{Tr}A}{\mathrm{Tr}B}\right),$$

where $A = \sum_i \pi_i A_i$ and $B = \sum_i \pi_i B_i$.

Proof. Suppose first that $H(B) < +\infty$. Then by using inequality (2) and concavity of the functions H, h_2 and η we obtain

$$\begin{gathered}
\sum_i \pi_i H(B_i \,\|\, B) = S(B) - \sum_i \pi_i S(B_i) = [H(B) - \sum_i \pi_i H(B_i)] \\
+[\eta(\mathrm{Tr}B) - \sum_i \pi_i \eta(\mathrm{Tr}B_i)] \geq [H(A) - \sum_i \pi_i H(A_i)] - \sum_i \pi_i \mathrm{Tr}B_i h_2\left(\frac{\mathrm{Tr}A_i}{\mathrm{Tr}B_i}\right) \\
+[\eta(\mathrm{Tr}B) - \sum_i \pi_i \eta(\mathrm{Tr}B_i)] + [H(B-A) - \sum_i \pi_i H(B_i - A_i)] \\
\geq [S(A) - \sum_i \pi_i S(A_i)] - [\eta(\mathrm{Tr}A) - \sum_i \pi_i \eta(\mathrm{Tr}A_i)] - \sum_i \pi_i \mathrm{Tr}B_i h_2\left(\frac{\mathrm{Tr}A_i}{\mathrm{Tr}B_i}\right) \\
\geq \sum_i \pi_i H(A_i \,\|\, A) - \mathrm{Tr}B h_2\left(\frac{\mathrm{Tr}A}{\mathrm{Tr}B}\right) - \eta(\mathrm{Tr}A).
\end{gathered}$$

In the case $H(B) = +\infty$ the above observation applied to the ensembles $\{\pi_i, P_nA_iP_n\}$ and $\{\pi_i, P_nB_iP_n\}$ for each n, where $\{P_n\}$ is an arbitrary sequence of finite rank projectors increasing to the unit operator $I_{\mathcal{H}}$, implies inequality

$$\sum_i \pi_i H(P_nA_iP_n \,\|\, P_nAP_n)$$
$$\leq \sum_i \pi_i H(P_nB_iP_n \,\|\, P_nBP_n) + \eta(\mathrm{Tr}P_nA) + \mathrm{Tr}P_nBh_2\left(\frac{\mathrm{Tr}P_nA}{\mathrm{Tr}P_nB}\right).$$

By using lemma 4 in [11] we can take the limit in this inequality and obtain the assertion of the lemma. $\square$

Remark 3. Proposition 7 in [16] and proposition 6A imply that the χ-function (corresp. the CCoOE) of arbitrary quantum channel can be represented as the least upper bound of increasing sequence of concave (corresp. convex) *continuous* bounded functions.

5 Applications

5.1 On continuity of the χ-capacity as a function of a channel

The χ-capacity of a quantum channel $\Phi \in \mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ constrained by an arbitrary subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ can be defined by (cf.[5],[8])

$$\bar{C}(\Phi,\mathcal{A}) = \sup_{\{\pi_i,\rho_i\}\in\mathcal{P}^{\mathrm{f}}_{\mathcal{A}}} \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\bar{\rho})) = \sup_{\rho\in\mathcal{A}} \chi_{\Phi}(\rho). \tag{17}$$

By using lower semicontinuity of the relative entropy it is easy to show that the function $\mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}') \ni \Phi \mapsto \bar{C}(\Phi,\mathcal{A})$ is lower semicontinuous, t.i.

$$\liminf_{n\to+\infty} \bar{C}(\Phi_n,\mathcal{A}) \geq \bar{C}(\Phi_0,\mathcal{A}) \tag{18}$$

for arbitrary sequence $\{\Phi_n\}$ of channels in $\mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ strongly converging to the channel Φ_0. There exist examples showing that $>$ can take place in (18) even in the case of uniform convergence of the sequence $\{\Phi_n\}$ to the channel Φ_0 and that the difference between the left and the right sides can be arbitrary large [15].

