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Entropy characteristics of subsets of states. II

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Abstract. We study properties of the χ-capacity (regarded as a function of sets of quantum states) in the infinite-dimensional case. We consider various subsets of states and determine their χ-capacity and optimal average. We construct counterexamples that illustrate general results. The possibility of "finite-dimensional approximations" of the χ-capacity and optimal average is shown for an arbitrary set of quantum states.

§ 1. Introduction

This paper is devoted to a systematic study of properties of the quantum entropy and χ-capacity.[1] It is a continuation of [11].

In § 2 we study general properties of the χ-capacity as a function of sets of states (Theorem 1 and Corollaries 1, 4, 5). In particular, we consider the question of the continuity of the χ-capacity with respect to monotone families of sets and the problem of the existence of a minimal closed set of a given χ-capacity. We prove that the χ-capacity and optimal average are stable with respect to quantum noise. We obtain lower and upper bounds for the χ-capacity of finite unions (Proposition 1, Remark 3). The results related to χ-capacity enable us to make several observations on general properties of sets of states and quantum entropy (Corollaries 2, 3, Remark 2, and the observation after Corollary 4).

In § 3 we apply general results of [11] and § 2 of this paper to study various sets of states. This yields conditions for the boundedness and continuity of the restriction of the quantum entropy to various sets of states (Propositions 2, 8, 10 and Corollary 6). We determine the χ-capacity and optimal average of several sets of states and study related questions, such as the existence of an optimal measure and regularity (Propositions 3–7, 9, 10). The following examples of sets of finite χ-capacity are constructed in §§ 3.1–3.3 and 3.5 respectively:

i) a closed countable set having no optimal measure,

ii) a closed set having no minimal closed subset of the same χ-capacity,

iii) a decreasing sequence of closed sets of the same positive χ-capacity whose intersection has χ-capacity zero,

iv) a closed set that has an optimal measure but no atomic optimal measure.

[1]This notion is referred to as *the Holevo capacity* in the Western literature and is usually associated with the notion of a quantum channel (see, for example, [9]).

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In § 4 we consider a "constructive" approach to the definition of χ-capacity and optimal average for an arbitrary set of quantum states. We show that one can define these notions for sets of states in a finite-dimensional Hilbert space and then extend them to sets of states in an infinite-dimensional Hilbert space by a limiting procedure, as in the case of entropy and relative entropy (Theorem 2). This definition leads in principle to the possibility of numerical approximations for the χ-capacity and optimal average of any set of quantum states.

The notation in this paper corresponds to that used in [11].

§ 2. Properties of the χ-capacity

In this section we consider general properties of the χ-capacity regarded as a function of sets of quantum states and the special role of the optimal average. A sufficient condition for the existence of an optimal measure for a closed set of states is obtained in [11] (Theorem 2). It requires one of two continuity conditions to hold. We shall see that these conditions also guarantee other properties related to χ-capacity. It is thus convenient to introduce the following notion.

Definition 1. An arbitrary set $\mathcal{A}$ of finite χ-capacity is said to be *regular* if one of the following conditions holds.

1) $H\big(\Omega(\mathcal{A})\big) < +\infty$ and $\lim_{n\to+\infty} H(\rho_n) = H\big(\Omega(\mathcal{A})\big)$ for any sequence $\{\rho_n\}$ of states in $\mathrm{co}(\mathcal{A})$ that H-converges [2] to the state $\Omega(\mathcal{A})$.

2) The function $\rho \mapsto H\big(\rho \,\|\, \Omega(\mathcal{A})\big)$ is continuous [3] on the set $\overline{\mathcal{A}}$.

Note that the continuity of the entropy on the set $\overline{\mathrm{co}}(\mathcal{A})$ guarantees the regularity of $\mathcal{A}$, but this requirement is very restrictive. In a sense, the conditions of the definition are the minimal continuity requirements that guarantee "good" properties of the χ-capacity. In particular, these conditions imply that an optimal measure exists by Theorem 2 of [11]. The two conditions are different: there are sets such that the first condition holds but the second does not, and vice versa. In the examples in § 3, most sets of finite χ-capacity are regular. Examples of irregular sets of finite χ-capacity and consequences of their irregularity are considered in §§ 3.1–3.3.

The following theorem summarizes the properties of the χ-capacity and optimal average to be used later.

Theorem 1. *The following properties hold.*[4]

1) $\overline{C}(\mathcal{A}) \geqslant 0$ *for any set* $\mathcal{A}$*, and equality holds if and only if* $\mathcal{A}$ *consists of a single point.*

2) $\overline{C}(\mathcal{A}) = \overline{C}\big(\overline{\mathrm{co}}(\mathcal{A})\big)$ *and* $\Omega(\mathcal{A}) = \Omega\big(\overline{\mathrm{co}}(\mathcal{A})\big)$ *for any set* $\mathcal{A}$.

3) *If* $\mathcal{A} \subseteq \mathcal{B}$*, then* $\overline{C}(\mathcal{A}) \leqslant \overline{C}(\mathcal{B})$*, and equality*[5] *implies that* $\Omega(\mathcal{A}) = \Omega(\mathcal{B})$.

4) *If* $\overline{C}(\mathcal{A}) < +\infty$*, then* $\mathcal{A}$ *is relatively compact and hence* $\overline{C}(\mathcal{A}) = \overline{C}\big(\mathrm{Ext}(\overline{\mathcal{A}})\big)$.

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[2] This means that $\lim_{n\to+\infty} H\big(\rho_n \,\|\, \Omega(\mathcal{A})\big) = 0$.

[3] Here and below we mean continuity with respect to the trace norm topology.

[4] In all statements concerning the optimal average state of any particular set it is assumed that this set has finite χ-capacity.

[5] Note that $\mathcal{A} \subsetneq \mathcal{B}$ does not imply that $\overline{C}(\mathcal{A}) < \overline{C}(\mathcal{B})$, even when $\mathcal{A}$ and $\mathcal{B}$ are closed and convex (see the examples in § 3).

5) *If* $\mathrm{d}\bigl(\Omega(\mathcal{A})\bigr)<1$, *then the set* $\mathcal{A}$ *is regular and the entropy is bounded on* $\overline{\mathrm{co}}(\mathcal{A})$. *If* $\mathrm{d}\bigl(\Omega(\mathcal{A})\bigr)=0$, *then the entropy is continuous on* $\overline{\mathrm{co}}(\mathcal{A})$.

6) *Let* $\{\mathcal{A}_n\}$ *be a sequence of sets such that* $\mathcal{A}_n\subseteq\mathcal{A}_{n+1}$ *for all* n. *Then*

$$\lim_{n\to+\infty}\overline{C}(\mathcal{A}_n)=\overline{C}\biggl(\bigcup_n\mathcal{A}_n\biggr),\qquad \lim_{n\to+\infty}\Omega(\mathcal{A}_n)=\Omega\biggl(\bigcup_n\mathcal{A}_n\biggr).$$

7) *Let* $\{\mathcal{A}_n\}$ *be a sequence of closed sets such that* $\mathcal{A}_n\supseteq\mathcal{A}_{n+1}$ *for all* n. *Then*

$$\lim_{n\to+\infty}\overline{C}(\mathcal{A}_n)=\overline{C}\biggl(\bigcap_n\mathcal{A}_n\biggr),\qquad \lim_{n\to+\infty}\Omega(\mathcal{A}_n)=\Omega\biggl(\bigcap_n\mathcal{A}_n\biggr)$$

provided that one of the following conditions holds.[6]

a) *The set* $\mathcal{A}_1$ *is regular and* $\Omega(\mathcal{A}_n)=\Omega(\mathcal{A}_1)$ *for all* n.

b) *The restriction of the entropy* $H(\rho)$ *to the set* $\overline{\mathrm{co}}(\mathcal{A}_1)$ *is continuous at some limit point*[7] ω *of the sequence* $\{\Omega(\mathcal{A}_n)\}$.

c) *The function* $\rho\mapsto H(\rho\,\|\,\omega)$ *is continuous on* $\mathcal{A}_1$ *for some limit point* ω *of the sequence* $\{\Omega(\mathcal{A}_n)\}$.

8) *Every set* $\mathcal{A}$ *of finite* χ*-capacity is contained in a maximal set* $\mathcal{V}_{\Omega(\mathcal{A}),\overline{C}(\mathcal{A})}$ *of the same* χ*-capacity.*[8]

9) *Every regular closed set* $\mathcal{A}$ *of finite* χ*-capacity contains a minimal closed subset of the same* χ*-capacity.*[9]

10) *If* $\overline{C}(\mathcal{A})<+\infty$ *and* $\overline{C}(\mathcal{B})<+\infty$, *then* $\overline{C}(\mathcal{A}\cup\mathcal{B})<+\infty$. *In particular, we have* $\overline{C}(\mathcal{A}\cup\mathcal{B})=\max\bigl(\overline{C}(\mathcal{A}),\overline{C}(\mathcal{B})\bigr)$ *when* $\Omega(\mathcal{A})=\Omega(\mathcal{B})$.

11) *If* $\Phi\colon\mathfrak{S}(\mathcal{H})\mapsto\mathfrak{S}(\mathcal{H}')$ *is an arbitrary channel, then* $\overline{C}\bigl(\Phi(\mathcal{A})\bigr)\leqslant\overline{C}(\mathcal{A})$, *and* $\overline{C}\bigl(\Phi(\mathcal{A})\bigr)=\overline{C}(\mathcal{A})$ *implies that* $\Omega\bigl(\Phi(\mathcal{A})\bigr)=\Phi\bigl(\Omega(\mathcal{A})\bigr)$.

12) *If* $\{\Phi_t\}_{t\in\mathbb{R}_+}$ *is an arbitrary family of channels from* $\mathfrak{S}(\mathcal{H})$ *to* $\mathfrak{S}(\mathcal{H})$ *such that* $\lim_{t\to+0}\Phi_t(\rho)=\rho$ *for all states* ρ *in* $\mathcal{A}$, *then*[10]

$$\lim_{t\to+0}\overline{C}\bigl(\Phi_t(\mathcal{A})\bigr)=\overline{C}(\mathcal{A}),\qquad \lim_{t\to+0}\Omega\bigl(\Phi_t(\mathcal{A})\bigr)=\Omega(\mathcal{A}).$$

Remark 1. The regularity and continuity requirements are essential in assertions 7) and 9) of Theorem 1. There are sequences of sets for which assertion 7) does not hold (see the example at the end of § 3.3). In § 3.2 we consider an example of a closed set of finite χ-capacity having no minimal closed subset of the same χ-capacity.

Proof. Assertions 1), 2) and the first part of 3) follow directly from the definition of χ-capacity since the relative entropy is lower semicontinuous and convex. The second part of 3) is proved as follows. Suppose that $\mathcal{A}\subseteq\mathcal{B}$ and $\overline{C}(\mathcal{A})=\overline{C}(\mathcal{B})$.

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[6] These conditions are essential (see Remark 1 below).

[7] The set of limit points of $\{\Omega(\mathcal{A}_n)\}$ is non-empty by assertion 4).

[8] A set is called a *maximal set of a given* χ*-capacity* if it is not a proper subset of a set of the same χ-capacity.

[9] A set is called a *minimal closed set of a given* χ*-capacity* if it has no proper closed subsets of the same χ-capacity.

[10] This assertion may be regarded as a stability property of the χ-capacity and optimal average state with respect to quantum noise.

By Theorem 1 of [11], the inequality $H\big(\rho\,\|\,\Omega(\mathcal{B})\big) \leqslant \overline{C}(\mathcal{B}) = \overline{C}(\mathcal{A})$ holds for all states ρ in $\mathcal{B}$. Since $\mathcal{A} \subseteq \mathcal{B}$, this inequality holds for all states ρ in $\mathcal{A}$. Thus the uniqueness assertion of Theorem 1 in [11] yields that $\Omega(\mathcal{A}) = \Omega(\mathcal{B})$.

The first part of 4) is obtained in [11], Corollary 5. The second part follows from assertion 2) and the Krein–Mil'man theorem.

Theorem 1 of [11] implies that $\overline{\mathrm{co}}(\mathcal{A}) \subseteq \mathcal{V}_{\Omega(\mathcal{A}),\overline{C}(\mathcal{A})}$. Therefore assertion 5) follows from Propositions 2 and 3 of [11].

To prove 6), we note that 3) yields the existence of the limit and the inequality

$$\lim_{n\to+\infty} \overline{C}(\mathcal{A}_n) \leqslant \overline{C}\biggl(\bigcup_n \mathcal{A}_n\biggr). \tag{1}$$

Let $\{\{\pi_i^k, \rho_i^k\}\}_k$ be an arbitrary approximating sequence of ensembles for the set $\bigcup_n \mathcal{A}_n$. This means that

$$\lim_{k\to+\infty} \chi(\{\pi_i^k, \rho_i^k\}) = \overline{C}\biggl(\bigcup_n \mathcal{A}_n\biggr). \tag{2}$$

Since each ensemble is a *finite* set of states, we see that for every k there is an $n(k)$ such that $\rho_i^k \in \mathcal{A}_{n(k)}$ for all i and, therefore, $\overline{C}(\mathcal{A}_{n(k)}) \geqslant \chi(\{\pi_i^k, \rho_i^k\})$. This and (2) imply that an equality holds in (1).

Suppose that $\overline{C}\big(\bigcup_n \mathcal{A}_n\big) = \overline{C}\big(\overline{\mathrm{co}}\big(\bigcup_n \mathcal{A}_n\big)\big) < +\infty$. The set $\overline{\mathrm{co}}\big(\bigcup_n \mathcal{A}_n\big)$ is compact by assertion 4). Hence the sequence $\{\Omega(\mathcal{A}_n)\}$ has limit points. Let $\omega = \lim_{k\to+\infty} \Omega(\mathcal{A}_{n_k})$ for some subsequence n_k.

