On singular Bosonic linear channels*

M.E. Shirokov†
Steklov Mathematical Institute, RAS, Moscow

Abstract

Properties of Bosonic linear (quasi-free) channels, in particular, Bosonic Gaussian channels with two types of degeneracy are considered.

The first type of degeneracy can be interpreted as existence of noise-free canonical variables (for Gaussian channels it means that $\det\alpha = 0$). It is shown that this degeneracy implies existence of (infinitely many) "direct sum decompositions" of Bosonic linear channel, which clarifies reversibility properties of this channel (described in arXiv:1212.2354) and provides explicit construction of reversing channels.

The second type of degeneracy consists in rank deficiency of the operator describing transformations of canonical variables. It is shown that this degeneracy implies existence of (infinitely many) decompositions of input space into direct sum of orthogonal subspaces such that the restriction of Bosonic linear channel to each of these subspaces is a discrete classical-quantum channel.

1 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ – the Banach spaces of all bounded operators in $\mathcal{H}$ and of all trace-class operators in $\mathcal{H}$ correspondingly, $\mathfrak{S}(\mathcal{H})$ – the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of positive operators with unit trace called *states* [5].

*This work was partially supported by the program "Mathematical control theory and dynamical systems" of RAS and by the RFBR grants 12-01-00319a and 13-01-00295a.

†email:msh@mi.ras.ru

Denote by $I_{\mathcal{H}}$ and $\mathrm{Id}_{\mathcal{H}}$ the unit operator in a Hilbert space $\mathcal{H}$ and the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$ correspondingly.

A completely positive trace preserving linear map $\Phi:\mathfrak{T}(\mathcal{H}_A)\to\mathfrak{T}(\mathcal{H}_B)$ is called *quantum channel* [5].

A channel $\Phi:\mathfrak{T}(\mathcal{H}_A)\to\mathfrak{T}(\mathcal{H}_B)$ is called *classical-quantum of discrete type* (briefly, *discrete c-q channel*) if it has the following representation

$$\Phi(\rho)=\sum_{k=1}^{\dim\mathcal{H}_A}\langle k|\rho|k\rangle\sigma_k,\quad \rho\in\mathfrak{T}(\mathcal{H}_A), \tag{1}$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_A$ and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$.[1]

Let $\mathcal{H}_X$ ($X=A,B,...$) be the space of irreducible representation of the Canonical Commutation Relations (CCR)

$$W_X(z)W_X(z')=\exp\left[-\frac{\mathrm{i}}{2}\,z^\top\Delta_X z'\right]W_X(z'+z)$$

with a symplectic space (Z_X,Δ_X) and the Weyl operators $W_X(z)$ [5, Ch.12]. Denote by s_X the number of modes of the system X, i.e. $2s_X=\dim Z_X$.

We will use the Schrodinger representation of CCR: for a given symplectic basis $\{e_i,h_i\}$ in Z_X, we can identify the space $\mathcal{H}_X$ with the space $L_2(\mathbb{R}^{s_X})$ of complex-valued functions of s_X variables (which will be denoted $\xi_1,...,\xi_{s_X}$) and the Weyl operators $W_X(e_i)$ and $W_X(h_i)$ with the operators

$$\psi(\xi_1,...,\xi_{s_X})\mapsto e^{\mathrm{i}\xi_i}\psi(\xi_1,...,\xi_{s_X})\quad\text{and}\quad\psi(\xi_1,...,\xi_{s_A})\mapsto\psi(\xi_1,...,\xi_i+1,...,\xi_{s_X}).$$

A Bosonic linear channel $\Phi:\mathfrak{T}(\mathcal{H}_A)\to\mathfrak{T}(\mathcal{H}_B)$ is defined via the action of its dual $\Phi^*:\mathfrak{B}(\mathcal{H}_B)\to\mathfrak{B}(\mathcal{H}_A)$ on the Weyl operators:

$$\Phi^*(W_B(z))=W_A(Kz)f(z),\quad z\in Z_B, \tag{2}$$

where K is a linear operator $Z_B\to Z_A$, and $f(z)$ is a complex continuous function on Z_B such that $f(0)=1$ and the matrix with the elements $f(z_s-z_r)\exp\left(\frac{\mathrm{i}}{2}\,z_s^\top[\Delta_B-K^\top\Delta_A K]z_r\right)$ is positive for any finite subset $\{z_s\}$ of Z_B [2, 4, 5].[2] This channel will be denoted $\Phi_{K,f}$.