If the sequence $\{\Phi_n\}$ is such that the inequality $\bar{C}(\Phi_n,\mathcal{A}) \leq \bar{C}(\Phi_0,\mathcal{A})$ can be proved for each n then (18) implies that

$$\lim_{n\to+\infty} \bar{C}(\Phi_n,\mathcal{A}) = \bar{C}(\Phi_0,\mathcal{A}). \tag{19}$$

For example, by the monotonicity property of the relative entropy this holds if $\Phi_n = \Pi_n \circ \Phi_0$ for each n, where $\{\Pi_n\}$ is a sequence of channels in $\mathfrak{F}_{=1}(\mathcal{H}', \mathcal{H}')$ strongly converging to the noiseless channel.

The results of the previous section make possible to prove the following continuity condition for the χ-capacity.

Proposition 7. *Let $\{\Phi_n\}$ be a sequence of channels in $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$, strongly converging to the channel Φ_0, and $\mathcal{A}$ be a compact subset of $\mathfrak{S}(\mathcal{H})$.*

If $\lim_{n\to+\infty} H_{\Phi_n}(\rho_n) = H_{\Phi_0}(\rho_0) < +\infty$ for arbitrary sequence $\{\rho_n\}$ of states in $\mathcal{A}$, converging to the state ρ_0, then (19) holds.

Proof. To prove (19) it is sufficient to show that the assumption

$$\lim_{n\to+\infty} \bar{C}(\Phi_n, \mathcal{A}) > \bar{C}(\Phi_0, \mathcal{A})$$

leads to a contradiction. For each n let ρ_n be such state in $\mathcal{A}$ that

$$\chi_{\Phi_n}(\rho_n) > \bar{C}(\Phi_n, \mathcal{A}) - 1/n. \tag{20}$$

Compactness of the set $\mathcal{A}$ implies existence of subsequence $\{\rho_{n_k}\}$ converging to some state $\rho_0 \in \mathcal{A}$. By the condition $\lim_{k\to+\infty} H_{\Phi_{n_k}}(\rho_{n_k}) = H_{\Phi_0}(\rho_0) < +\infty$ and proposition 5 implies

$$\lim_{k\to+\infty} \chi_{\Phi_{n_k}}(\rho_{n_k}) = \chi_{\Phi_0}(\rho_0) \leq \bar{C}(\Phi_0, \mathcal{A}).$$

This coming with (20) leads to a contradiction.

By using proposition 7 it is possible to show that the χ-capacity of a channel with energy constraint is continuous on the set of channels with bounded energy amplification factor, considered in example 2.

Corollary 4. *Let H and H' be $\mathfrak{H}$-operators (Hamiltonians) in the spaces $\mathcal{H}$ and $\mathcal{H}'$ correspondingly such that $\mathrm{Tr}\exp(-\lambda H') < +\infty$ for all $\lambda > 0$. The function $\Phi \mapsto \bar{C}(\Phi, \mathcal{K}_{H,h})$ is continuous on the set $\mathfrak{F}_{H,H',k}$ (defined by (12)).*

Proof. By the lemma in [5] the set $\mathcal{K}_{H,h}$ is compact. Let h and k be fixed positive numbers. For arbitrary sequences $\{\Phi_n\} \subset \mathfrak{F}_{H,H',k}$ and $\{\rho_n\} \subset \mathcal{K}_{H,h}$ the sequence $\{\Phi_n(\rho_n)\}$ belongs to the set $\mathcal{K}_{H',kh}$, on which the entropy is continuous by the observation in [20].$\square$

For arbitrary quantum channel $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$ and arbitrary *convex* subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ such that $\bar{C}(\Phi, \mathcal{A}) < +\infty$ there exists the unique state $\Omega(\Phi, \mathcal{A})$ in $\mathfrak{S}(\mathcal{H}')$ called output optimal average for the $\mathcal{A}$-constrained channel Φ (proposition 1 in [15][2]). This state inherits the main properties of

[2]the case of noncompact set $\mathcal{A}$ is considered in quant-ph/0408009.

the image of the average state of optimal ensemble for a finite dimensional $\mathcal{A}$-constrained channel Φ [7]. If there exists an optimal measure μ for the $\mathcal{A}$-constrained channel Φ (see definition in [8]) then $\Omega(\Phi,\mathcal{A}) = \Phi(\bar{\rho}(\mu))$. It is interesting to note that continuity of the function $\Phi \mapsto \bar{C}(\Phi,\mathcal{A})$ on some set of channels implies continuity of the function $\Phi \mapsto \Omega(\Phi,\mathcal{A})$ on this set.