By Theorem 1 of [11], for every n there is an ensemble $\{\pi_i^n, \rho_i^n\}$ of states in $\mathcal{A}_n$ with average state $\overline{\rho}_n$ such that

$$\chi(\{\pi_i^n, \rho_i^n\}) \geqslant \overline{C}(\mathcal{A}_n) - \frac{1}{n}\,, \qquad \|\overline{\rho}_n - \Omega(\mathcal{A}_n)\|_1 \leqslant \frac{1}{n}\,. \tag{3}$$

Using the fact that (1) is an equality (already proved) and the first inequality in (3), the sequence $\{\{\pi_i^n, \rho_i^n\}\}_n$ is approximating for the set $\bigcup_n \mathcal{A}_n$. Hence Theorem 1 of [11] shows that the sequence $\{\overline{\rho}_n\}_n$ converges to the state $\Omega\big(\bigcup_n \mathcal{A}_n\big)$ as $n \to +\infty$. By the second inequality in (3), the subsequence $\{\overline{\rho}_{n_k}\}_k$ converges to the state ω. Therefore $\omega = \Omega\big(\bigcup_n \mathcal{A}_n\big)$. Thus every limit point of the sequence $\{\Omega(\mathcal{A}_n)\}$ coincides with $\Omega\big(\bigcup_n \mathcal{A}_n\big)$.

To prove 7), we note that 3) implies the existence of the limit and the inequality

$$\lim_{n\to+\infty} \overline{C}(\mathcal{A}_n) \geqslant \overline{C}\biggl(\bigcap_n \mathcal{A}_n\biggr). \tag{4}$$

We claim that the additional conditions in 7) provide different ways of proving that equality holds in (4).

We first consider conditions b) and c). Without loss of generality, we may assume that the limit

$$\lim_{n\to+\infty} \Omega(\mathcal{A}_n) = \omega \tag{5}$$

exists.

Theorem 1 of [11] yields that for every positive integer n there is a measure μ_n finitely supported by $\mathcal{A}_n$ such that

$$\chi(\mu_n) \geqslant \overline{C}(\mathcal{A}_n) - \frac{1}{n}, \qquad \|\overline{\rho}(\mu_n) - \Omega(\mathcal{A}_n)\|_1 \leqslant \frac{1}{n}. \tag{6}$$

The supports of all the measures in $\{\mu_n\}$ lie in the set $\mathcal{A}_1$, which is compact by assertion 4). Hence the sequence $\{\mu_n\}$ is compact in the weak topology and contains a subsequence $\{\mu_{n_k}\}$ weakly converging to some measure μ_*. Since the map $\mu \mapsto \overline{\rho}(\mu)$ is continuous, relations (5) and (6) imply that $\omega = \overline{\rho}(\mu_*) = \lim_{k\to+\infty} \overline{\rho}(\mu_{n_k})$. Using Theorem 6.1 of [7], we easily see that $\operatorname{supp}\mu_* \subseteq \bigcap_n \mathcal{A}_n$.

Suppose that condition b) holds in 7). Then the limit

$$\lim_{k\to+\infty} H\bigl(\overline{\rho}(\mu_{n_k})\bigr) = H\bigl(\overline{\rho}(\mu_*)\bigr) = H(\omega) < +\infty \tag{7}$$

exists. Using formula (2) of [11], we get

$$\chi(\mu_{n_k}) = H\bigl(\overline{\rho}(\mu_{n_k})\bigr) - \widehat{H}(\mu_{n_k})$$

for all sufficiently large k. Using (7) and the lower semicontinuity of the functional $\widehat{H}(\mu)$, we get

$$\begin{aligned}\lim_{n\to+\infty} \overline{C}(\mathcal{A}_n) &= \limsup_{k\to+\infty} \chi(\mu_{n_k}) = \lim_{k\to+\infty} H\bigl(\overline{\rho}(\mu_{n_k})\bigr) - \liminf_{k\to+\infty} \widehat{H}(\mu_{n_k}) \\ &\leqslant H\bigl(\overline{\rho}(\mu_*)\bigr) - \widehat{H}(\mu_*) = \chi(\mu_*) \leqslant \overline{C}\biggl(\bigcap_n \mathcal{A}_n\biggr),\end{aligned}$$

which guarantees that equality holds in (4).

Suppose that condition c) holds in 7). Since this condition means that the function $H(\rho\,\|\,\omega)$ is continuous on the compact set $\mathcal{A}_1$, the definition of weak convergence yields that

$$\lim_{k\to+\infty} \int H(\rho\,\|\,\omega)\mu_{n_k}(d\rho) = \int H(\rho\,\|\,\omega)\mu_*(d\rho) = \chi(\mu_*) \leqslant \overline{C}\biggl(\bigcap_n \mathcal{A}_n\biggr).$$

Using the generalized Donald identity (formula (4) of [11]), we have

$$\int H(\rho\,\|\,\omega)\mu_{n_k}(d\rho) = \chi(\mu_{n_k}) + H(\overline{\rho}(\mu_{n_k})\,\|\,\omega) \geqslant \chi(\mu_{n_k})$$

and the above inequality yields that

$$\overline{C}\biggl(\bigcap_n \mathcal{A}_n\biggr) \geqslant \lim_{k\to+\infty} \int H(\rho\,\|\,\omega)\mu_{n_k}(d\rho) \geqslant \lim_{k\to+\infty} \chi(\mu_{n_k}) = \lim_{n\to+\infty} \overline{C}(\mathcal{A}_n),$$

which implies equality in (4).

To complete the consideration of conditions b) and c) in assertion 7), it suffices to show that the limit state ω in (5) is the optimal average state of the set $\bigcap_n \mathcal{A}_n$. By Theorem 1 in [11] we have $H\bigl(\rho\,\|\,\Omega(\mathcal{A}_n)\bigr) \leqslant \overline{C}(\mathcal{A}_n)$ for any state ρ in $\bigcap_n \mathcal{A}_n$

and arbitrary n. Using (5), the equality in (4) (already proved) and the lower semicontinuity of the relative entropy, we get

$$H(\rho\,\|\,\omega)\leqslant\liminf_{n\to+\infty}H\bigl(\rho\,\|\,\Omega(\mathcal{A}_n)\bigr)\leqslant\liminf_{n\to+\infty}\overline{C}(\mathcal{A}_n)=\overline{C}\biggl(\bigcap_n\mathcal{A}_n\biggr)$$

for all such ρ. Theorem 1 of [11] implies that $\omega=\Omega\bigl(\bigcap_n\mathcal{A}_n\bigr)$.

Now consider condition a) in 7). Since we are assuming that $\mathcal{A}_1$ is regular and $\Omega(\mathcal{A}_n)=\Omega(\mathcal{A}_1)$ for all n, it follows that the sets $\mathcal{A}_n$ are regular for all n. By Theorem 2 of [11], for every n there is an optimal measure μ_n supported by $\mathcal{A}_n$ such that $\chi(\mu_n)=\overline{C}(\mathcal{A}_n)$ and $\overline{\rho}(\mu_n)=\Omega(\mathcal{A}_n)$. If condition 1) in Definition 1 holds, then formula (7) holds trivially and the proof is completed by repeating the arguments used in the case of condition b) in assertion 7). If condition 2) in Definition 1 holds, then the arguments used in the case of condition c) in assertion 7) can be applied directly.

Assertion 8) follows from Theorem 1 of [11].

To prove assertion 9), we consider the non-empty set $\mathfrak{A}$ of all closed subsets of $\mathcal{A}$ whose χ-capacity equals that of $\mathcal{A}$. We define a partial order $\prec$ on $\mathfrak{A}$ by

$$\mathcal{B}\prec\mathcal{C}\iff\mathcal{B}\supseteq\mathcal{C}.$$

Clearly, assertion 9) implies the existence of a maximal element in $\mathfrak{A}$. By Zorn's lemma, 9) will be proved if we can show that any chain in $\mathfrak{A}$ has a maximal element. The role of this maximal element will be played by the intersection of all elements of the chain provided that this intersection is an element of $\mathfrak{A}$. Since $\mathcal{A}$ is compact by assertion 4), the intersection of an *arbitrary* decreasing family of subsets of $\mathcal{A}$ coincides with the intersection of some *countable* subfamily. Hence it suffices to show that

$$\overline{C}\biggl(\bigcap_n\mathcal{B}_n\biggr)=\overline{C}(\mathcal{A})$$

for any decreasing sequence $\{\mathcal{B}_n\}$ of closed subsets of $\mathcal{A}$ such that $\overline{C}(\mathcal{B}_n)=\overline{C}(\mathcal{A})$ for all n. But this property follows from the regularity of $\mathcal{A}$ and assertion 7) with the first condition (because assertion 3) guarantees that $\Omega(\mathcal{B}_n)=\Omega(\mathcal{A})$ for all n).

The first part of assertion 10) follows from Proposition 1 below. The second part is a corollary of assertion 3) and Theorem 1 in [11] since

$$H\bigl(\rho\,\|\,\Omega(\mathcal{A})\bigr)\leqslant\max\bigl(\overline{C}(\mathcal{A}),\overline{C}(\mathcal{B})\bigr),\qquad\Omega(\mathcal{A})=\Omega(\mathcal{B})$$

for all ρ in $\mathcal{A}\cup\mathcal{B}$.

The first part of assertion 11) follows directly from the definition of χ-capacity and the monotonicity property of the relative entropy. To prove the second part of 11), we suppose that $\overline{C}\bigl(\Phi(\mathcal{A})\bigr)=\overline{C}(\mathcal{A})$. Using the monotonicity of the relative entropy and Theorem 1 of [11], we obtain that

$$H\bigl(\Phi(\rho)\,\|\,\Phi\bigl(\Omega(\mathcal{A})\bigr)\bigr)\leqslant H\bigl(\rho\,\|\,\Omega(\mathcal{A})\bigr)\leqslant\overline{C}(\mathcal{A})=\overline{C}\bigl(\Phi(\mathcal{A})\bigr)$$

for any state ρ in $\mathcal{A}$. By Theorem 1 of [11] it follows that $\Omega\bigl(\Phi(\mathcal{A})\bigr)=\Phi\bigl(\Omega(\mathcal{A})\bigr)$.

Assertion 12) follows from the first part of assertion 11) and Lemma 1 below.

Assertion 4) of Theorem 1 and Definition 1 yield the following modification of Theorem 2 in [11].

Corollary 1. *Let $\mathcal{A}$ be a closed set of finite χ-capacity. If the set* $\operatorname{Ext}(\mathcal{A})$ *is regular, then there is an optimal measure for $\mathcal{A}$.*

Assertion 5) of Theorem 1 yields the following observation.

Corollary 2. *Let $\mathcal{A}$ be a closed convex set of finite χ-capacity.*

1) *If* $\mathrm{d}(\rho) < 1$ *for all ρ in $\mathcal{A}$, then $\mathcal{A}$ is regular and the entropy is bounded on $\mathcal{A}$.*
2) *If* $\mathrm{d}(\rho) = 0$ *for all ρ in $\mathcal{A}$, then the entropy is continuous on $\mathcal{A}$.*

Remark 2. Corollary 2 shows that boundedness of the entropy on a closed convex set of states with zero coefficient of decrease (such as Gaussian states) implies that the entropy is continuous on this set.

Assertions 4) and 6) of Theorem 1 provide a sufficient condition for a union of subsets of states to be compact.

Corollary 3. *If $\{\mathcal{A}_n\}$ is a sequence of sets such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ and $\overline{C}(\mathcal{A}_n) \leqslant M < +\infty$ for all n, then the set $\bigcup_n \mathcal{A}_n$ is relatively compact.*

Assertion 11) of Theorem 1 yields the following property of optimal average states.

Corollary 4. *Let $\mathcal{A}$ be a set of finite χ-capacity $\overline{C}(\mathcal{A})$. Then $\Omega(\mathcal{A})$ is an invariant state for any channel Φ such that $\Phi(\mathcal{A}) \subseteq \overline{\mathrm{co}}(\mathcal{A})$ and $\overline{C}\bigl(\Phi(\mathcal{A})\bigr) = \overline{C}(\mathcal{A})$. In particular, $\Omega(\mathcal{A})$ is an invariant state for any automorphism*[11] *α of $\mathfrak{S}(\mathcal{H})$ such that $\alpha(\mathcal{A}) \subseteq \overline{\mathrm{co}}(\mathcal{A})$.*

Let $\mathfrak{F}(\mathcal{A})$ be the set of all channels Φ from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}(\mathcal{H})$ such that $\Phi(\mathcal{A}) \subseteq \overline{\mathrm{co}}(\mathcal{A})$ and $\overline{C}\bigl(\Phi(\mathcal{A})\bigr) = \overline{C}(\mathcal{A})$. This set is non-empty and contains all automorphisms α of $\mathfrak{S}(\mathcal{H})$ such that $\alpha(\mathcal{A}) \subseteq \overline{\mathrm{co}}(\mathcal{A})$.

Corollary 4 yields the following observation. *Let $\mathcal{A}$ be any set of finite χ-capacity. Then the set $\overline{\mathrm{co}}(\mathcal{A})$ contains at least one common invariant state for all channels in $\mathfrak{F}(\mathcal{A})$.*

Using Theorem 1 of [11] and Corollary 4, we get the following result.

Corollary 5. *Let $\mathcal{A}$ be an arbitrary set of states, $\mathfrak{F}_0$ an arbitrary subset of $\mathfrak{F}(\mathcal{A})$ and* $\operatorname{Inv}\mathfrak{F}_0$ *the set of all invariant states common to all channels in $\mathfrak{F}_0$. Then the χ-capacity of $\mathcal{A}$ is given by*

$$\overline{C}(\mathcal{A}) = \inf_{\sigma \in \operatorname{Inv}\mathfrak{F}_0 \cap \overline{\mathrm{co}}(\mathcal{A})} \sup_{\rho\in\mathcal{A}} H(\rho \,\|\, \sigma),$$

where we take $\overline{C}(\mathcal{A}) = +\infty$ if $\operatorname{Inv}\mathfrak{F}_0 \cap \overline{\mathrm{co}}(\mathcal{A}) = \varnothing$.

In particular, if $\overline{\mathrm{co}}(\mathcal{A})$ contains a unique state σ_0 which is invariant for all channels in $\mathfrak{F}_0$, then $\overline{C}(\mathcal{A}) = \sup_{\rho\in\mathcal{A}} H(\rho \,\|\, \sigma_0)$. Moreover, if $\overline{C}(\mathcal{A}) < +\infty$, then $\Omega(\mathcal{A}) = \sigma_0$.

[11] By Wigner's theorem, every automorphism of $\mathfrak{S}(\mathcal{H})$ is given by $U(\,\cdot\,)U^*$, where U is either a unitary or an anti-unitary operator in $\mathcal{H}$ [10].

Corollaries 4 and 5 enable us to determine (or at least localize) the optimal average state and calculate the χ-capacity for any set $\mathcal{A}$ of states by finding a sufficiently large family $\mathfrak{F}_0$ of channels in $\mathfrak{F}(\mathcal{A})$. This will be used in the next section.