[1]We use the term "discrete" here, since in infinite dimensions there exist channels naturally called classical-quantum, which has no representation (1) [6].

[2]In [2] this channel is called quasi-free.

A very important class of Bosonic linear channels consists of Bosonic Gaussian channels defined by (2) with the Gaussian function

$$f(z) = \exp\left[\,\mathrm{i}l^\top z - \tfrac{1}{2} z^\top \alpha z\,\right],$$

where l is a $2s_B$-dimensional real row and α is a real symmetric $(2s_B)\times(2s_B)$ matrix satisfying the inequality

$$\alpha \geq \pm\frac{\mathrm{i}}{2}\left[\,\Delta_B - K^\top \Delta_A K\,\right].$$

Bosonic Gaussian channels play a central role in infinite-dimensional quantum information theory [3, 5].

Denote by Z_f the subset $f^{-1}(1) = \{z \in Z_B \,|\, f(z) = 1\}$. One can show that Z_f is a linear subspace of Z_B coinciding with $\ker\alpha$ in the case of Gaussian function f [5, Ch.12].

In this paper we consider properties of a Bosonic linear channel $\Phi_{K,f}$ with the following two types of degeneracy:

- $Z_f \doteq f^{-1}(1) \neq \{0\}$;
- $\mathrm{rank}K < \dim Z_A$ $\;(\mathrm{Ran}K \neq Z_A)$.

These types of degeneracy are related via the notion of a weak complementary channel (see detailed definition in [5, Ch.6]). Indeed, under the assumption of existence of Bosonic linear unitary dilation[3] for a channel $\Phi_{K,f}$ a weak complementary channel to $\Phi_{K,f}$ is a Bosonic linear channel $\Phi_{L,g}$ from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_E)$, where E is a Bosonic system-environment [1, 5, 9]. Lemma 2 in [9] implies[4]

$$\dim[\mathrm{Ran}L]^\perp = \dim Z_f, \quad \dim[\mathrm{Ran}K]^\perp = \dim Z_g.$$

Hence a channel $\Phi_{K,f}$ has the first type of degeneracy if and only if any[5] weak complementary channel to $\Phi_{K,f}$ has the second type of degeneracy and vice versa.

[3] A sufficient condition for existence of such dilation is given in [2]. For Bosonic Gaussian channels it is proved in [1] (see also [5, Ch.12]).

[4] Here " $\perp$ " denotes the skew-orthogonal complementary subspace. We will always use this sense of the symbol " $\perp$ " dealing with a subspace of a symplectic space.

[5] In contrast to complementary channel a weak complementary channel is not uniquely defined.

2 The case $Z_f \doteq f^{-1}(1) \neq \{0\}$ $\big(\det\alpha = 0\big)$

Physically, the condition $Z_f \neq \{0\}$ means (in the Heisenberg picture) that the channel $\Phi^*_{K,f}$ injects no noise in some canonical variables of the system B (which can be called noise-free canonical variables).

We will use the following simple observation (see e.g. Lemma 2 in [9]).