Proposition 8. *Let $\{\Phi_n\}$ be a sequence of channels in $\mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$, strongly converging to the channel Φ_0, and $\mathcal{A}$ be a convex subset of $\mathfrak{S}(\mathcal{H})$.*

If $\lim\limits_{n\to+\infty}\bar{C}(\Phi_n,\mathcal{A}) = \bar{C}(\Phi_0,\mathcal{A}) < +\infty$ then $\lim\limits_{n\to+\infty}\Omega(\Phi_n,\mathcal{A}) = \Omega(\Phi_0,\mathcal{A})$.

Proof. By proposition 1 in [15] for arbitrary $\varepsilon > 0$ there exists ensemble $\{\pi_i,\rho_i\}$ with the average state in $\mathcal{A}$ such that

$$\chi_{\Phi_0}(\{\pi_i,\rho_i\}) \geq \bar{C}(\Phi_0,\mathcal{A}) - \varepsilon \ \text{ and } \ \|\sum_i \pi_i\Phi_0(\rho_i) - \Omega(\Phi_0,\mathcal{A})\|_1 < \varepsilon. \tag{21}$$

Lower semicontinuity of the relative entropy implies

$$\chi_{\Phi_n}(\{\pi_i,\rho_i\}) \geq \chi_{\Phi_0}(\{\pi_i,\rho_i\}) - \varepsilon$$

for all sufficiently large n. By the assumption

$$\bar{C}(\Phi_n,\mathcal{A}) \leq \bar{C}(\Phi_0,\mathcal{A}) + \varepsilon$$

for all sufficiently large n.

Thus for all sufficiently large n we have

$$0 \leq \bar{C}(\Phi_n,\mathcal{A}) - \chi_{\Phi_n}(\{\pi_i,\rho_i\}) \leq \bar{C}(\Phi_0,\mathcal{A}) - \chi_{\Phi_0}(\{\pi_i,\rho_i\}) + 2\varepsilon \leq 3\varepsilon$$

and by using corollary 1 in [15] we obtain

$$\begin{aligned}\frac{1}{2}\|\sum_i \pi_i\Phi_n(\rho_i) - \Omega(\Phi_n,\mathcal{A})\|_1^2 &\leq H\left(\sum_i \pi_i\Phi_n(\rho_i) \,\|\, \Omega(\Phi_n,\mathcal{A})\right) \\ &\leq \bar{C}(\Phi_n,\mathcal{A}) - \chi_{\Phi_n}(\{\pi_i,\rho_i\}) \leq 3\varepsilon.\end{aligned} \tag{22}$$

By strong convergence of the sequence $\{\Phi_n\}$ to the channel Φ_0 we have

$$\|\sum_i \pi_i\Phi_n(\rho_i) - \sum_i \pi_i\Phi_0(\rho_i)\|_1 \leq \varepsilon \tag{23}$$

for all sufficiently large n.

By using (21),(22) and (23) we obtain

$$\|\Omega(\Phi_n,\mathcal{A})-\Omega(\Phi_0,\mathcal{A})\|_1 \leq \|\Omega(\Phi_n,\mathcal{A})-\sum_i \pi_i\Phi_n(\rho_i)\|_1$$

$$+\|\sum_i \pi_i\Phi_n(\rho_i)-\sum_i \pi_i\Phi_0(\rho_i)\|_1+\|\sum_i \pi_i\Phi_0(\rho_i)-\Omega(\Phi_0,\mathcal{A})\|_1 \leq 2\varepsilon+\sqrt{6\varepsilon}$$

for all sufficiently large n. $\square$

5.2 On additivity of the χ-capacity

The approximation procedure is the essential part of the proof that additivity conjecture in finite dimensions implies strong additivity of the χ-capacity for all infinite dimensional channels [15]. It also provides possibility to derive strong additivity of the χ-capacity for two infinite dimensional channels with one of them noiseless or entanglement-breaking from the corresponding finite dimensional results [7], [19].[3]