We consider bounds for the χ-capacity of a finite union of sets.

Proposition 1. *Let $\{\mathcal{A}_k\}_{k=1}^n$ be a finite family of sets. Then*

$$\max_{\{\lambda_k\}}\biggl(\sum_{k=1}^n \lambda_k\overline{C}(\mathcal{A}_k)+\chi\bigl(\{\lambda_k,\Omega(\mathcal{A}_k)\}\bigr)\biggr)\leqslant \overline{C}\biggl(\bigcup_{k=1}^n\mathcal{A}_k\biggr)\leqslant \max_{1\leqslant k\leqslant n}\overline{C}(\mathcal{A}_k)+\log n,$$

where the first maximum is taken over all probability distributions with n outcomes.

If $\overline{C}(\mathcal{A}_k)=C$ for all $k=1,\dots,n$, then we have

$$C+\overline{C}\bigl(\{\Omega(\mathcal{A}_1),\dots,\Omega(\mathcal{A}_n)\}\bigr)\leqslant \overline{C}\biggl(\bigcup_{k=1}^n\mathcal{A}_k\biggr)\leqslant C+\log n.$$

Proof. By Theorem 1 of [11], for each positive integer m and every $k=1,\dots,n$ there is an ensemble μ_k^m such that

$$\chi(\mu_k^m)\geqslant \overline{C}(\mathcal{A}_k)-\frac{1}{m}\,,\qquad \bigl\|\overline{\rho}(\mu_k^m)-\Omega(\mathcal{A}_k)\bigr\|_1\leqslant\frac{1}{m}\,. \tag{8}$$

We take an arbitrary probability distribution $\{\lambda_k\}_{k=1}^n$ and consider the ensemble $\mu_m=\sum_{k=1}^n\lambda_k\mu_k^m$ of states in $\bigcup_{k=1}^n\mathcal{A}_k$. Using Lemma 1 of [11], the lower semicontinuity of the relative entropy and the inequality (8), we get

$$\begin{aligned}\overline{C}\biggl(\bigcup_{k=1}^n\mathcal{A}_k\biggr)&\geqslant \liminf_{m\to+\infty}\chi(\mu_m)=\liminf_{m\to+\infty}\biggl(\sum_{k=1}^n\lambda_k\chi(\mu_k^m)+\chi\bigl(\{\lambda_k,\overline{\rho}(\mu_k^m)\}\bigr)\biggr)\\&=\sum_{k=1}^n\lambda_k\overline{C}(\mathcal{A}_k)+\liminf_{m\to+\infty}\chi\bigl(\{\lambda_k,\overline{\rho}(\mu_k^m)\}\bigr)\geqslant\sum_{k=1}^n\lambda_k\overline{C}(\mathcal{A}_k)+\chi\bigl(\{\lambda_k,\Omega(\mathcal{A}_k)\}\bigr).\end{aligned}$$

This yields the lower bound for the χ-capacity of the union.

To prove the upper bound, we note that any ensemble μ of states in $\bigcup_{k=1}^n\mathcal{A}_k$ can be represented as $\sum_{k=1}^n\lambda_k\mu_k$, where μ_k is an ensemble of states in $\mathcal{A}_k$ for each $k=1,\dots,n$ and $\{\lambda_k\}_{k=1}^n$ is a probability distribution. Using Lemma 1 of [11] and Proposition 3 from the next section, we obtain that

$$\chi(\mu)=\sum_{k=1}^n\lambda_k\chi(\mu_k)+\chi\bigl(\{\lambda_k,\overline{\rho}(\mu_k)\}\bigr)\leqslant\max_{1\leqslant k\leqslant n}\overline{C}(\mathcal{A}_k)+\log n.$$

Remark 3. Proposition 1 shows that the χ-capacity of a union of sets with given χ-capacities depends on the relative positions of their optimal average states. By assertion 10) of Theorem 1, if all optimal average states coincide, then the χ-capacity of the union is minimal: it equals the maximal χ-capacity of the sets being united. The greater the diversity of the optimal average states, the higher the χ-capacity of the union. This is most obvious for the union of two sets since Proposition 1 and the lower bound for the relative entropy (see [11], inequality (1)) imply that

$$\overline{C}(\mathcal{A}\cup\mathcal{B}) \geqslant \max_{\lambda\in[0,1]}\biggl(\lambda\overline{C}(\mathcal{A})+(1-\lambda)\overline{C}(\mathcal{B})+\frac{1}{2}\,\lambda(1-\lambda)\|\Omega(\mathcal{A})-\Omega(\mathcal{B})\|_1^2\biggr).$$

We also note that the lower and upper bounds in Proposition 1 coincide if and only if $\overline{C}(\mathcal{A}_i)=\overline{C}(\mathcal{A}_j)$ and

$$\bigcup_{\rho\in\mathcal{A}_i}\operatorname{supp}\rho \perp \bigcup_{\rho\in\mathcal{A}_j}\operatorname{supp}\rho \quad \text{for all } i\neq j.$$

The following result is used in the proof of Theorem 1 as well as in the proof of Theorem 2 in § 4.

Lemma 1. *Let $\{\Psi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous maps from $\mathfrak{S}(\mathcal{H})$ to itself, where Λ is an ordered set. Suppose that $\lim_\lambda \Psi_\lambda(\rho)=\rho$ for all states ρ in some set $\mathcal{A}$. Then the following assertions hold.*

1) $\liminf_\lambda \overline{C}\bigl(\Psi_\lambda(\mathcal{A})\bigr)\geqslant\overline{C}(\mathcal{A})$.

2) *If* $\lim_\lambda \overline{C}\bigl(\Psi_\lambda(\mathcal{A})\bigr)=\overline{C}(\mathcal{A})<+\infty$, *then* $\lim_\lambda \Omega\bigl(\Psi_\lambda(\mathcal{A})\bigr)=\Omega(\mathcal{A})$.

Proof. Assertion 1) of the lemma follows from the lower semicontinuity of the relative entropy. Indeed, for every $\varepsilon>0$ there is an ensemble $\{\pi_i,\rho_i\}$ such that

$$\chi(\{\pi_i,\rho_i\})\geqslant C(\varepsilon)=\begin{cases}\overline{C}(\mathcal{A})-\varepsilon, & \overline{C}(\mathcal{A})<+\infty,\\ \varepsilon, & \overline{C}(\mathcal{A})=+\infty.\end{cases}$$

Using the lower semicontinuity of the relative entropy, we obtain that

$$\liminf_\lambda \overline{C}\bigl(\Psi_\lambda(\mathcal{A})\bigr)\geqslant\liminf_\lambda \chi\bigl(\{\pi_i,\Psi_\lambda(\rho_i)\}\bigr)\geqslant\chi(\{\pi_i,\rho_i\})\geqslant C(\varepsilon).$$

Since ε can be arbitrary, this proves assertion 1) of the lemma.

Suppose that $\lim_\lambda \overline{C}\bigl(\Psi_\lambda(\mathcal{A})\bigr)=\overline{C}(\mathcal{A})<+\infty$. By Theorem 1 of [11], for every $\varepsilon>0$ there is an ensemble $\{\pi_i,\rho_i\}$ such that

$$\chi\bigl(\{\pi_i,\rho_i\}\bigr)\geqslant\overline{C}(\mathcal{A})-\varepsilon,\qquad \biggl\|\sum_i\pi_i\rho_i-\Omega(\mathcal{A})\biggr\|_1<\varepsilon. \tag{9}$$

Arguments from the first part of the proof show that there is a λ_ε^1 such that

$$\chi\bigl(\{\pi_i,\Psi_\lambda(\rho_i)\}\bigr)\geqslant\chi\bigl(\{\pi_i,\rho_i\}\bigr)-\varepsilon\quad\forall\,\lambda\geqslant\lambda_\varepsilon^1.$$

By assumption, there is a λ_ε^2 such that

$$\overline{C}\bigl(\Psi_\lambda(\mathcal{A})\bigr)\leqslant\overline{C}(\mathcal{A})+\varepsilon\quad\forall\,\lambda\geqslant\lambda_\varepsilon^2.$$

Thus for all $\lambda\geqslant\max(\lambda_\varepsilon^1,\lambda_\varepsilon^2)$ we have

$$0\leqslant\overline{C}\bigl(\Psi_\lambda(\mathcal{A})\bigr)-\chi\bigl(\{\pi_i,\Psi_\lambda(\rho_i)\}\bigr)\leqslant\overline{C}(\mathcal{A})-\chi\bigl(\{\pi_i,\rho_i\}\bigr)+2\varepsilon\leqslant3\varepsilon.$$

Using Corollary 4 of [11], we obtain that

$$\frac{1}{2}\bigg\|\sum_i \pi_i\Psi_\lambda(\rho_i)-\Omega\big(\Psi_\lambda(\mathcal{A})\big)\bigg\|_1^2 \leqslant H\bigg(\sum_i \pi_i\Psi_\lambda(\rho_i)\,\|\,\Omega\big(\Psi_\lambda(\mathcal{A})\big)\bigg) \leqslant \overline{C}\big(\Psi_\lambda(\mathcal{A})\big)-\chi\big(\{\pi_i,\Psi_\lambda(\rho_i)\}\big)\leqslant 3\varepsilon. \tag{10}$$

The continuity property of the family $\{\Psi_\lambda\}$ implies the existence of a λ^3_ε such that

$$\bigg\|\sum_i \pi_i\Psi_\lambda(\rho_i)-\sum_i \pi_i\rho_i\bigg\|_1 \leqslant \varepsilon \quad \forall\,\lambda\geqslant\lambda^3_\varepsilon. \tag{11}$$

Using (9)–(11), we obtain that

$$\big\|\Omega\big(\Psi_\lambda(\mathcal{A})\big)-\Omega(\mathcal{A})\big\|_1 \leqslant \bigg\|\Omega\big(\Psi_\lambda(\mathcal{A})\big)-\sum_i \pi_i\Psi_\lambda(\rho_i)\bigg\|_1 + \bigg\|\sum_i \pi_i\Psi_\lambda(\rho_i)-\sum_i \pi_i\rho_i\bigg\|_1 + \bigg\|\sum_i \pi_i\rho_i-\Omega(\mathcal{A})\bigg\|_1 \leqslant 2\varepsilon+\sqrt{6\varepsilon}$$

for all $\lambda\geqslant\max(\lambda^1_\varepsilon,\lambda^2_\varepsilon,\lambda^3_\varepsilon)$. Since ε is arbitrary, this inequality proves assertion 2) of the lemma.

§ 3. Examples

In this section we study some types of sets of states using the general results obtained in [11] and the previous section.

3.1. Finite sets of states and convergent sequences. By assertion 4) of Theorem 1, every set of finite χ-capacity is relatively compact. Consider the following elementary examples of relatively compact sets:

1) a finite set of states $\{\rho_n\}_{n=1}^N$,

2) a sequence of states $\{\rho_n\}_{n=1}^{+\infty}$ converging to some state ρ_*,

3) a sequence of states $\{\rho_n\}_{n=1}^{+\infty}$ H-converging[12] to some state ρ_*.

In the following proposition we consider properties of the entropy restricted to the convex closures of these sets.

Proposition 2. 1) *Let $\{\rho_n\}_{n=1}^N$ be a finite set of states in $\mathfrak{S}(\mathcal{H})$. The entropy is continuous on the* (*closed*) *set* $\mathrm{co}(\{\rho_n\}_{n=1}^N)$ *if and only if*

$$H(\rho_n)<+\infty \quad \textit{for all} \quad n=1,2,\dots,N.$$

2) *Let $\{\rho_n\}_{n=1}^{+\infty}$ be a sequence of states converging to a state ρ_*. The entropy is bounded on the set* $\overline{\mathrm{co}}(\{\rho_n\}_{n=1}^{+\infty})$ *if and only if there is an $\mathfrak{H}$-operator H with* $\mathrm{g}(H)<+\infty$ *such that*

$$\sup_n \mathrm{Tr}\,\rho_n H<+\infty.$$

[12]This means that $\lim_{n\to+\infty} H(\rho_n\,\|\,\rho_*)=0$.

The entropy is continuous on $\overline{\mathrm{co}}\,(\{\rho_n\}_{n=1}^{+\infty})$ *if one of the following equivalent conditions holds.*

(i) $H(\rho_n)<+\infty$ *for all* n, $\lim_{n\to+\infty}H(\rho_n)=H(\rho_*)<+\infty$ *and there is a state* σ *such that* $H(\rho_n\,\|\,\sigma)<+\infty$ *for all* n *and*

$$\lim_{n\to+\infty}H(\rho_n\,\|\,\sigma)=H(\rho_*\,\|\,\sigma)<+\infty.$$

(ii) *There is an* $\mathfrak{H}$*-operator* H *with* $\mathrm{g}(H)=0$ *such that*

$$\sup_n \mathrm{Tr}\,\rho_n H<+\infty.$$

(iii) *There is an* $\mathfrak{H}$*-operator* H *with* $\mathrm{g}(H)<+\infty$ *such that* $\mathrm{Tr}\,\rho_n H<+\infty$ *for all* n *and*

$$\lim_{n\to+\infty}\mathrm{Tr}\,\rho_n H=\mathrm{Tr}\,\rho_* H<+\infty.$$

3) *Let* $\{\rho_n\}_{n=1}^{+\infty}$ *be a sequence of states* H*-converging to a state* ρ_*. *The entropy is bounded on the set* $\overline{\mathrm{co}}(\{\rho_n\}_{n=1}^{+\infty})$ *if and only if*

$$\sup_n H(\rho_n)<+\infty.$$

The entropy is continuous on the set $\overline{\mathrm{co}}(\{\rho_n\}_{n=1}^{+\infty})$ *if and only if* $H(\rho_n)<+\infty$ *for all* n *and*

$$\lim_{n\to+\infty}H(\rho_n)=H(\rho_*)<+\infty.$$

Remark 4. It is interesting to compare the boundedness and continuity conditions for convergent and H-convergent sequences. The conditions for H-convergent sequences look like natural generalizations of the corresponding conditions for finite sets while the conditions for convergent sequences include additional requirements. These requirements are essential since there is a convergent sequence $\{\rho_n\}_{n=1}^{+\infty}$ of states such that $H(\rho_n)$ is finite for all n and

$$\lim_{n\to+\infty}H(\rho_n)=H\Bigl(\lim_{n\to+\infty}\rho_n\Bigr)<+\infty$$

but the entropy is unbounded on the set $\overline{\mathrm{co}}(\{\rho_n\}_{n=1}^{+\infty})$ (see the example at the end of this subsection).