Lemma 1. *The restriction of the operator K to the subspace Z_f is non-degenerate and $\Delta_A(Kz_1, Kz_2) = \Delta_B(z_1, z_2)$ for all $z_1, z_2 \in Z_f$.*

Let Z_{A_0} and Z_{B_0} be minimal symplectic subspaces containing respectively $K(Z_f)$ and Z_f. By Lemma 1 $\dim Z_{A_0} = \dim Z_{B_0}$. We have

$$Z_X = Z_{X_0} \oplus Z_{X_*} \ \ (Z_{X_*} = [Z_{X_0}]^{\perp}), \quad \mathcal{H}_X = \mathcal{H}_{X_0} \otimes \mathcal{H}_{X_*}, \quad (X = A, B).$$

Since $W_B(z) = W_{B_0}(z) \otimes I_{B_*}$ and $W_A(Kz) = W_{A_0}(Kz) \otimes I_{A_*}$ for all $z \in Z_f$, the von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ generated respectively by the families $\{W_A(Kz)\}_{z \in Z_f}$ and $\{W_B(z)\}_{z \in Z_f}$ have the following forms

$$\mathcal{A} = \mathcal{A}_0 \otimes I_{A_*}, \quad \mathcal{B} = \mathcal{B}_0 \otimes I_{B_*},$$

where $\mathcal{A}_0$ and $\mathcal{B}_0$ are algebras acting respectively on $\mathcal{H}_{A_0}$ and on $\mathcal{H}_{B_0}$.

By Lemma 1 and Lemma 2 in the Appendix there exists a symplectic transformation $T : Z_{B_0} \to Z_{A_0}$ such that $Kz = Tz$ for all $z \in Z_f$. Hence $W_{A_0}(Kz) = U_T W_{B_0}(z) U_T^*$ for all $z \in Z_f$, where U_T is the unitary operator implementing T. It follows that the algebras $\mathcal{A}_0$ and $\mathcal{B}_0$ are unitary equivalent, i.e. $\mathcal{A}_0 = U_T \mathcal{B}_0 U_T^*$.

Since $\Phi^*_{K,f}(W_{B_0}(z) \otimes I_{B_*}) = W_{A_0}(Kz) \otimes I_{A_*}$ for all $z \in Z_f$ (by definition of the channel $\Phi_{K,f}$), the restriction of the dual channel $\Phi^*_{K,f}$ to the algebra $\mathcal{B}$ coincides with the isomorphism

$$\mathcal{B} = \mathcal{B}_0 \otimes I_{B_*} \ni Y \otimes I_{B_*} \ \mapsto \ [U_T Y U_T^*] \otimes I_{A_*} \in \mathcal{A}_0 \otimes I_{A_*} = \mathcal{A}$$

between the algebras $\mathcal{B}$ and $\mathcal{A}$.

It follows, in particular, that for an arbitrary projector $P \in \mathcal{A}$ there exists a projector $Q \in \mathcal{B}$ such that $P = \Phi^*_{K,f}(Q)$. It is easy to see that this relation means that $\Phi_{K,f}\left(\mathfrak{T}(P(\mathcal{H}_A))\right) \subseteq \mathfrak{T}(Q(\mathcal{H}_B))$. So, we obtain the following observation.

Proposition 1. *Let $\Phi_{K,f}$ be a Bosonic channel with $Z_f \neq \{0\}$. For an arbitrary orthogonal resolution of the identity* [6] $\{P_k\} \subset \mathcal{A} \doteq \left[\{W_A(Kz)\}_{z\in Z_f}\right]''$ *there exists an orthogonal resolution of the identity* $\{Q_k\} \subset \mathcal{B} \doteq \left[\{W_B(z)\}_{z\in Z_f}\right]''$ *such that $P_k = \Phi^*_{K,f}(Q_k)$ for all k and hence*

$$\Phi_{K,f}\left(\mathfrak{T}(\mathcal{H}_A^k)\right) \subseteq \mathfrak{T}(\mathcal{H}_B^k) \quad \forall k, \tag{3}$$

where $\mathcal{H}_A^k = P_k(\mathcal{H}_A)$ and $\mathcal{H}_B^k = Q_k(\mathcal{H}_B)$ (so that $\mathcal{H}_X = \bigoplus_k \mathcal{H}_X^k, X = A, B$).