In [18] strong additivity of the χ-capacity for two infinite dimensional channels with one of them complementary to entanglement-breaking channel is proved *under the condition* that the output entropies of both channels are finite on the set of pure input states. This condition seems to be essential since it is coincidence of the output entropies of two comlementary channels on the set of pure states that provides "transition" of the additivity properties between pairs of complementary channels (see the proof of theorem 1 in [6]) and infinite values of these output entropies prevent this transition. But the condition of finiteness of the output entropy on the set of pure states for a given channel is difficult to verify in general, which is a real obstacle in application of the above result. Moreover, this condition is not valid for large class of infinite dimensional channels. We will show below that the approximation approach makes possible to overcome the problem of infinite output entropies and to prove strong additivity of the χ-capacity for two infinite dimensional channels with one of them complementary to entanglement-breaking channel even in the case when the output entropies of these channels are everywhere

[3] Note that direct generalization of the proofs of these results to the infinite dimensional case seems to be nontrivial. For example, the proof of theorem 2 in [19] is based on finiteness of the output entropy and on decomposition of an arbitrary separable state into *discrete* convex combination of pure product states, which is not valid in the infinite dimensional case [9].

infinite.

Proposition 9. *Let $\Phi \in \mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ be a channel such that its complementary channel is entanglement breaking and $\Psi \in \mathfrak{F}_{=1}(\mathcal{K},\mathcal{K}')$ be an arbitrary channel. Then strong additivity of the χ-capacity holds for the channels Φ and Ψ.*

Proof. By using lemma 5 and proposition 6 in [15] it is possible to reduce the proof to the case $\dim\mathcal{K} < +\infty$ and $\dim\mathcal{K}' < +\infty$. By proposition 6 in [15] it is sufficient to prove inequality

$$\chi_{\Phi\otimes\Psi}(\omega) \leq \chi_{\Phi}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) \tag{24}$$

for arbitrary state ω in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ such that $\mathrm{rank}\omega^{\mathcal{H}} < +\infty$. Let ω be such a state and $\mathcal{H}_{\omega} = \mathrm{supp}\omega^{\mathcal{H}}$ be the corresponding finite dimensional subspace.

Let $\Phi(\rho) = \mathrm{Tr}_{\mathcal{H}''}V\rho V^*$, where V is the Stinespring isometry from $\mathcal{H}$ into $\mathcal{H}'\otimes\mathcal{H}''$. By the condition the complementary channel $\widehat{\Phi}(\rho) = \mathrm{Tr}_{\mathcal{H}'}V\rho V^*$ is entanglement-breaking.

Let $\{P_n\}$ be an arbitrary sequence of finite rank projectors in $\mathfrak{B}(\mathcal{H}'')$, increasing to the unit operator $I_{\mathcal{H}''}$. Consider the quantum operations

$$\Phi_n(\rho) = \mathrm{Tr}_{\mathcal{H}''}I_{\mathcal{H}'}\otimes P_n\cdot V\rho V^*\cdot I_{\mathcal{H}'}\otimes P_n = \mathrm{Tr}_{\mathcal{H}''}I_{\mathcal{H}'}\otimes P_n\cdot V\rho V^*, \quad \rho\in\mathfrak{S}(\mathcal{H}),$$

and

$$\widehat{\Phi}_n(\rho) = \mathrm{Tr}_{\mathcal{H}'}I_{\mathcal{H}'}\otimes P_n\cdot V\rho V^*\cdot I_{\mathcal{H}'}\otimes P_n = P_n\widehat{\Phi}(\rho)P_n, \quad \rho\in\mathfrak{S}(\mathcal{H}).$$