Proof. 1) Let $\mathcal{A}=\{\rho_i\}_{i=1}^{N}$. The necessity of the continuity condition is obvious. To show its sufficiency, we note that this condition and general properties of entropy [14] imply that the entropy is bounded on the closed set $\mathrm{co}(\mathcal{A})$ and, therefore, the χ-capacity of this set is finite. By Theorem 1 of [11] there is a unique state $\Omega(\mathcal{A})$ such that

$$H\bigl(\rho_n\,\|\,\Omega(\mathcal{A})\bigr)=\mathrm{Tr}\,\rho_n\bigl(-\log\Omega(\mathcal{A})\bigr)-H(\rho_n)\leqslant\overline{C}(\mathcal{A})<+\infty.$$

Hence $\mathrm{Tr}\,\rho_n\bigl(-\log\Omega(\mathcal{A})\bigr)\leqslant\overline{C}(\mathcal{A})+\max_n H(\rho_n)<+\infty$ for all $n=1,\dots,N$. Thus the linear function $\mathrm{Tr}\,\rho\bigl(-\log\Omega(\mathcal{A})\bigr)$ is finite (and hence continuous) on the *finite* set $\mathcal{A}$. By Proposition 4 of [11], this means that the entropy is continuous on the set $\overline{\mathrm{co}}(\mathcal{A})$.

2) The boundedness condition in this case follows from Proposition 1 of [11] while the continuity condition follows from Proposition 4 of [11].

3) Let $\mathcal{A} = \{\rho_i\}_{i=1}^{+\infty}$. In this case, the necessity of the boundedness and continuity conditions is obvious. To prove the sufficiency of the boundedness condition, we note that the χ-capacity of $\mathcal{A}$ is finite (see Proposition 3 below). By Theorem 1 of [11] there is a unique state $\Omega(\mathcal{A})$ such that

$$H\big(\rho_n \,\|\, \Omega(\mathcal{A})\big) = \operatorname{Tr} \rho_n \big(-\log \Omega(\mathcal{A})\big) - H(\rho_n) \leqslant \overline{C}(\mathcal{A}) < +\infty$$

for all n. Hence,

$$\sup_n \operatorname{Tr} \rho_n \big(-\log \Omega(\mathcal{A})\big) \leqslant \overline{C}(\mathcal{A}) + \sup_n H(\rho_n) < +\infty.$$

By Proposition 1 of [11], it follows that the entropy is bounded on $\overline{\mathrm{co}}(\mathcal{A})$. The sufficiency of the continuity condition follows from the first continuity condition in case 2) with $\sigma = \rho_*$.

In the following proposition we consider questions concerning the χ-capacity of finite sets and convergent sequences.

Proposition 3. 1) *Let $\{\rho_n\}_{n=1}^N$ be a finite set of states in $\mathfrak{S}(\mathcal{H})$. Then the set $\{\rho_n\}_{n=1}^N$ is regular,*

$$\overline{C}(\{\rho_n\}_{n=1}^N) \leqslant \log N,$$

and there is an optimal ensemble $\mu_* = \{\pi_n, \rho_n\}_{n=1}^N$ for the set $\{\rho_n\}_{n=1}^N$.

2) *Let $\{\rho_n\}_{n=1}^{+\infty}$ be a sequence of states converging to a state ρ_*. The χ-capacity of the set $\{\rho_n\}_{n=1}^{+\infty}$ is finite if and only if there is a state σ such that*[13]

$$\sup_n H(\rho_n \,\|\, \sigma) < +\infty.$$

3) *Let $\{\rho_n\}_{n=1}^{+\infty}$ be a sequence of states H-converging to a state ρ_*. Then the χ-capacity of the set $\{\rho_n\}_{n=1}^{+\infty}$ is finite and*

$$\overline{C}(\{\rho_n\}_{n=1}^{+\infty}) \leqslant \inf_m \max\Big(\sup_{n>m} H(\rho_n \,\|\, \rho_*), \log m\Big) + \log 2.$$

In cases 1)–3), the existence of an optimal measure (an optimal ensemble $\mu_* = \{\pi_n, \rho_n\}$) for the set $\{\rho_n\}$ is equivalent to the existence of a probability distribution $\{\pi_n\}$ and a positive number C satisfying the system

$$\begin{aligned} H\Big(\rho_n \,\Big\|\, \sum_k \pi_k \rho_k\Big) &= C, \qquad \pi_n > 0, \\ H\Big(\rho_n \,\Big\|\, \sum_k \pi_k \rho_k\Big) &\leqslant C, \qquad \pi_n = 0. \end{aligned} \tag{12}$$

If this system has a solution, then $\overline{C}(\{\rho_n\}) = C$ and $\Omega(\{\rho_n\}) = \sum_n \pi_n \rho_n$.

[13] An example below shows that the χ-capacity of a convergent sequence can be infinite.

Proof. 1) To prove the upper bound for the χ-capacity of $\mathcal{A} = \{\rho_n\}_{n=1}^N$, it suffices to note that $\mathrm{co}(\mathcal{A})$ is an output set for the channel $\sigma \mapsto \sum_{n=1}^N \langle n|\sigma|n\rangle\rho_n$ from the state space in the N-dimensional Hilbert space with an orthonormal basis $\{|n\rangle\}_{n=1}^N$ and then use the monotonicity property of the relative entropy [5]. By Theorem 1 of [11], the finiteness of the χ-capacity implies that of $H\bigl(\rho_n \,\|\, \Omega(\mathcal{A})\bigr)$ for all n, and hence the regularity of $\mathcal{A}$. The existence of an optimal measure (optimal ensemble) follows from Theorem 2 of [11].

2) This assertion follows directly from Theorem 1 of [11].

3) To prove the upper bound for the χ-capacity of $\mathcal{A} = \{\rho_n\}_{n=1}^{+\infty}$, we write $\mathcal{A}$ as the union of the finite set $\mathcal{A}_1 = \{\rho_n\}_{n=1}^m$ and the set $\mathcal{A}_2 = \{\rho_n\}_{n=m+1}^{+\infty}$. Using Theorem 1 of [11], Proposition 1 and assertion 1) of the present proposition, we obtain that

$$\begin{aligned}\overline{C}(\mathcal{A}) = \overline{C}(\mathcal{A}_1 \cup \mathcal{A}_2) &\leqslant \max\bigl(\overline{C}(\mathcal{A}_1), \overline{C}(\mathcal{A}_2)\bigr) + \log 2\\ &\leqslant \max\Bigl(\sup_{n>m} H(\rho_n \,\|\, \rho_*), \log m\Bigr) + \log 2.\end{aligned}$$

If $\{\pi_n\}$ is an optimal probability distribution, then it satisfies (12) with $C = \overline{C}(\{\rho_n\})$ by Proposition 7 of [11]. Conversely, let $(\{\pi_n\}, C)$ be a solution of (12). Using the second part of Theorem 1 of [11], we easily see that the ensemble $\{\pi_n, \rho_n\}$ is optimal for the set $\{\rho_n\}$ and $C = \overline{C}(\{\rho_n\})$.

Consider the case of finite sets of states.

If $N = 2$, then $\Omega(\{\rho_1, \rho_2\}) = \pi\rho_1 + (1-\pi)\rho_2$, where the number π is uniquely determined by the equation

$$H\bigl(\rho_1 \,\|\, \pi\rho_1 + (1-\pi)\rho_2\bigr) = H\bigl(\rho_2 \,\|\, \pi\rho_1 + (1-\pi)\rho_2\bigr).$$

Both sides of this equation are equal to $\overline{C}(\{\rho_1, \rho_2\})$. If $N > 2$, then the situation can be more complicated. The set $\{\rho_1, \dots, \rho_N\}$ may contain a proper subset $\{\rho_{n_1}, \dots, \rho_{n_{N'}}\}$, $N' < N$, such that $\overline{C}(\{\rho_{n_1}, \dots, \rho_{n_{N'}}\}) = \overline{C}(\{\rho_1, \dots, \rho_N\})$. This means that some elements of the optimal probability distribution $\{\pi_n\}$ are equal to zero. Indeed, this situation holds if we add to the set $\{\rho_1, \rho_2\}$ an arbitrary state ρ_3 with $H\bigl(\rho_3 \,\|\, \Omega(\{\rho_1, \rho_2\})\bigr) \leqslant \overline{C}(\{\rho_1, \rho_2\})$. Using Theorem 1 of [11], we easily see that $\Omega(\{\rho_1, \rho_2\}) = \Omega(\{\rho_1, \rho_2, \rho_3\})$ and $\overline{C}(\{\rho_1, \rho_2\}) = \overline{C}(\{\rho_1, \rho_2, \rho_3\})$ in this case. This is the simplest example showing that $\mathcal{A} \subsetneq \mathcal{B}$ does not imply $\overline{C}(\mathcal{A}) < \overline{C}(\mathcal{B})$.

There are two cases when the optimal average state coincides with the uniform average:

$$\Omega\bigl(\{\rho_n\}_{n=1}^N\bigr) = N^{-1} \sum_{n=1}^N \rho_n.$$

The first case is when the states $\rho_1, \dots, \rho_N$ form an orbit of some group of automorphisms of $\mathfrak{S}(\mathcal{H})$ (see § 3.5). The second is when the supports of $\rho_1, \dots, \rho_N$ are orthogonal to each other. It is this case in which the χ-capacity attains its maximal value $\log N$ independently of the types of the states $\rho_1, \dots, \rho_N$ and the values of their entropies. Indeed, this follows from the equation

$$H\biggl(\rho_n \,\Big\|\, N^{-1} \sum_{k=1}^N \rho_k\biggr) = H(\rho_n \,\|\, N^{-1}\rho_n) + 1 - N^{-1} = \log N, \qquad n = 1, \dots, N,$$

which is obtained using general properties of the relative entropy [6], [14].

The following example illustrates the case of a convergent sequence. It shows that there are non-trivial cases when one can solve the system (12) directly, which determines the optimal probability distribution and the value of the χ-capacity.

Example 1 (a convergent sequence of states). Let $\{|n\rangle\}$ be an orthonormal basis in $\mathcal{H}$ and $\{q_n\}$ a sequence of numbers in $[0,1]$ converging to zero. Given $\varepsilon\in[0,1]$, we consider the countable set $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ consisting of the states

$$\rho^{\pm}_n=(1-q_n)|1\rangle\langle 1|+q_n|n\rangle\langle n|\pm\eta_n(q_n,\varepsilon)\sqrt{(1-q_n)q_n}\,(|1\rangle\langle n|+|n\rangle\langle 1|),\qquad n\geqslant 2,$$

where the parameter $\eta_n(q_n,\varepsilon)\in[0,1]$ is defined by the condition

$$H(\rho^{\pm}_n)=(1-\varepsilon)h_2(q_n)=-(1-\varepsilon)\big((1-q_n)\log(1-q_n)+q_n\log q_n\big).$$

Thus ε may be regarded as a purity parameter of the states in the sequence. If $\varepsilon=0$, then $\eta_n(q_n,\varepsilon)=0$ and all the states $\rho^{+}_n=\rho^{-}_n$, $n\geqslant 2$, are diagonal in the basis $\{|n\rangle\}$ and have maximal entropy. If $\varepsilon=1$, then $\eta_n(q_n,\varepsilon)=1$ and all the states $\rho^{\pm}_n$ with $n\geqslant 2$ are pure.

The set $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ may be regarded as a sequence converging to the state $\rho_1=|1\rangle\langle 1|$.

Proposition 4. 1) *The χ-capacity of $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ is finite if and only if there is a positive number λ such that*

$$\sum_n\exp\biggl(-\frac{\lambda}{q_n}\biggr)<+\infty. \tag{13}$$

2) *If condition (13) holds, then a necessary and sufficient condition for the existence of an optimal measure (optimal ensemble $\mu_*=\{\pi^{\pm}_n,\rho^{\pm}_n\}$) for the set $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ is given by the inequality*

$$\sum_{n>1}q_n^{-\varepsilon}(1-q_n)^{1+\frac{(1-q_n)(1-\varepsilon)}{q_n}}\exp\biggl(-\frac{\lambda^*_{\{q_n\}}}{q_n}\biggr)\geqslant 1, \tag{14}$$

where

$$\lambda^*_{\{q_n\}}=\inf\biggl\{\lambda\colon\sum_n\exp\biggl(-\frac{\lambda}{q_n}\biggr)<+\infty\biggr\}.$$

3) *If the sequence $\{q_n\}$ satisfies conditions (13) and (14) with a given ε, then the following assertions hold.*

(i) *The χ-capacity of $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ is given by*

$$\overline{C}\big(\mathcal{S}^{\varepsilon}_{\{q_n\}}\big)=\lambda^{\varepsilon}_{\{q_n\}}-\log\pi^{\varepsilon}_{\{q_n\}}.$$

(ii) *The optimal average state $\Omega\big(\mathcal{S}^{\varepsilon}_{\{q_n\}}\big)$ of the set $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ is given by*

$$\pi^{\varepsilon}_{\{q_n\}}|1\rangle\langle 1|+\pi^{\varepsilon}_{\{q_n\}}\sum_{n>1}\Bigl(q_n(1-q_n)^{\frac{(1-q_n)}{q_n}}\Bigr)^{(1-\varepsilon)}\exp\biggl(-\frac{\lambda^{\varepsilon}_{\{q_n\}}}{q_n}\biggr)|n\rangle\langle n|.$$

(iii) *The optimal probability distribution* $\{\pi_n^{\pm}\}$ *is given by*

$$\pi_1^{\pm}=0,\quad \pi_n^{\pm}=\frac{1}{2}\,\pi_{\{q_n\}}^{\varepsilon}q_n^{-\varepsilon}(1-q_n)^{\frac{(1-q_n)(1-\varepsilon)}{q_n}}\exp\biggl(-\frac{\lambda_{\{q_n\}}^{\varepsilon}}{q_n}\biggr),\quad n\geqslant 2,$$

where $\lambda_{\{q_n\}}^{\varepsilon}$ *is the unique solution of the equation*

$$\sum_{n>1}q_n^{-\varepsilon}(1-q_n)^{1+\frac{(1-q_n)(1-\varepsilon)}{q_n}}\exp\biggl(-\frac{\lambda}{q_n}\biggr)=1,$$

and $\pi_{\{q_n\}}^{\varepsilon}=\Bigl(\sum_{n>1}q_n^{-\varepsilon}(1-q_n)^{\frac{(1-q_n)(1-\varepsilon)}{q_n}}\exp\bigl(-\frac{\lambda_{\{q_n\}}^{\varepsilon}}{q_n}\bigr)\Bigr)^{-1}\in[0,1].$

4) *Condition* (13) *is equivalent to boundedness of the entropy on the set* $\overline{\mathrm{co}}\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)$ *for an arbitrary* ε.