Remark 1. If Z_f^c is a nontrivial isotropic subspace of Z_f then the algebra $\mathcal{A}^c = [\{W_A(Kz)\}_{z\in Z_f^c}]''$ is commutative and isomorphic to the algebra $L_\infty(\mathbb{R}^d)$, where $d = \dim Z_f^c$. It follows that any element of an orthogonal resolution of the identity $\{P_k\} \subset \mathcal{A}^c \subseteq \mathcal{A}$ can be represented as a sum of mutually orthogonal projectors in $\mathcal{A}^c$. Hence, by Proposition 1, any subspace of the corresponding decomposition $\mathcal{H}_A = \bigoplus_k \mathcal{H}_A^k$ can be, in turn, decomposed into direct sum of orthogonal subspaces, for each of which the invariance relation similar to (3) holds.

A quantum channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is called *reversible* (or sufficient) with respect to a family $\mathfrak{S}$ of states in $\mathfrak{S}(\mathcal{H}_A)$ if there exists a quantum channel $\Psi : \mathfrak{T}(\mathcal{H}_B) \to \mathfrak{T}(\mathcal{H}_A)$ such that $\Psi(\Phi(\rho)) = \rho$ for all $\rho \in \mathfrak{S}$ [7, 8].

The above family $\mathfrak{S}$ is naturally called *reversed family* for the channel Φ, while the channel Ψ may be called *reversing channel*.

Necessary and sufficient conditions for reversibility of Bosonic linear channels with respect to orthogonal and non-orthogonal families of pure states (as well as explicit forms of reversed families) are explored in [9] by using the "method of complementary channel".

Proposition 1 clarifies the sufficiency in the "orthogonal part" of these conditions. Moreover, it shows reversibility of the channel $\Phi_{K,f}$ such that $Z_f \neq \{0\}$ with respect to particular orthogonal families of states (not necessarily pure) and provides explicit description of reversing channels.

Corollary 1. *Let $\Phi_{K,f}$ be a Bosonic linear channel such that $Z_f \neq \{0\}$ and $\{P_k\}$ an orthogonal resolution of the identity in* $\mathcal{A} \doteq \left[\{W_A(Kz)\}_{z\in Z_f}\right]''$. *The channel $\Phi_{K,f}$ is reversible with respect to any family $\{\rho_k\}$ of states in $\mathfrak{S}(\mathcal{H}_A)$ such that $\mathrm{supp}\rho_k \subseteq \mathcal{H}_A^k = P_k(\mathcal{H}_A)$. The simplest reversing channel*

[6] An orthogonal resolution of the identity is a family of mutually orthogonal projectors whose sum coincides with the identity operator.

for the family $\{\rho_k\}$ *has the form*

$$\Psi(\sigma) = \sum_k [\mathrm{Tr} Q_k \sigma]\rho_k, \quad \sigma \in \mathfrak{S}(\mathcal{H}_B),$$

where $\{Q_k\}$ *is the orthogonal resolution of the identity in* $\mathcal{B} \doteq \left[\{W_B(z)\}_{z\in Z_f}\right]''$ *described in Proposition 1.*

Corollary 1 gives an explicit proof of the part of Theorem 2 in [9], which states reversibility of the channel $\Phi_{K,f}$ with respect to non-complete orthogonal families of pure states provided that Z_f is a nontrivial isotropic subspace of Z_B. It also shows sufficiency of the condition obtained in Section 4.3 in [9] describing all reversed families in this case.