Let $\widehat{\Psi}$ be the complementary channel to the channel Ψ. Note that the restriction of the quantum operation $\widehat{\Phi}_n$ to the set $\mathfrak{S}(\mathcal{H}_{\omega})$ is a finite dimensional entanglement-breaking operation[4]. By using proposition 2C and by repeating the arguments from the proof of theorem 2 in [19] it is possible to show existence of such sequence $\{\sigma_n\}\subset\mathfrak{S}(\mathcal{K})$ converging to the state $\omega^{\mathcal{K}}$ that for each n the following inequality holds

$$\overline{\mathrm{co}}S_{\widehat{\Phi}_n\otimes\widehat{\Psi}}(\omega) = \mathrm{co}S_{\widehat{\Phi}_n\otimes\widehat{\Psi}}(\omega) \geq \overline{\mathrm{co}}S_{\widehat{\Phi}_n}(\omega^{\mathcal{H}}) + \alpha_n\overline{\mathrm{co}}S_{\widehat{\Psi}}(\sigma_n), \tag{25}$$

where $\alpha_n = \inf_{\rho\in\mathfrak{S}(\mathcal{H}_{\omega})}\mathrm{Tr}\widehat{\Phi}_n(\rho)$.

Since

$$S_{\widehat{\Phi}_n}(\rho) = S_{\Phi_n}(\rho), \quad \forall\rho\in\mathrm{extr}\mathfrak{S}(\mathcal{H}), \quad S_{\widehat{\Psi}}(\sigma) = S_{\Psi}(\sigma), \quad \forall\sigma\in\mathrm{extr}\mathfrak{S}(\mathcal{K})$$

[4]This is an obvious generalization of the notion of entanglement-breaking channel.

and

$$S_{\widehat{\Phi}_n\otimes\widehat{\Psi}}(\omega)=S_{\Phi_n\otimes\Psi}(\omega),\quad \forall\omega\in\mathrm{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}),$$

proposition 2A implies that inequality (25) is equivalent to the following one

$$\overline{\mathrm{co}}S_{\Phi_n\otimes\Psi}(\omega)\geq\overline{\mathrm{co}}S_{\Phi_n}(\omega^{\mathcal{H}})+\alpha_n\overline{\mathrm{co}}S_{\Psi}(\sigma_n). \tag{26}$$

Note that inequality (3) implies

$$S_{\Phi_n\otimes\Psi}(\omega)\leq S_{\Phi_n}(\omega^{\mathcal{H}})+S(\mathrm{Tr}_{\mathcal{H}'}\Phi_n\otimes\Psi(\omega))-\varepsilon_n, \tag{27}$$

where $\varepsilon_n=\eta(\mathrm{Tr}\Phi_n(\omega^{\mathcal{H}}))$.

By using (8), (26), (27) and proposition 2B we obtain

$$\chi_{\Phi_n\otimes\Psi}(\omega)=S_{\Phi_n\otimes\Psi}(\omega)-\overline{\mathrm{co}}S_{\Phi_n\otimes\Psi}(\omega)$$

$$\leq S_{\Phi_n}(\omega^{\mathcal{H}})-\overline{\mathrm{co}}S_{\Phi_n}(\omega^{\mathcal{H}})+S(\mathrm{Tr}_{\mathcal{H}'}\Phi_n\otimes\Psi(\omega))-\overline{\mathrm{co}}S_{\Psi}(\sigma_n)$$

$$+(1-\alpha_n)\overline{\mathrm{co}}S_{\Psi}(\sigma_n)\leq\chi_{\Phi_n}(\omega^{\mathcal{H}})+\chi_{\Psi}(\omega^{\mathcal{K}})+[(1-\alpha_n)S_{\Psi}(\sigma_n)]$$

$$+[S(\mathrm{Tr}_{\mathcal{H}'}\Phi_n\otimes\Psi(\omega))-S_{\Psi}(\omega^{\mathcal{K}})]+[\overline{\mathrm{co}}S_{\Psi}(\omega^{\mathcal{K}})-\overline{\mathrm{co}}S_{\Psi}(\sigma_n)].$$

The sequence of quantum operations $\{\Phi_n\}$ strongly converges to the channel Φ and satisfies condition B in proposition 6. This proposition and proposition 4 make possible to prove inequality (24) by taking the limit in the above inequality since the terms in the square brackets tends to zero as $n\rightarrow+\infty$ due to assumed finite dimensionality of the spaces $\mathcal{H}_\omega$ and $\mathcal{K}'$.$\square$

Example 3. By proposition 9 strong additivity of the χ-capacity holds for arbitrary channel Ψ and the channel Φ^a_p considered in the example in [18] with arbitrary probability density function $p(t)$ and $a\leq+\infty$. This implies in particular that the classical capacity the channel Φ^a_p with arbitrary constraint coincides with the χ-capacity.