5) *The maximal entropy state of the set* $\overline{\mathrm{co}}\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)$ *exists for some* ε (*and hence for every* ε) *if and only if the sequence* $\{q_n\}$ *satisfies conditions* (13) *and* (14) *with* $\varepsilon=1$. *Then we have*

$$\Gamma\bigl(\overline{\mathrm{co}}\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)\bigr)=\pi_{\{q_n\}}^{1}|1\rangle\langle1|+\pi_{\{q_n\}}^{1}\sum_{n>1}\exp\biggl(-\frac{\lambda_{\{q_n\}}^{1}}{q_n}\biggr)|n\rangle\langle n|$$

for every ε, *where* $\pi_{\{q_n\}}^{1}$ *and* $\lambda_{\{q_n\}}^{1}$ *are the parameters defined above.*[14]

6) *If condition* (13) *holds for every* $\lambda>0$, *then the entropy is continuous on the set* $\overline{\mathrm{co}}\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)$ *for every* ε.

Fig. 1 shows the results of a numerical calculation of the χ-capacity of $\mathcal{S}_{\{q_n\}}^{\varepsilon}$ as a function of ε for various sequences $\{q_n\}$.

Proof. By Theorem 1 of [11], the finiteness of the χ-capacity of $\mathcal{S}_{\{q_n\}}^{\varepsilon}$ means the existence of an optimal average state $\Omega\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)$ in $\overline{\mathrm{co}}\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)$ such that

$$\sup_{n>1}H\bigl(\rho_n^{\pm}\,\|\,\Omega\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)\bigr)<+\infty. \tag{15}$$

By Lemma 1 of [1], the optimal average state can be represented as

$$\Omega\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)=\pi_1\rho_1+\sum_{n>1,\pm}\pi_n^{\pm}\rho_n^{\pm}. \tag{16}$$

Since the set $\mathcal{S}_{\{q_n\}}^{\varepsilon}$ is invariant under the action of the automorphism $U(\,\cdot\,)U^*$, where U is a unitary operator diagonal in the basis $\{|n\rangle\}$ and having eigenvalues ±1, Corollary 4 implies that the state $\Omega\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)$ is invariant under the action of this automorphism. Hence this state is diagonal in the basis $\{|n\rangle\}$. This means that $\pi_n^{+}=\pi_n^{-}=\frac{1}{2}\pi_n$ for all $n>1$ in (16), where $\{\pi_n\}_{n=1}^{+\infty}$ is some probability distribution. Thus we have

$$\Omega\bigl(\mathcal{S}_{\{q_n\}}^{\varepsilon}\bigr)=\pi|1\rangle\langle1|+\sum_{n>1}\pi_nq_n|n\rangle\langle n|, \tag{17}$$

[14] It is interesting to compare this observation with the results of Proposition 1 in [11] for the $\mathfrak{H}$-operator $H=\sum_{n=2}^{+\infty}q_n^{-1}|n\rangle\langle n|$.

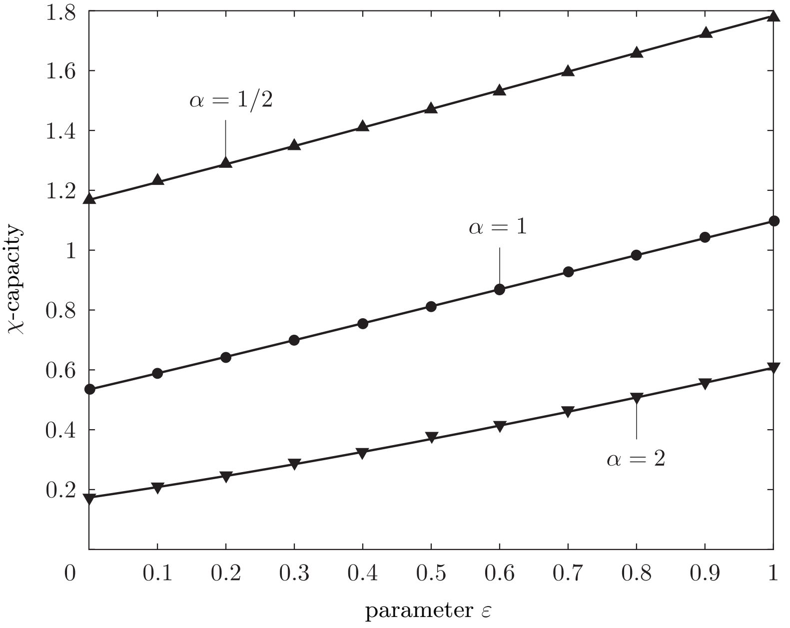

Figure 1. The χ-capacity of the set $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ as a function of ε for the sequences $\{q_n = n^{-\alpha}\}_{n\geqslant 2}$, $\alpha = 1/2, 1, 2$

where $\pi = \pi_1 + \sum_{n>1}(1-q_n)\pi_n$. Thus,

$$H\big(\rho_1 \,\|\, \Omega\big(\mathcal{S}^{\varepsilon}_{\{q_n\}}\big)\big) = -\log\pi, \tag{18}$$

$$\begin{aligned} H\big(\rho_n^{\pm} \,\|\, \Omega\big(\mathcal{S}^{\varepsilon}_{\{q_n\}}\big)\big) &= -(1-q_n)\log\pi - q_n\log(\pi_n q_n) \\ &\quad + (1-\varepsilon)\big((1-q_n)\log(1-q_n) + q_n\log q_n\big) = -(1-q_n)\log\pi \\ &\quad - q_n\log\pi_n - \varepsilon q_n\log q_n + (1-\varepsilon)(1-q_n)\log(1-q_n), \qquad n>1. \end{aligned} \tag{19}$$

Since $q_n \to 0$ as $n \to +\infty$, condition (15) means that $\sup_{n>1} q_n(-\log\pi_n)$ is finite. It is easy to see that the existence of a probability distribution $\{\pi_n\}$ satisfying this condition is equivalent to the existence of a positive number λ such that $\sum_n \exp\big(-\frac{\lambda}{q_n}\big) < +\infty$.

We note that (15) and (19) imply that $\pi_n > 0$ for all $n > 1$. Using this along with (18) and (19), we can rewrite the system (12) as

$$\begin{gathered} -\log\pi \leqslant C, \qquad \pi_1(C+\log\pi) = 0, \\ (1-q_n)\big((1-\varepsilon)\log(1-q_n) - \log\pi\big) - q_n\log\pi_n - \varepsilon q_n\log q_n = C. \end{gathered} \tag{20}$$

The second part of the system (20) implies that

$$\pi_n = \pi q_n^{-\varepsilon}(1-q_n)^{\frac{(1-q_n)(1-\varepsilon)}{q_n}}\exp\bigg(-\frac{C+\log\pi}{q_n}\bigg), \qquad n\geqslant 2. \tag{21}$$

Since the sequence $\{\pi_n\}$ must converge to zero as $n \to +\infty$, we conclude that $-\log\pi < C$. Hence the first part of the system (20) implies that $\pi_1 = 0$.

It is easy to see that if there is a probability distribution $\{\pi_n\}$ satisfying (20), then $\pi = \sum_{n>1}(1-q_n)\pi_n$ and C form a solution of the system

$$
\begin{aligned}
&\sum_{n>1} q_n^{-\varepsilon}(1-q_n)^{1+\frac{(1-q_n)(1-\varepsilon)}{q_n}} \exp\biggl(-\frac{C+\log\pi}{q_n}\biggr) = 1,\\
&\sum_{n>1} q_n^{-\varepsilon}(1-q_n)^{\frac{(1-q_n)(1-\varepsilon)}{q_n}} \exp\biggl(-\frac{C+\log\pi}{q_n}\biggr) = \pi^{-1}.
\end{aligned}
\tag{22}
$$

Conversely, any solution (π, C) of (22) determines (by formula (21)) a probability distribution $\{\pi_n\}$ that satisfies (20).

We claim that the system (22) has a solution (π, C) if and only if the inequality (14) holds. Indeed, consider the functions

$$
\begin{aligned}
F(x) &= \sum_{n>1} q_n^{-\varepsilon}(1-q_n)^{1+\frac{(1-q_n)(1-\varepsilon)}{q_n}} \exp\biggl(-\frac{x}{q_n}\biggr),\\
G(x) &= \sum_{n>1} q_n^{-\varepsilon}(1-q_n)^{\frac{(1-q_n)(1-\varepsilon)}{q_n}} \exp\biggl(-\frac{x}{q_n}\biggr).
\end{aligned}
$$

They are continuous and strictly decreasing on $\bigl(\lambda^*_{\{q_n\}}, +\infty\bigr)$. Moreover, $F(x)$ does not exceed $G(x)$. Hence there are the inverse functions $F^{-1}(y)$ and $G^{-1}(y)$, which are continuous and strictly decreasing on $F\bigl((\lambda^*_{\{q_n\}}, +\infty)\bigr)$ and $G\bigl((\lambda^*_{\{q_n\}}, +\infty)\bigr)$ respectively. Using these functions, we can rewrite the system (22) as

$$
\begin{aligned}
F(C+\log\pi) &= 1\\
G(C+\log\pi) &= \pi^{-1}.
\end{aligned}
$$

The inequality (14) is equivalent to the inequality $\lim_{x\to\lambda^*_{\{q_n\}}+0} F(x) \geqslant 1$. By the previous observation, this means that $F^{-1}(1)$ is well defined. Thus, if the inequality (14) holds, then $C + \log\pi = F^{-1}(1)$. Hence $\pi = \bigl(G(F^{-1}(1))\bigr)^{-1} \leqslant \bigl(F(F^{-1}(1))\bigr)^{-1} = 1$ and $C = F^{-1}(1) + \log G\bigl(F^{-1}(1)\bigr)$ form a unique solution of (22). Denoting $F^{-1}(1)$ and π by $\lambda^{\varepsilon}_{\{q_n\}}$ and $\pi^{\varepsilon}_{\{q_n\}}$ respectively, we obtain all the assertions concerning the χ-capacity of $\mathcal{S}^{\varepsilon}_{\{q_n\}}$. If the inequality (14) does not hold,[15] then the system (22) has no solution and hence there is no optimal probability distribution $\{\pi_n\}$. Thus the set $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ is irregular in this case.

Since the boundedness of the entropy on the set $\overline{\mathrm{co}}\bigl(\mathcal{S}^{\varepsilon}_{\{q_n\}}\bigr)$ implies that the χ-capacity of $\mathcal{S}^{\varepsilon}_{\{q_n\}}$ is finite, it also implies that (13) holds by the observation above. The converse assertion follows from the boundedness condition in assertion 2) of Proposition 2 with the $\mathfrak{H}$-operator $\sum_{n=2}^{+\infty} q_n^{-1}|n\rangle\langle n|$. Thus condition (13) is equivalent to the boundedness of the entropy on the set $\overline{\mathrm{co}}\bigl(\mathcal{S}^{\varepsilon}_{\{q_n\}}\bigr)$.

[15] It is easy to construct a sequence $\{q_n\}$ for which (13) holds but (14) does not (see the end of this subsection).

Suppose that the sequence $\{q_n\}$ satisfies condition (14) with $\varepsilon = 1$. Since the closed set $\mathcal{S}^1_{\{q_n\}}$ consists of pure states, the existence of an optimal measure for this set (which is guaranteed by the condition) implies that the optimal average state $\Omega(\mathcal{S}^1_{\{q_n\}})$ coincides with the maximal entropy state $\Gamma(\overline{\mathrm{co}}(\mathcal{S}^1_{\{q_n\}}))$. Noting that $\Omega(\mathcal{S}^1_{\{q_n\}})$ lies in $\overline{\mathrm{co}}(\mathcal{S}^0_{\{q_n\}})$ and that $\overline{\mathrm{co}}(\mathcal{S}^0_{\{q_n\}}) \subseteq \overline{\mathrm{co}}(\mathcal{S}^\varepsilon_{\{q_n\}})$ for every ε, we conclude that

$$\Gamma(\overline{\mathrm{co}}(\mathcal{S}^\varepsilon_{\{q_n\}})) = \Omega(\mathcal{S}^1_{\{q_n\}})$$

for every ε in this case.

Suppose that the maximal entropy state $\Gamma(\overline{\mathrm{co}}(\mathcal{S}^\varepsilon_{\{q_n\}}))$ exists for some ε. Using the observation at the end of § 3 of [11], we easily see that this yields the existence of a maximal entropy state $\Gamma(\overline{\mathrm{co}}(\mathcal{S}^\varepsilon_{\{q_n\}}))$ for all ε, including $\varepsilon = 1$. Since the set $\mathcal{S}^1_{\{q_n\}}$ consists of pure states, the state $\Gamma(\overline{\mathrm{co}}(\mathcal{S}^1_{\{q_n\}}))$ coincides with the state $\Omega(\mathcal{S}^1_{\{q_n\}})$. By Lemma 2 of [11], the restriction of the entropy to the set $\overline{\mathrm{co}}(\mathcal{S}^1_{\{q_n\}})$ is continuous at $\Omega(\mathcal{S}^1_{\{q_n\}}) = \Gamma(\overline{\mathrm{co}}(\mathcal{S}^1_{\{q_n\}}))$. It follows that the set $\mathcal{S}^1_{\{q_n\}}$ is regular. By Theorem 2 of [11] there is an optimal measure for the set $\mathcal{S}^1_{\{q_n\}}$. Hence the observation above shows that the sequence $\{q_n\}$ satisfies condition (14) with $\varepsilon = 1$.

By the second continuity condition in assertion 2) of Proposition 2 with the $\mathfrak{H}$-operator $\sum_{n=2}^{+\infty} q_n^{-1}|n\rangle\langle n|$, the finiteness of the series in (13) for arbitrary λ implies that the entropy is continuous on the set $\overline{\mathrm{co}}(\mathcal{S}^\varepsilon_{\{q_n\}})$ for arbitrary ε.