Indeed, if Z_f is a nontrivial isotropic subspace of Z_B then the above-defined algebras $\mathcal{A} = \mathcal{A}_0 \otimes I_{A_*}$ and $\mathcal{B} = \mathcal{B}_0 \otimes I_{B_*}$ are commutative and in the Schrodinger representation (described in Section 1) $\mathcal{A}_0 \cong \mathcal{B}_0 \cong L_\infty(\mathbb{R}^d)$, where $d = \dim Z_f$. For $X = A, B$ the algebra $\mathcal{X} = \mathcal{X}_0 \otimes I_{X_*}$ acts on the space $\mathcal{H}_X \cong L_2(\mathbb{R}^{s_X})$ as follows:

$$(F \otimes I_{X_*}\psi)(\xi_1, \ldots, \xi_{s_X}) = F(\xi_1, \ldots, \xi_d)\psi(\xi_1, \ldots, \xi_{s_X}),\ F \in \mathcal{X}_0 \cong L_\infty(\mathbb{R}^d). \quad (4)$$

Since projectors in $L_\infty(\mathbb{R}^d)$ correspond to indicator functions of subsets of $\mathbb{R}^d$, any orthogonal resolutions of the identity $\{P_k\}$ and $\{Q_k\}$ involved in Proposition 1 correspond to a decomposition $\{D_k\}$ of $\mathbb{R}^d$ into disjoint measurable subsets. So, $\mathcal{H}_A^k = P_k(\mathcal{H}_A) = L_2(D_k \times \mathbb{R}^{s_A - d})$ is the subspace of $\mathcal{H}_A = L_2(\mathbb{R}^{s_A})$ consisting of functions vanishing almost everywhere outside the cylinder

$$D_k \times \mathbb{R}^{s_A - d} = \{(\xi_1, \ldots, \xi_{s_A}) \,|\, (\xi_1, \ldots, \xi_d) \in D_k\}.$$

while $\mathcal{H}_B^k = Q_k(\mathcal{H}_B) = L_2(D_k \times \mathbb{R}^{s_B - d})$ is the subspace of $\mathcal{H}_B = L_2(\mathbb{R}^{s_B})$ consisting of functions vanishing almost everywhere outside the cylinder

$$D_k \times \mathbb{R}^{s_B - d} = \{(\xi_1, \ldots, \xi_{s_B}) \,|\, (\xi_1, \ldots, \xi_d) \in D_k\}.$$

Proposition 1 asserts that all states supported by $\mathcal{H}_A^k$ are transformed by the channel $\Phi_{K,f}$ into states supported by $\mathcal{H}_B^k$.

It follows (as stated in Corollary 1) that any family $\{|\psi_k\rangle\langle\psi_k|\}$ such that $\psi_k \in L_2(D_k \times \mathbb{R}^{s_A - d})$ for each k is a reversed family for the channel $\Phi_{K,f}$ and the role of reversing channel is played by the map $\sigma \mapsto \sum_k [\mathrm{Tr} Q_k \sigma] |\psi_k\rangle\langle\psi_k|$. The arguments from Subsection 4.3 in [9] (the case $\mathrm{ri}(\Phi_{K,f}) = 01$) show that *all* reversed families for the channel $\Phi_{K,f}$ have the above-described form.

3 The case $\mathrm{rank}K < \dim Z_A$

It is shown in [9, Proposition 3] that the condition $\mathrm{rank}K < \dim Z_A$ is equivalent to existence of discrete c-q subchannels of the channel $\Phi_{K,f}$. The following proposition strengthens this observation.

Proposition 2. *If* $\mathrm{rank}K < \dim Z_A$ *then for arbitrary given* $m \leq +\infty$ *there exists a decomposition* $\mathcal{H}_A = \bigoplus_{k=1}^{+\infty} \mathcal{H}_A^k$ *such that* $\dim \mathcal{H}_A^k = m$ *and*

$$\Phi_{K,f}|_{\mathfrak{T}(\mathcal{H}_A^k)} \textit{ is a discrete c-q channel for each } k.$$

Any such decomposition is determined by the following parameters:

- *non-trivial isotropic subspace* $Z_0 \subseteq [\mathrm{Ran}K]^{\perp}$*;*
- *non-degenerate decomposition* $\{D_i\}_{i=1}^m$ *of* $\mathbb{R}^d$*, where* $d = \dim Z_0$*;*
- *collection* $\{E_i\}_{i=1}^m$*, where* $E_i = \{|e_k^i\rangle\}_{k=1}^{+\infty}$ *is an ONB in* $L_2(D_i \times \mathbb{R}^{s_A - d})$*;*
- *collection* $\{\pi_i\}_{i=1}^m$*, where* π_i *is a permutation of* $\mathbb{N}$.