5.3 Representation for the CCoOE

The convex closure of the output entropy (CCoOE) of a quantum channel is an important characteristics related to the classical capacity of this channel [16]. This notion also plays essential role in the theory of entanglement: an important entanglement measure of a state of a composite quantum system

– the Entanglement of Formation (EoF) – can be defined as the CCoOE of a partial trace [1].

By proposition 2 the CCoOE of a quantum channel $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$ is represented by the expression

$$\overline{\mathrm{co}} H_{\Phi}(\rho) = \inf_{\mu \in \widehat{\mathcal{P}}_{\{\rho\}}} \int_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})} H_{\Phi}(\sigma)\mu(d\sigma), \quad \rho \in \mathfrak{S}(\mathcal{H}). \tag{28}$$

In [16] it is shown that for arbitrary state ρ with finite output entropy $H_{\Phi}(\rho)$ the infimum in this expression can be taken only over atomic measures, which means that

$$\overline{\mathrm{co}} H_{\Phi}(\rho) = \inf_{\{\pi_i, \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_{\Phi}(\rho_i), \tag{29}$$

(where the infimum is over all countable ensembles $\{\pi_i, \rho_i\}$ of pure states with the average state ρ.)

But validity of expression (29) for arbitrary state ρ remains open question. The second example in remark 2 in [17] shows that positive answer on this question can not be obtained by using only general analytical properties of the (output) entropy. For given channel Φ validity of expression (29) for arbitrary state ρ is equivalent to lower semicontinuity of the right side of this expression on the input state space $\mathfrak{S}(\mathcal{H})$.

Thus in the case of general quantum channel Φ it is necessary to use representation (28), which involves optimization over *all* measures with given barycenter ρ. This provides some technical problems in dealing with CCoOE. Moreover this expression looks unnatural from the physical point of view since for given state ρ with *finite* mean energy, produced in a physical experiment, the above optimization involves measures supported by states with *infinite* mean energy.[5]

In this subsection we obtain representation for the CCoOE of an arbitrary quantum channel $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$ as a limit of increasing sequence of *continuous* bounded convex functions on $\mathfrak{S}(\mathcal{H})$ defined via the expressions similar to (29).

Let $n > 1$ be fixed natural number. Consider the function

$$H^n_{\Phi}(\rho) = -\sum_{i=1}^{n} \lambda_i \log \lambda_i + \left(\sum_{i=1}^{n} \lambda_i\right) \log \left(\sum_{i=1}^{n} \lambda_i\right) = H\left(\left\{\frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}\right\}_{i=1}^{n}\right),$$

[5] Any countable ensemble having the average state with finite mean energy consists of states with finite mean energy.

where $\{\lambda_i\}_{i=1}^n$ is the set of n maximal eigenvalues of the state $\Phi(\rho)$, which can be called truncated output entropy. By lemma 4 in [11] the sequence $\{H_\Phi^n\}$ of continuous bounded functions on $\mathfrak{S}(\mathcal{H})$ is nondecreasing and pointwise converges to the output entropy H_Φ.

Let

$$\check{H}_\Phi^n(\rho) = \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi^n(\rho_i).$$

By proposition 5 in [17] the function $\check{H}_\Phi^n (= (H_\Phi^n)_*)$ is the convex *continuous* extension of the function $\mathrm{extr}\mathfrak{S}(\mathcal{H}) \ni \rho \mapsto H_\Phi^n(\rho)$ to the set $\mathfrak{S}(\mathcal{H})$.[6]

The sequence $\{\check{H}_\Phi^n\}_n$ of convex continuous bounded functions on $\mathfrak{S}(\mathcal{H})$ is increasing and majorized by the function $\overline{\mathrm{co}}H_\Phi$. The results of the previous section make possible to prove the following observation.[7]

Proposition 10. *For arbitrary channel $\Phi \in \mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ and arbitrary state $\rho \in \mathfrak{S}(\mathcal{H})$ the following relation holds*

$$\overline{\mathrm{co}}H_\Phi(\rho) = \lim_{n\to+\infty} \check{H}_\Phi^n(\rho) = \lim_{n\to+\infty} \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi^n(\rho_i).$$

Remark 4. This proposition does not imply validity of expression (29). There exists an increasing sequence $\{f_n\}$ of concave continuous bounded functions on $\mathfrak{S}(\mathcal{H})$ converging to the (concave lower semicontinuous) bounded function f such that

$$\lim_{n\to+\infty} \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f_n(\rho_i) = 0 \quad \text{and} \quad \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i) = 1$$

for some state $\rho \in \mathfrak{S}(\mathcal{H})$ (see the second example in remark 2 in [17]).