To conclude this subsection, we give an example of a sequence $\{q_n\}$ for which condition (13) holds while condition (14) with arbitrary ε does not. Put $q_n = 1/\log(n\log^3(2n+1))$ for $n \geqslant 2$. Then $\lambda^*_{\{q_n\}} = 1$ and the left-hand side of (14) with $\varepsilon = 1$ is approximately equal to 0.89. It follows that condition (14) does not hold with arbitrary ε. The observation above shows that the entropy is bounded on $\overline{\mathrm{co}}(\mathcal{S}^\varepsilon_{\{q_n\}})$ and the χ-capacity of $\mathcal{S}^\varepsilon_{\{q_n\}}$ is finite for every ε, but there is neither a maximal entropy state for $\overline{\mathrm{co}}(\mathcal{S}^\varepsilon_{\{q_n\}})$ nor an optimal measure $\mu_* = \{\pi^\pm_n, \rho^\pm_n\}$ for $\mathcal{S}^\varepsilon_{\{q_n\}}$.

3.2. The sets $\mathcal{L}(\sigma)$ and $\mathcal{K}_{H,h}$. Let $\sigma = \sum_k \lambda_k |k\rangle\langle k|$ be an arbitrary state. In [11] we defined the set $\mathcal{L}(\sigma)$ of all states whose diagonal elements in the basis $\{|k\rangle\}$ coincide with those of σ. By Proposition 6 of [11], the entropy is continuous on the set $\mathcal{L}(\sigma)$ if and only if $\sup_{\rho\in\mathcal{L}(\sigma)} H(\rho) = H(\sigma) < +\infty$. In the following proposition we study questions concerning the χ-capacity of the set $\mathcal{L}(\sigma)$.

Proposition 5. *Let σ be an arbitrary state in $\mathfrak{S}(\mathcal{H})$. Then the following assertions hold.*

1) *The χ-capacity of $\mathcal{L}(\sigma)$ is equal to $H(\sigma)$.*

2) *If $\overline{C}(\mathcal{L}(\sigma)) = H(\sigma) < +\infty$, then the set $\mathcal{L}(\sigma)$ is regular and admits an optimal measure with barycentre $\Omega(\mathcal{L}(\sigma)) = \sigma$ and support in the set of pure states in $\mathcal{L}(\sigma)$.*

Proof. Suppose that the χ-capacity $\overline{C}(\mathcal{L}(\sigma))$ is finite. Let G be the group of all unitary operators on $\mathcal{H}$ that are diagonal in the basis $\{|k\rangle\}$. Since $\mathcal{L}(\sigma)$ is invariant under the action of the automorphism $U(\,\cdot\,)U^*$ for every $U \in G$, Corollary 5 implies that $\Omega(\mathcal{L}(\sigma)) = \sigma$. Let ρ be an arbitrary pure state in $\mathcal{L}(\sigma)$, for example, the state corresponding to the vector $\sum_k \sqrt{\lambda_k}\,|k\rangle$. Theorem 1 of [11] and Proposition 6 of [11] imply that

$$\overline{C}\big(\mathcal{L}(\sigma)\big) \geqslant H\big(\rho \,\|\, \Omega(\mathcal{L}(\sigma))\big) = H(\rho \,\|\, \sigma) = H(\sigma).$$

Since $\sup_{\rho\in\mathcal{L}(\sigma)} H(\rho) = H(\sigma)$, this inequality is an equation. To complete the proof of assertion 1), we note that $\overline{C}\big(\mathcal{L}(\sigma)\big) = +\infty$ implies that $H(\sigma) = +\infty$ by Proposition 6 of [11].

The regularity assertion follows from Proposition 6 in [11].

Since $\overline{C}\big(\mathcal{L}(\sigma)\big) = H\big(\Omega(\mathcal{L}(\sigma))\big)$, the assertion on the existence of an optimal measure follows from Theorem 2 of [11] and Propositions 6, 7 of [11].

The set $\mathcal{K}_{H,h}$ was introduced in [11]. It is defined by the inequality $\operatorname{Tr}\rho H \leqslant h$, where H is an $\mathfrak{H}$-operator and h is a positive number. Proposition 1 in [11] gives necessary and sufficient conditions for the boundedness and continuity of the entropy on $\mathcal{K}_{H,h}$ in terms of the coefficient $\mathrm{g}(H)$ of increase of the $\mathfrak{H}$-operator H. This proposition also shows that the maximal entropy state exists for $\mathcal{K}_{H,h}$ if and only if $h \leqslant h_*(H)$. (The parameters $\mathrm{g}(H)$ and $h_*(H)$ are defined in [11] before Proposition 1.) In the following proposition we consider questions concerning the χ-capacity of the set $\mathcal{K}_{H,h}$.

Proposition 6. *Let H be an $\mathfrak{H}$-operator on the Hilbert space $\mathcal{H}$ and h a positive number such that $h > h_{\mathrm{m}}(H)$. Then the following assertions hold.*

1) *The χ-capacity of the set $\mathcal{K}_{H,h}$ coincides with $\sup_{\rho\in\mathcal{K}_{H,h}} H(\rho)$. Hence it is finite if and only if $\mathrm{g}(H) < +\infty$. If this condition holds, then*

$$\Omega(\mathcal{K}_{H,h}) = \begin{cases} \Gamma(\mathcal{K}_{H,h}) = \big(\operatorname{Tr}\exp(-\lambda^* H)\big)^{-1}\exp(-\lambda^* H), & h \leqslant h_*(H), \\ \big(\operatorname{Tr}\exp(-\,\mathrm{g}(H)H)\big)^{-1}\exp(-\,\mathrm{g}(H)H), & h > h_*(H), \end{cases}$$

where $\lambda^* = \lambda^*(H,h)$ is uniquely determined by the equation

$$\operatorname{Tr} H\exp(-\lambda H) = h\operatorname{Tr}\exp(-\lambda H).$$

2) *The following statements are equivalent.*

(i) *The inequality $h \leqslant h_*(H)$ holds.*

(ii) *The set $\mathcal{K}_{H,h}$ is regular.*

(iii) $\overline{C}(\mathcal{K}_{H,h}) \leqslant H\big(\Omega(\mathcal{K}_{H,h})\big)$ *or, equivalently,* $\overline{C}(\mathcal{K}_{H,h}) = H\big(\Omega(\mathcal{K}_{H,h})\big)$.

(iv) $\overline{C}(\mathcal{K}_{H,h}) = \overline{C}\big(\mathcal{K}_{H,h}\cap\mathcal{L}(\Omega(\mathcal{K}_{H,h}))\big)$.

(v) *There is an optimal measure for the set $\mathcal{K}_{H,h}$.*

Fig. 2 shows the results of a numerical calculation of the χ-capacity of $\mathcal{K}_{H,h}$ as a function of $h = c$ for the $\mathfrak{H}$-operator $H = -\log\sigma$ with $h_*(H) < +\infty$.

Proof. Write $H = \sum_k h_k |k\rangle\langle k|$ and let $\mathcal{K}^{c}_{H,h}$ be the subset of $\mathcal{K}_{H,h}$ consisting of states that are diagonal in the basis $\{|k\rangle\}$. Then $\mathcal{K}_{H,h} = \bigcup_{\rho\in\mathcal{K}^c_{H,h}} \mathcal{L}(\rho)$ and, therefore,

$$\overline{C}(\mathcal{K}_{H,h}) \geqslant \sup_{\rho\in\mathcal{K}^c_{H,h}} \overline{C}\big(\mathcal{L}(\rho)\big) = \sup_{\rho\in\mathcal{K}^c_{H,h}} H(\rho) = \sup_{\rho\in\mathcal{K}_{H,h}} H(\rho),$$

where the last equation follows from inequality (22) of [11]. This proves the first part of assertion 1) since the reverse inequality is obvious.

In the proof of Proposition 1 in [11], we constructed a sequence $\{\rho_n\}$ of states in $\mathcal{K}^c_{H,h}$ such that $\lim_{n\to\infty} H(\rho_n) = \sup_{\rho\in\mathcal{K}_{H,h}} H(\rho)$ and $\lim_{n\to\infty}\rho_n = \rho_*(\mathcal{K}_{H,h})$. By Proposition 5, for every n there is an optimal measure μ_n for the set $\mathcal{L}(\rho_n)$ such that $\overline{\rho}(\mu_n) = \rho_n$ and $\chi(\mu_n) = H(\rho_n)$. The first part of assertion 1) of the proposition shows that the sequence $\{\mu_n\}$ of measures is an approximating sequence for the set $\mathcal{K}_{H,h}$. By Theorem 1 in [11], the limit $\rho_*(\mathcal{K}_{H,h})$ of the corresponding sequence of barycentres $\{\rho_n\}$ is the optimal average state of $\mathcal{K}_{H,h}$. Assertion 1) is proved.

The equivalence of statements (i)–(v) will be proved in the following order: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii). By the observation above, (i) implies that $\Omega(\mathcal{K}_{H,h}) = \Gamma(\mathcal{K}_{H,h})$. By Lemma 2 in [11], the restriction of the entropy to $\mathcal{K}_{H,h}$ is continuous at the state $\Omega(\mathcal{K}_{H,h})$. This guarantees that the set $\mathcal{K}_{H,h}$ is regular.

(ii) $\Longrightarrow$ (v). This follows directly from Theorem 2 in [11].

(v) $\Longrightarrow$ (iii). This follows directly from Corollary 8 in [11].

(iii) $\Longrightarrow$ (iv). This follows from Proposition 5 and assertion 1).

(iv) $\Longrightarrow$ (i). If $h > h_*(H)$, then Proposition 1 in [11] and Proposition 5 show that

$$\overline{C}\big(\mathcal{L}(\Omega(\mathcal{K}_{H,h}))\big) = H\big(\Omega(\mathcal{K}_{H,h})\big) < \sup_{\rho\in\mathcal{K}_{H,h}} H(\rho) = \overline{C}(\mathcal{K}_{H,h}).$$

The constructions in the proofs of Proposition 1 in [11] and Proposition 6 enable us to construct the following example, which shows that the regularity condition in assertion 9) of Theorem 1 is essential.

Example 2 (a closed set of finite χ-capacity having no minimal closed subset of the same χ-capacity). Let H be an $\mathfrak{H}$-operator such that

$$h_*(H) = \frac{\operatorname{Tr} H\exp(-\operatorname{g}(H)H)}{\operatorname{Tr}\exp(-\operatorname{g}(H)H)} < +\infty.$$

For example, take

$$H = \sum_{k=1}^{+\infty} \log\big((k+1)\log^3(k+1)\big)|k\rangle\langle k|.$$

As observed in the proof of Proposition 1 in [11], for every given $h > h_*(H)$ there is a positive integer n_0 such that the state ρ_n is well defined by formula (8) in [11] for all $n \geqslant n_0$ and the sequence $\{\rho_n\}_{n\geqslant n_0}$ converges to the state $\rho_*(\mathcal{K}_{H,h})$ defined by formula (13) in [11]. We put $\mathcal{A}_0 = \bigcup_{n\geqslant n_0}\mathcal{L}(\rho_n)$ and $\mathcal{A} = \overline{\mathcal{A}}_0 = \mathcal{A}_0 \cup \mathcal{L}\big(\rho_*(\mathcal{K}_{H,h})\big)$. The proof of Proposition 1 in [11] and Proposition 6 in [11] yield that

$$\overline{C}(\mathcal{A}) = \lim_{n\to+\infty} H(\rho_n) > H\big(\rho_*(\mathcal{K}_{H,h})\big) = \sup_{\rho\in\mathcal{L}(\rho_*(\mathcal{K}_{H,h}))} H(\rho).$$

We claim that the closed set $\mathcal{A}$ has no minimal closed subset of the same χ-capacity. Indeed, let $\mathcal{B}$ be a minimal subset of $\mathcal{A}$. Since $\overline{C}\big(\mathcal{L}(\rho_*(\mathcal{K}_{H,h}))\big)$ is smaller than $\overline{C}(\mathcal{A}) = \overline{C}(\mathcal{B})$, the set $\mathcal{B}$ has non-empty intersection with $\mathcal{L}(\rho_{n_*})$ for some $n_* \geqslant n_0$. We shall show that the closed set $\mathcal{B}\setminus\mathcal{L}(\rho_{n_*}) \subsetneq \mathcal{B}$ has the same χ-capacity as $\mathcal{B}$, contrary to the assumed minimality of $\mathcal{B}$.

Since $\sup_{\rho\in\mathcal{L}(\rho_*(\mathcal{K}_{H,h}))} H(\rho) < \overline{C}(\mathcal{B})$, there is an approximating sequence of ensembles $\{\{\pi_i^k,\rho_i^k\}\}_k$ for the set $\mathcal{B}$ such that the corresponding sequence $\{\overline{\rho}_k\}_k$ of average states is disjoint from $\mathcal{L}\bigl(\rho_*(\mathcal{K}_{H,h})\bigr)$ and, therefore, $\Pi_{\{|k\rangle\}}(\overline{\rho}_k) = \rho_{n_k}$ for some sequence $\{n_k\}$ of positive integers. Since $\sup_{\rho\in\mathcal{L}(\rho_n)} H(\rho) < \overline{C}(\mathcal{B})$ for every $n \geqslant n_0$, the sequence $\{n_k\}$ tends to $+\infty$. We note that for every ρ in $\mathcal{B}$ the state $\Pi_{\{|k\rangle\}}(\rho)$ coincides with either $\rho_*(\mathcal{K}_{H,h})$ or ρ_n for some n. Moreover, if $\overline{\rho}_k = \sum_i \pi_i^k \rho_i^k$, then $\Pi_{\{|k\rangle\}}(\overline{\rho}_k) = \sum_i \pi_i^k \Pi_{|k\rangle}(\rho_i^k)$ for every k. Using this observation and definitions (8), (13) of [11], we conclude that $\Pi_{\{|k\rangle\}}(\rho_i^k) = \rho_{n_k}$ for all i and k. Thus the states $\{\pi_i^k,\rho_i^k\}$ are not contained in $\mathcal{L}(\rho_{n_*})$ for all sufficiently large k and, therefore, the "tail" of the sequence $\{\{\pi_i^k,\rho_i^k\}\}_k$ is an approximating sequence of the set $\mathcal{B}\setminus\mathcal{L}(\rho_{n_*})$. It follows that $\overline{C}(\mathcal{B}) = \overline{C}\bigl(\mathcal{B}\setminus\mathcal{L}(\rho_{n_*})\bigr)$.