For a given choice of these parameters,

$$\Phi_{K,f}(\rho) = \sum_{i=1}^{m} \langle e_{\pi_i(k)}^i|\rho|e_{\pi_i(k)}^i\rangle \sigma_k^i, \quad \rho \in \mathfrak{S}(\mathcal{H}_A^k), \tag{5}$$

for each k*, where* $\{\sigma_k^i\}$ *is a collection of states in* $\mathcal{H}_B$.

Proof. Let Z_0 be a non-trivial isotropic subspace of $[\mathrm{Ran}K]^{\perp}$. Then the commutative von Neumann algebra $\mathcal{A}_0 = [\{W_A(z)\}_{z\in Z_0}]''$ is contained in the algebra $[\{W_A(Kz)\}_{z\in Z_B}]'$. In the Schrodinger representation the algebra $\mathcal{A}_0$ coincides with the algebra $L_\infty(\mathbb{R}^d)$, where $d = \dim Z_0$, acting on the space $\mathcal{H}_A = L_2(\mathbb{R}^{s_A})$ in accordance with formula (4). Hence an arbitrary orthogonal resolution of the identity $\{P_i\}_{i=1}^m$ in $\mathcal{A}_0$ corresponds to a decomposition $\{D_i\}_{i=1}^m$ of $\mathbb{R}^d$ into disjoint measurable subsets (in the sense that the projector P_i corresponds to the indicator function of the set D_i).

For a given decomposition $\{D_i\}_{i=1}^m$ of $\mathbb{R}^d$ choose collections $\{E_i\}_{i=1}^m$ and $\{\pi_i\}_{i=1}^m$, where $E_i = \{|e_k^i\rangle\}_{k=1}^{+\infty}$ is an orthonormal basis in $L_2(D_i \times \mathbb{R}^{s_A-d}) = P_i(\mathcal{H}_A)$ and π_i is a permutation of $\mathbb{N}$. For each $k \in \mathbb{N}$ let $\mathcal{H}_A^k$ be the subspace of $\mathcal{H}_A$ generated by the family $\{|e_{\pi_i(k)}^i\rangle\}_{i=1}^m$. Since $\{P_i\}_{i=1}^m \subset [\{W_A(Kz)\}_{z\in Z_B}]'$, we have

$$\langle e_{\pi_i(k)}^i|W_A(Kz)|e_{\pi_j(k)}^j\rangle = 0 \quad \text{for all } i \neq j \text{ and all } k.$$

By Lemma 3 in [9, Appendix 6.1] the subchannel of the channel $\Phi_{K,f}$ corresponding to the subspace $\mathcal{H}_A^k$ is a discrete c-q channel for each k having representation (5). $\square$

Corollary 2. *Let $\mathcal{H}_A = \bigoplus_{k=1}^{+\infty} \mathcal{H}_A^k$ be a given decomposition from Proposition 2 and P_k the projector onto $\mathcal{H}_A^k$ for each k. The channel $\Phi_{K,f}$ coincides with the discrete c-q channel*

$$\rho \mapsto \sum_{k=1}^{+\infty} \sum_{i=1}^{m} \langle e^i_{\pi_i(k)} | \rho | e^i_{\pi_i(k)} \rangle \sigma^i_k$$

on the set $\{\rho \in \mathfrak{S}(\mathcal{H}_A) \,|\, \rho = \Pi(\rho)\}$, where $\Pi(\rho) = \sum_{k=1}^{+\infty} P_k \rho P_k$.