Proof. By the above observation it is sufficient to show that

$$\liminf_{n\to+\infty} \check{H}_\Phi^n(\rho) \geq \overline{\mathrm{co}}H_\Phi(\rho) \tag{30}$$

for arbitrary state $\rho \in \mathfrak{S}(\mathcal{H})$.

Let $\{P_n\}$ be a sequence of projectors in $\mathfrak{B}(\mathcal{H}')$, increasing to the unit operator $I_{\mathcal{H}}$, such that $\mathrm{rank}P_n = n$. Consider the sequence $\{\Phi_n(\cdot) = P_n\Phi(\cdot)P_n\}$ of operations in $\mathfrak{F}_{\leq 1}(\mathcal{H},\mathcal{H}')$.

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[6] Since in general case the function H_Φ^n is not concave on $\mathfrak{S}(\mathcal{H})$ we can not assert that $\check{H}_\Phi^n = \overline{\mathrm{co}}H_\Phi^n$.

[7] It is nontrivial since the set $\mathfrak{S}(\mathcal{H})$ is not compact.

Let ρ be an arbitrary pure state in $\mathfrak{S}(\mathcal{H})$. If $\{\lambda_i\}_{i=1}^n$ and $\{\lambda_i^n\}_{i=1}^n$ are sets of maximal eigenvalues (in decreasing order) of the operators $\Phi(\rho)$ and $\Phi_n(\rho)$ then the Ritz principle implies $\lambda_i \geq \lambda_i^n$ for each $i = \overline{1,n}$. Hence by using (1) we obtain

$$H_\Phi^n(\rho) = H\left(\left\{\frac{\lambda_i}{\sum_{i=1}^n \lambda_i}\right\}_{i=1}^n\right) \geq H(\{\lambda_i\}_{i=1}^n) \geq H(\{\lambda_i^n\}_{i=1}^n) = H_{\Phi_n}(\rho).$$

It follows that

$$\inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi^n(\rho_i) \geq \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_{\Phi_n}(\rho_i), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

Since the function H_{Φ_n} is concave continuous and bounded on $\mathfrak{S}(\mathcal{H})$ corollary 10 in [17] implies that the right side of the above inequality coincides with $\overline{\mathrm{co}}H_{\Phi_n}(\rho)$.

The sequence $\{\Phi_n\}$ satisfies condition A in proposition 6. Hence for arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ we obtain

$$\liminf_{n\to+\infty} \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi^n(\rho_i) \geq \lim_{n\to+\infty} \overline{\mathrm{co}}H_{\Phi_n}(\rho) = \overline{\mathrm{co}}H_\Phi(\rho),$$

which means (30). $\square$

Corollary 5. *Let $\Phi \in \mathfrak{F}_{=1}(\mathcal{H},\mathcal{H}')$ be an arbitrary channel and $\mathcal{A}$ be such compact subset of $\mathfrak{S}(\mathcal{H})$ that the output entropy H_Φ is continuous on $\mathcal{A}$. Then the increasing sequence $\{\check{H}_\Phi^n\}$ of continuous functions converges to the function $\overline{\mathrm{co}}H_\Phi$ uniformly on $\mathcal{A}$.*

Proof. Proposition 7 in [16] implies continuity of the function $\overline{\mathrm{co}}H_\Phi$ on the set $\mathcal{A}$. Hence the assertion of the corollary follows from proposition 10 and Dini's lemma. $\square$

Corollary 5 shows that for arbitrary Gaussian channel Φ the sequence $\{\check{H}_\Phi^n\}$ provides uniform approximation of the function $\overline{\mathrm{co}}H_\Phi$ on the set of states with bounded mean energy (see the remark after proposition 3 in [8]).