3.3. The set $\boldsymbol{\mathcal{V}_{\sigma,c}}$. The set $\mathcal{V}_{\sigma,c}$ was introduced in [11]. It is defined by the inequality $H(\rho\,\|\,\sigma) \leqslant c$, where σ is a state and c is a non-negative number. If σ is a state of infinite rank, then the family $\{\mathcal{V}_{\sigma,c}\}_{c\in\mathbb{R}_+}$ of non-empty sets is strictly increasing and $\mathcal{V}_{\sigma,0} = \{\sigma\}$.

By Theorem 1 of [11], every set $\mathcal{A}$ of finite χ-capacity is contained in the compact convex set $\mathcal{V}_{\Omega(\mathcal{A}),\overline{C}(\mathcal{A})}$ such that $\Omega\bigl(\mathcal{V}_{\Omega(\mathcal{A}),\overline{C}(\mathcal{A})}\bigr) = \Omega(\mathcal{A})$ and $\overline{C}\bigl(\mathcal{V}_{\Omega(\mathcal{A}),\overline{C}(\mathcal{A})}\bigr) = \overline{C}(\mathcal{A})$. Below we shall find the χ-capacity and optimal average state for the set $\mathcal{V}_{\sigma,c}$ for arbitrary σ and c.

Proposition 3 of [11] gives necessary and sufficient conditions for the boundedness and continuity of the entropy on the set $\mathcal{V}_{\sigma,c}$ in terms of the coefficient $\mathrm{d}(\sigma)$ of decrease of the state σ. This proposition also shows that the maximal entropy state exists for $\mathcal{V}_{\sigma,c}$ if and only if $c \leqslant c_*(\sigma)$. (The parameters $\mathrm{d}(\sigma)$ and $c_*(\sigma)$ are defined before Proposition 3 in [11].) In the following proposition we consider questions concerning the χ-capacity of the set $\mathcal{V}_{\sigma,c}$. We put

$$c^*(\sigma) = \frac{\operatorname{Tr}\sigma^{\mathrm{d}(\sigma)}(-\log\sigma)}{\operatorname{Tr}\sigma^{\mathrm{d}(\sigma)}}$$

if $\operatorname{Tr}\sigma^{\mathrm{d}(\sigma)} < +\infty$, and $c^*(\sigma) = +\infty$ otherwise. Note that

$$c^*(\sigma) = \frac{c_*(\sigma) + \log\operatorname{Tr}\sigma^{\mathrm{d}(\sigma)}}{1-\mathrm{d}(\sigma)} \geqslant c_*(\sigma)$$

if $\mathrm{d}(\sigma) < 1$, and $c^*(\sigma) = H(\sigma)$ if $\mathrm{d}(\sigma) = 1$.

Proposition 7. *Let σ be a state of infinite rank in $\mathfrak{S}(\mathcal{H})$.*

1) *If $c \leqslant H(\sigma) \leqslant +\infty$, then $\overline{C}(\mathcal{V}_{\sigma,c}) = c$ and $\Omega(\mathcal{V}_{\sigma,c}) = \sigma$.*

2) *If $H(\sigma) < c \leqslant c^*(\sigma)$, then*

$$\overline{C}(\mathcal{V}_{\sigma,c}) = \lambda^* c + \log\operatorname{Tr}\sigma^{\lambda^*}, \qquad \Omega(\mathcal{V}_{\sigma,c}) = \bigl(\operatorname{Tr}\sigma^{\lambda^*}\bigr)^{-1}\sigma^{\lambda^*},$$

where $\lambda^* = \lambda^*(\sigma,c)$ is uniquely determined by the equation $\operatorname{Tr}\sigma^{\lambda}(-\log\sigma) = c\operatorname{Tr}\sigma^{\lambda}$.

3) *If $c^*(\sigma) < +\infty$ and $c \geqslant c^*(\sigma)$, then*

$$\overline{C}(\mathcal{V}_{\sigma,c}) = \mathrm{d}(\sigma)c + \log\operatorname{Tr}\sigma^{\mathrm{d}(\sigma)}, \qquad \Omega(\mathcal{V}_{\sigma,c}) = \bigl(\operatorname{Tr}\sigma^{\mathrm{d}(\sigma)}\bigr)^{-1}\sigma^{\mathrm{d}(\sigma)}.$$

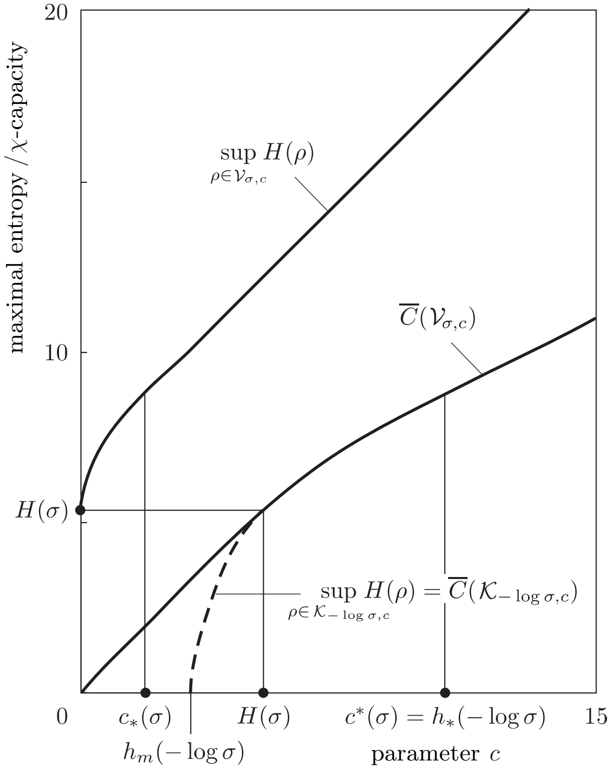

Figure 2. The least upper bound for the entropy and χ-capacity of the sets $\mathcal{K}_{-\log\sigma,c}$ and $\mathcal{V}_{\sigma,c}$ as functions of c for the state $\sigma \sim \sum_{k=1}^{+\infty} \frac{|k\rangle\langle k|}{(k+100)\log^3(k+100)}$ with $\mathrm{d}(\sigma) = 1/2$

In cases 1)–3) *we have*

$$\overline{C}(\mathcal{V}_{\sigma,c}) = \inf_{\lambda\in[\mathrm{d}(\sigma),1]} \bigl(\lambda c + \log \operatorname{Tr}\sigma^{\lambda}\bigr).$$

4) *The following statements are equivalent.*
  (i) *The inequality* $c \leqslant c^*(\sigma)$ *holds.*
  (ii) $\overline{C}(\mathcal{V}_{\sigma,c}) \leqslant H\bigl(\Omega(\mathcal{V}_{\sigma,c})\bigr)$.
  (iii) $\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}\bigl(\mathcal{V}_{\sigma,c} \cap \mathcal{L}\bigl(\Omega(\mathcal{V}_{\sigma,c})\bigr)\bigr)$.
  (iv) *There is an optimal measure for the set* $\mathcal{V}_{\sigma,c}$.

5) *The set* $\mathcal{V}_{\sigma,c}$ *is regular if and only if* $\mathrm{d}(\sigma) < 1$ *and* $c < c^*(\sigma)$.

Fig. 2 shows the results of numerical calculations of the χ-capacity of $\mathcal{V}_{\sigma,c}$ as a function of c for a state σ with $c^*(\sigma) < +\infty$.

Proof. Let $\sigma = \sum_k \lambda_k |k\rangle\langle k|$ be a state of full rank. Then $-\log\sigma$is an $\mathfrak{H}$-operator. Theorem 1, 2) of [11] implies that

$$\overline{C}(\mathcal{V}_{\sigma,c}) \leqslant c. \tag{23}$$

Let $c \leqslant H(\sigma) \leqslant +\infty$. We consider the subset $\mathcal{T} = \mathcal{V}_{\sigma,c} \cap \mathcal{L}(\sigma)$ of $\mathcal{V}_{\sigma,c}$. Since the χ-capacity is monotone, we see from (23) that $\overline{C}(\mathcal{T}) \leqslant \overline{C}(\mathcal{V}_{\sigma,c}) \leqslant c < +\infty$. Thus, to prove the equation $\overline{C}(\mathcal{V}_{\sigma,c}) = c$, it suffices to show that $\overline{C}(\mathcal{T}) \geqslant c$.

Let G be the group of all unitary operators on $\mathcal{H}$ that are diagonal in the basis $\{|k\rangle\}$. Since the set $\mathcal{T}$ is invariant under the action of the automorphism $U(\,\cdot\,)U^*$ for each $U \in G$, Corollary 5 implies that $\Omega(\mathcal{T}) = \sigma$. By Theorem 1, 2) of [11], to prove the inequality $\overline{C}(\mathcal{T}) \geqslant c$, it suffices to find a state σ_c in the set $\mathcal{T}$ such that $H\bigl(\sigma_c \,\|\, \Omega(\mathcal{T})\bigr) = H(\sigma_c \,\|\, \sigma) = c$.

In the case $H(\sigma) < +\infty$, the relative entropy $H(\rho \,\|\, \sigma)$ is a continuous function on $\mathcal{L}(\sigma)$ with range $[0, H(\sigma)]$ by Proposition 6 of [11]. This guarantees the existence of a state σ_c with the desired properties.

In the case $H(\sigma) = +\infty$, the existence of σ_c follows from Lemma 2 below (with $n = 1$).

Thus $\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{T}) = c$, and assertion 3) of Theorem 1 implies that $\Omega(\mathcal{V}_{\sigma,c}) = \Omega(\mathcal{T}) = \sigma$.

Let $c > H(\sigma)$. Since $\mathcal{K}_{-\log\sigma,c} \subset \mathcal{V}_{\sigma,c}$ and the χ-capacity is monotone, we have

$$\overline{C}(\mathcal{K}_{-\log\sigma,c}) \leqslant \overline{C}(\mathcal{V}_{\sigma,c}). \tag{24}$$

We note that $c^*(\sigma) = h_*(-\log\sigma)$. By Proposition 6, to prove all assertions concerning the cases $H(\sigma) < c \leqslant c^*(\sigma)$ and $c \geqslant c^*(\sigma)$, it suffices to show that

$$\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{K}_{-\log\sigma,c}) \tag{25}$$

since this equation implies that $\Omega(\mathcal{V}_{\sigma,c}) = \Omega(\mathcal{K}_{-\log\sigma,c})$ by assertion 3) of Theorem 1.

Suppose that $\mathrm{d}(\sigma) = \mathrm{g}(-\log\sigma) = 1$. Then $c^*(\sigma) = h_*(-\log\sigma) = H(\sigma)$. We have $\overline{C}(\mathcal{K}_{-\log\sigma,c}) = c$ for all $c \geqslant H(\sigma)$ by Proposition 6. Thus inequalities (23) and (24) yield equation (25).

Suppose that $\mathrm{d}(\sigma) = \mathrm{g}(-\log\sigma) < 1$. Then Lemma 3 of [11] implies that

$$H\bigl(\rho \,\|\, \bigl(\operatorname{Tr}\sigma^\lambda\bigr)^{-1}\sigma^\lambda\bigr) \leqslant \lambda H(\rho \,\|\, \sigma) + \log\operatorname{Tr}\sigma^\lambda \leqslant \lambda c + \log\operatorname{Tr}\sigma^\lambda$$

for all ρ in $\mathcal{V}_{\sigma,c}$ and all $\lambda \in (\mathrm{d}(\sigma), 1]$. Using Theorem 1, 2) of [11], we obtain that

$$\overline{C}(\mathcal{V}_{\sigma,c}) \leqslant \inf_{\lambda\in(\mathrm{d}(\sigma),1]} \sup_{\rho\in\mathcal{V}_{\sigma,c}} H\bigl(\rho \,\|\, \bigl(\operatorname{Tr}\sigma^\lambda\bigr)^{-1}\sigma^\lambda\bigr) \leqslant \inf_{\lambda\in(\mathrm{d}(\sigma),1]} \bigl(\lambda c + \log\operatorname{Tr}\sigma^\lambda\bigr).$$

By Proposition 1 in [11] and Proposition 6 we have

$$\overline{C}(\mathcal{K}_{-\log\sigma,c}) = \inf_{\lambda\in(\mathrm{d}(\sigma),+\infty)} \bigl(\lambda c + \log\operatorname{Tr}\sigma^\lambda\bigr).$$

We easily see from the condition $c > H(\sigma)$ that this infimum is attained at some $\lambda^* \leqslant 1$. Hence this infimum coincides with the previous one and equation (25) holds in this case.

The equivalence of statements (i)–(iv) will be established by proving the following implications: (i) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) and (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (iv). In the case $H(\sigma) < +\infty$ we prove the existence of an optimal measure for $\mathcal{V}_{\sigma,c}$ (provided that $c \leqslant c^*(\sigma)$) separately for $c \leqslant H(\sigma)$ and $H(\sigma) < c \leqslant c^*(\sigma)$.

If $c \leqslant H(\sigma)$, then the observation above shows that $\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{T})$, where $\mathcal{T} = \mathcal{V}_{\sigma,c} \cap \mathcal{L}(\sigma)$. By Proposition 6 of [11], the entropy is continuous on $\mathcal{T}$. Hence $\mathcal{T}$ is regular. Combining this with Theorem 2 of [11], we see that there is an optimal measure for $\mathcal{T}$. Since $\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{T})$ and $\mathcal{T} \subset \mathcal{V}_{\sigma,c}$, this measure is also an optimal measure for $\mathcal{V}_{\sigma,c}$.

If $\mathrm{d}(\sigma) < 1$ and $H(\sigma) < c \leqslant c^*(\sigma)$, then the observation above shows that $\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{K}_{-\log\sigma,c})$ and $c^*(\sigma) = h_*(-\log\sigma)$. By Proposition 6 there is an optimal measure for $\mathcal{K}_{-\log\sigma,c}$. Since $\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{K}_{-\log\sigma,c})$ and $\mathcal{K}_{-\log\sigma,c} \subset \mathcal{V}_{\sigma,c}$, this measure is also an optimal measure for $\mathcal{V}_{\sigma,c}$.

In the case $H(\sigma) = +\infty$ we prove the existence of an optimal measure by the following direct construction.