Corollary 2 implies, in particular, that $\Phi_{K,f} \circ \Pi$ is a discrete c-q channel.

Appendix

Lemma 2. *Let Z_0 be an arbitrary subspace of Z and $K : Z_0 \to Z$ a linear map such that $Kz_1 \neq Kz_2$ and $\Delta(Kz_1, Kz_2) = \Delta(z_1, z_2)$ for all $z_1 \neq z_2$ in Z_0. There exists a symplectic transformation $T : Z \to Z$ such that $Kz = Tz$ for all $z \in Z_0$.*[7]

Proof. We may assume that the subspace Z_0 is not symplectic (since otherwise the assertion of the lemma is trivial).

By Lemma 6 in [9, Appendix 6.2] there exists a symplectic basis $\{e_i, h_i\}$ in Z such that $\{e_i, h_i\}_{i=1}^d \cup \{e_i\}_{i=d+1}^p$ is a basis in Z_0. To prove the lemma it suffices to show that the set of vectors $\{Ke_i, Kh_i\}_{i=1}^d \cup \{Ke_i\}_{i=d+1}^p$ can be extended to a symplectic basis in Z. By the property of the map K the set $\{Ke_i, Kh_i\}_{i=1}^d$ is a symplectic basis in the linear hull Z_d of this set and $\{Ke_i\}_{i=d+1}^p \subset [Z_d]^\perp$. So, we have to find a set of vectors $\{\tilde{h}_i\}_{i=d+1}^p$ in $[Z_d]^\perp$ such that $\Delta(Ke_i, \tilde{h}_j) = \delta_{ij}$, $i, j = \overline{d+1, p}$. This set can be constructed sequentially: the vector $\tilde{h}_{d+1}$ can be chosen in the subspace

$$[Z_d]^\perp \cap \left[\left[\{Ke_i\}_{i=d+2}^p \right]^\perp \setminus \left[\{Ke_i\}_{i=d+1}^p \right]^\perp \right],$$

the vector $\tilde{h}_{d+2}$ – in the subspace

$$[Z_{d+1}]^\perp \cap \left[\left[\{Ke_i\}_{i=d+3}^p \right]^\perp \setminus \left[\{Ke_i\}_{i=d+2}^p \right]^\perp \right],$$

[7] I would be grateful for a direct reference on this result.

where $Z_{d+1} = Z_d \oplus \mathrm{lin}[Ke_{d+1}, \tilde{h}_{d+1}]$, etc. $\square$

I am grateful to A.S.Holevo for useful discussion.

References

[1] F.Caruso, J.Eisert, V.Giovannetti, A.S.Holevo "Multi-mode bosonic Gaussian channels", New Journal of Physics 10, 083030, 2008; arXiv:0804.0511.

[2] B.Demoen, P.Vanheuverzwijn, A.Verbeure "Completely positive quasi-free maps on the CCR algebra", Rep. Math. Phys. V.15, P.27-39, 1979.

[3] J.Eisert, M.M.Wolf "Gaussian quantum channels", Quantum Information with Continuous Variables of Atoms and Light, P.23-42 (Imperial College Press, London, 2007); arXiv:quant-ph/0505151.

[4] A.S.Holevo "Towards the mathematical theory of quantum communication channels", Probl. Inform. Transm. V.8, N.1, P.63-71, 1972.

[5] A.S.Holevo "Quantum systems, channels, information. A mathematical introduction", Berlin, DeGruyter, 2012.

[6] A.S.Holevo "Gaussian classical-quantum channels: gain of entanglement-assistance", arXiv:1211.4774.

[7] A.Jencova, D.Petz "Sufficiency in quantum statistical inference", Commun. Math. Phys. 263, P.259-276, 2006; arXiv:math-ph/0412093.

[8] A.Jencova "Reversibility conditions for quantum operations", arXiv:1107.0453.

[9] M.E.Shirokov "On reversibility of Bosonic linear channels with respect to families of pure states", arXiv:1212.2354.