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Consider the channel $\Theta : \mathfrak{S}(\mathcal{H}\otimes\mathcal{K}) \ni \omega \mapsto \mathrm{Tr}_\mathcal{K}\omega \in \mathfrak{S}(\mathcal{H})$. The Entanglement of Formation of a state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ can be defined by (cf.[16])

$$E_\mathrm{F}(\omega) = \overline{\mathrm{co}}H_\Theta(\omega) = \inf_{\mu\in\widehat{\mathcal{P}}_{\{\omega\}}} \int_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})} H_\Theta(\sigma)\mu(d\sigma).$$

Proposition 10 implies that

$$E_{\mathrm{F}}(\omega)=\lim_{n\to+\infty}\check{H}^{n}_{\Theta}(\omega)=\lim_{n\to+\infty}\inf_{\{\pi_i,\omega_i\}\in\widehat{\mathcal{P}}_{\{\omega\}}}\sum_i \pi_i H^{n}_{\Theta}(\omega_i).$$

This proves the conjecture that E_{F} is a function of class $\widehat{P}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$ (cf.[17]).

Corollary 5 implies that the above convergence is uniform on the set of states of composite system with bounded mean energy.

6 Appendix

The following compactness criterion for subsets of $\mathfrak{T}_1(\mathcal{H})$ can be proved by simple modification of the arguments used in the proof of the compactness criterion for subsets of $\mathfrak{S}(\mathcal{H})$, presented in the Appendix in [8].

Proposition. *The closed subset $\mathcal{A}$ of $\mathfrak{T}_1(\mathcal{H})$ is (trace norm) compact if and only if for arbitrary $\varepsilon>0$ there exists a finite rank projector P_ε such that* $\mathrm{Tr}(I_{\mathcal{H}}-P_\varepsilon)A<\varepsilon$ *for all $A\in\mathcal{A}$.*

Corollary. *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathfrak{T}_1(\mathcal{H})$ and $\mathfrak{T}_1(\mathcal{K})$ correspondingly. The subset $\mathcal{A}\otimes\mathcal{B}$ of $\mathfrak{T}_1(\mathcal{H}\otimes\mathcal{K})$ consisting of all operators C such that* $\mathrm{Tr}_{\mathcal{K}}C\in\mathcal{A}$ *and* $\mathrm{Tr}_{\mathcal{H}}C\in\mathcal{B}$ *is compact if and only if the sets $\mathcal{A}$ and $\mathcal{B}$ are compact.*

Proof. Compactness of the set $\mathcal{A}\otimes\mathcal{B}$ implies compactness of the sets $\mathcal{A}$ and $\mathcal{B}$ due to continuity of partial trace.

Let $\mathcal{A}$ and $\mathcal{B}$ be compact. By the above proposition for arbitrary $\varepsilon>0$ there exist finite rank projectors P_ε and Q_ε such that

$$\mathrm{Tr}P_\varepsilon A>\mathrm{Tr}A-\varepsilon,\ \forall A\in\mathcal{A},\quad \text{and}\quad \mathrm{Tr}Q_\varepsilon B>\mathrm{Tr}B-\varepsilon,\ \forall B\in\mathcal{B}.$$

Since $C^{\mathcal{H}}=\mathrm{Tr}_{\mathcal{K}}C\in\mathcal{A}$ and $C^{\mathcal{K}}=\mathrm{Tr}_{\mathcal{H}}C\in\mathcal{B}$ for arbitrary $C\in\mathcal{A}\otimes\mathcal{B}$ we have

$$\mathrm{Tr}((P_\varepsilon\otimes Q_\varepsilon)\cdot C)=\mathrm{Tr}((P_\varepsilon\otimes I_{\mathcal{K}})\cdot C)-\mathrm{Tr}(P_\varepsilon\otimes(I_{\mathcal{K}}-Q_\varepsilon))\cdot C)$$

$$\geq \mathrm{Tr}P_\varepsilon C^{\mathcal{H}}-\mathrm{Tr}(I_{\mathcal{K}}-Q_\varepsilon)C^{\mathcal{K}}>\mathrm{Tr}C-2\varepsilon.$$

The above proposition implies compactness of the set $\mathcal{A}\otimes\mathcal{B}$.$\square$

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