Given c, we consider the positive integer m and the state $\rho_{c,1,m}$ that are provided by Lemma 2 below. We put $P_m = \sum_{k=1}^{m} |k\rangle\langle k|$. Let G_m be the *compact* group of all unitary operators in $\mathfrak{B}\big(P_m(\mathcal{H})\big)$ that are diagonal in the basis $\{|k\rangle\}_{k=1}^{m}$ of the subspace $P_m(\mathcal{H})$. For arbitrary U in G_m we consider the unitary operator $\widehat{U} = U \oplus I_{\mathcal{H}\ominus P_m(\mathcal{H})}$ on $\mathcal{H}$. Using the properties of $\rho_{c,1,m}$, we easily see that

$$\int_{G_m} \widehat{U}\rho_{c,1,m}\widehat{U}^*\mu_H(dU) = \sigma,$$

where μ_H is the Haar measure on G_m. Since

$$H\big(\widehat{U}\rho_{c,1,m}\widehat{U}^* \,\|\, \sigma\big) = H\big(\rho_{c,1,m} \,\|\, \widehat{U}^*\sigma\widehat{U}\big) = H(\rho_{c,1,m} \,\|\, \sigma) = c,$$

the image of μ_H under the map $U \mapsto \widehat{U}\rho_{c,1,m}\widehat{U}^*$ is an optimal measure for $\mathcal{V}_{\sigma,c}$. The support of this measure lies in $\mathcal{L}(\sigma)$ by construction.

(iv) $\Longrightarrow$ (ii). This follows directly from Corollary 8 in [11].

(ii) $\Longrightarrow$ (i). If $c^*(\sigma) < +\infty$ and $c > c^*(\sigma)$, then the proof of assertion 3) along with Proposition 1 of [11] and Proposition 6 yields that

$$\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{K}_{-\log\sigma,c}) > H\big(\Omega(\mathcal{K}_{-\log\sigma,c})\big), \qquad \Omega(\mathcal{V}_{\sigma,c}) = \Omega(\mathcal{K}_{-\log\sigma,c}). \tag{26}$$

(i) $\Longrightarrow$ (iii). If $c \leqslant H(\sigma)$, then the observation above shows that $\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{T})$ and $\Omega(\mathcal{V}_{\sigma,c}) = \sigma$, where $\mathcal{T} = \mathcal{V}_{\sigma,c} \cap \mathcal{L}(\sigma)$. If $H(\sigma) < c \leqslant c^*(\sigma)$, then the proof of assertion 2) along with Propositions 5, 6 yields that

$$\overline{C}(\mathcal{V}_{\sigma,c}) = \overline{C}(\mathcal{K}_{-\log\sigma,c}) = H\big(\Omega(\mathcal{K}_{-\log\sigma,c})\big) = \overline{C}\big(\mathcal{L}(\Omega(\mathcal{K}_{-\log\sigma,c}))\big). \tag{27}$$

Since $\mathcal{L}\big(\Omega(\mathcal{K}_{-\log\sigma,c})\big) \subset \mathcal{K}_{-\log\sigma,c} \subset \mathcal{V}_{\sigma,c}$ and $\Omega(\mathcal{V}_{\sigma,c}) = \Omega(\mathcal{K}_{-\log\sigma,c})$, we obtain (iii).

(iii) $\Longrightarrow$ (i). If $c^*(\sigma) < +\infty$ and $c > c^*(\sigma)$, then inequality (26) holds. This contradicts (iii) by Proposition 5.

If $\mathrm{d}(\sigma) < 1$ and $c < c^*(\sigma)$, then $\mathrm{d}\big(\Omega(\mathcal{V}_{\sigma,c})\big) < 1$ by the observation above. Hence the regularity of $\mathcal{V}_{\sigma,c}$ follows from assertion 5) of Theorem 1.

To prove the converse assertion, we note that Lemma 3 below and the part of the proposition already proved imply that the second regularity condition does not hold for $\mathcal{V}_{\sigma,c}$ for any state σ of infinite rank and any $c > 0$. Thus it suffices to show that the first regularity condition does not hold for $\mathcal{V}_{\sigma,c}$ if either $\mathrm{d}(\sigma) = 1$ or $c \geqslant c^*(\sigma)$.

If $\mathrm{d}(\sigma)=1$, then the observation above shows that $\Omega(\mathcal{V}_{\sigma,c})=\sigma$ for all c. In the case $H(\sigma)<+\infty$, Proposition 2 of [11] implies that there is a sequence $\{\rho_n\}$ of states such that

$$\lim_{n\to+\infty} H(\rho_n\,\|\,\sigma)=0, \qquad \lim_{n\to+\infty} H(\rho_n)>H(\sigma).$$

Thus the state ρ_n belongs to $\mathcal{V}_{\sigma,c}$ for all sufficiently large n and, therefore, the first regularity condition does not hold. In the case $H(\sigma)=+\infty$, it is clear that the first regularity condition does not hold.

If $\mathrm{d}(\sigma)<1$ and $c\geqslant c^*(\sigma)$, then $\Omega(\mathcal{V}_{\sigma,c})=\bigl(\operatorname{Tr}\sigma^{\mathrm{d}(\sigma)}\bigr)^{-1}\sigma^{\mathrm{d}(\sigma)}$ by the observation above. As shown in the proof of Proposition 3 in [11], for every m there is a sequence $\{\rho_n^m\}_n$ of states that satisfy formulae (19) in [11] and lie in $\mathcal{V}_{\sigma,c}$ for all sufficiently large n. Thus the first regularity condition does not hold in this case.

The set $\mathcal{V}_{\sigma,c}$ with $H(\sigma)=+\infty$ is a non-trivial example of an irregular set of finite χ-capacity containing states with infinite entropy and having an optimal measure.

Lemma 2. *Let $\sigma=\sum_{k=1}^{\infty}\lambda_k|k\rangle\langle k|$ be a state with infinite entropy. For every positive integer n let $\mathcal{L}_n(\sigma)$ be the closed convex subset of $\mathcal{L}(\sigma)$ consisting of all states ρ such that $\langle i|\rho|j\rangle=0$ if $i\neq j$ and either $i<n$ or $j<n$. Then for every $c\geqslant 0$ and every $n\in\mathbb{N}$ one can find a positive integer m and a state $\rho_{c,n,m}$ in $\mathcal{L}_n(\sigma)$ such that*

$$H(\rho_{c,n,m}\,\|\,\sigma)=c$$

and $\langle i|\rho_{c,n,m}|j\rangle=0$ if $i\neq j$ and either $i>m$ or $j>m$.

Proof. Let $c\geqslant 0$ and $n\in\mathbb{N}$ be arbitrary numbers. We consider the state

$$\sigma_n=\mu_n^{-1}\sum_{k=n}^{+\infty}\lambda_k|k\rangle\langle k|,$$

where $\mu_n=\sum_{k=n}^{+\infty}\lambda_k$. Consider the sequence of states

$$\biggl\{\rho_n^m=\mu_n^{-1}\sum_{n\leqslant i,j\leqslant m}\sqrt{\lambda_i}\,\sqrt{\lambda_j}\,|i\rangle\langle j|+\mu_n^{-1}\sum_{k>m}\lambda_k|k\rangle\langle k|\biggr\}_m$$

converging to the pure state $\rho_n^*=\mu_n^{-1}\sum_{i,j\geqslant n}\sqrt{\lambda_i}\,\sqrt{\lambda_j}\,|i\rangle\langle j|$ as $m\to+\infty$. Since $H(\sigma_n)=+\infty$, it follows from Proposition 6 in [11] that $H(\rho_n^*\,\|\,\sigma_n)=+\infty$. Using this and general properties of the relative entropy, we obtain

$$H(\rho_n^m\,\|\,\sigma_n)<+\infty \quad \forall\, m\in\mathbb{N}, \qquad \lim_{m\to+\infty} H(\rho_n^m\,\|\,\sigma_n)=+\infty.$$

Thus there is a positive integer $m(c)$ such that

$$c\mu_n^{-1}\leqslant H(\rho_n^{m(c)}\,\|\,\sigma_n)<+\infty.$$

The convex lower semicontinuous function $f(\lambda)=H\bigl(\lambda\rho_n^{m(c)}+(1-\lambda)\sigma_n\,\|\,\sigma_n\bigr)$ does not exceed $\lambda H\bigl(\rho_n^{m(c)}\,\|\,\sigma_n\bigr)$ on $[0,1]$. Hence it is continuous on $[0,1]$ (see [2]).

Since $f(0) = 0$ and $f(1) = H\big(\rho_n^{m(c)} \,\|\, \sigma_n\big) \geqslant c\mu_n^{-1}$, there is a $\lambda^* \in [0,1]$ such that $f(\lambda^*) = c\mu_n^{-1}$.

Put $m = m(c)$ and $\rho_{c,n,m} = \sum_{k=1}^{n-1} \lambda_k |k\rangle\langle k| + \mu_n\big(\lambda^* \rho_n^m + (1-\lambda^*)\sigma_n\big)$. It is easy to see that $H(\rho_{c,n,m} \,\|\, \sigma) = \mu_n H\big(\lambda^* \rho_n^{m(c)} + (1-\lambda^*)\sigma_n \,\|\, \sigma_n\big) = c$ and that $\rho_{c,n,m} \in \mathcal{L}_n(\sigma)$. By construction, $\langle i|\rho_{c,n,m}|j\rangle = 0$ if $i \neq j$ and either $i > m$ or $j > m$.

Lemma 3. *Suppose that σ is a state of infinite rank. Then the relative entropy $H\big(\rho \,\|\, \big(\operatorname{Tr}\sigma^\lambda\big)^{-1}\sigma^\lambda\big)$ is a discontinuous function of ρ on the set $\mathcal{V}_{\sigma,c}$ for every $c > 0$ and every λ such that $\operatorname{Tr}\sigma^\lambda < +\infty$.*

Proof. We may assume without loss of generality that σ is a full rank state. Let ϱ be a pure state with $H(\varrho \,\|\, \sigma) = +\infty$ and P_n the spectral projector of σ corresponding to its maximal n eigenvalues. Then the sequence $\varrho_n = (\operatorname{Tr} P_n\varrho)^{-1} P_n\varrho P_n$ of pure states converges to the pure state ϱ. Using general properties of the relative entropy, we see that $H(\varrho_n \,\|\, \sigma) < +\infty$ for all n and $\lim_{n\to+\infty} H(\varrho_n \,\|\, \sigma) = +\infty$.

Consider the sequence $\big\{\eta_n = c\big(H(\varrho_n \,\|\, \sigma)\big)^{-1}\big\}_{n \geqslant n_0}$, where we choose n_0 to be such that $H(\varrho_n \,\|\, \sigma) > c$ for all $n \geqslant n_0$. Put $\rho_n = \eta_n\varrho_n + (1-\eta_n)\sigma$ for all $n \geqslant n_0$. Using general properties of the relative entropy, we get

$$c - h_2(\eta_n) = \eta_n H(\varrho_n \,\|\, \sigma) - h_2(\eta_n) \leqslant H(\rho_n \,\|\, \sigma) \leqslant \eta_n H(\varrho_n \,\|\, \sigma) = c,$$

where $h_2(x) = -x\log x - (1-x)\log(1-x)$.

Since $\eta_n \to 0$ as $n \to 0$, this inequality implies that

$$\rho_n \in \mathcal{V}_{\sigma,c} \text{ for all } n \quad \text{and} \quad \lim_{n\to+\infty} H(\rho_n \,\|\, \sigma) = c. \tag{28}$$

Let λ be an arbitrary positive number such that $\operatorname{Tr}\sigma^\lambda < +\infty$. By Lemma 3 of [11] we have

$$H\big(\rho_n \,\|\, \big(\operatorname{Tr}\sigma^\lambda\big)^{-1}\sigma^\lambda\big) = \lambda H(\rho_n \,\|\, \sigma) + \log\operatorname{Tr}\sigma^\lambda - (1-\lambda)H(\rho_n). \tag{29}$$

Using general properties of the entropy, we obtain

$$(1-\eta_n)H(\sigma) \leqslant H(\rho_n) \leqslant (1-\eta_n)H(\sigma) + h_2(\eta_n)$$

for all $n \geqslant n_0$. Hence $\lim_{n\to+\infty} H(\rho_n) = H(\sigma)$.

Thus (28) and (29) imply that

$$\lim_{n\to+\infty} H\big(\rho_n \,\|\, \big(\operatorname{Tr}\sigma^\lambda\big)^{-1}\sigma^\lambda\big) = c\lambda + \log\operatorname{Tr}\sigma^\lambda - (1-\lambda)H(\sigma).$$

By construction, the sequence $\{\rho_n\}$ of states in $\mathcal{V}_{\sigma,c}$ converges to σ. Since

$$H\big(\sigma \,\|\, \big(\operatorname{Tr}\sigma^\lambda\big)^{-1}\sigma^\lambda\big) = \log\operatorname{Tr}\sigma^\lambda - (1-\lambda)H(\sigma),$$

the above equation means that

$$\lim_{n\to+\infty} H\big(\rho_n \,\|\, \big(\operatorname{Tr}\sigma^\lambda\big)^{-1}\sigma^\lambda\big) = H\big(\sigma \,\|\, \big(\operatorname{Tr}\sigma^\lambda\big)^{-1}\sigma^\lambda\big) + c\lambda.$$

Hence the function $H\big(\rho \,\|\, \big(\operatorname{Tr}\sigma^\lambda\big)^{-1}\sigma^\lambda\big)$ is discontinuous on $\mathcal{V}_{\sigma,c}$.

The irregularity of the set $\mathcal{V}_{\sigma,c}$ for every state σ with infinite entropy leads to the discontinuity of the χ-capacity with respect to monotone decreasing sequences of subsets of this set (assertion 7) of Theorem 1). We illustrate this by the following example.

Example 3 (a decreasing sequence of closed sets of the same positive χ-capacity such that their intersection has χ-capacity zero). Let σ be a state with infinite entropy. For arbitrary positive integer n let $\mathcal{L}_n(\sigma)$ be the convex closed subset of $\mathfrak{S}(\mathcal{H})$ introduced in Lemma 2. For any given $c > 0$ we consider the decreasing sequence of closed convex sets $\mathcal{A}_n = \mathcal{L}_n(\sigma) \cap \mathcal{V}_{\sigma,c}$. Since σ is the only state in $\mathcal{A}_n$ invariant under the action of all automorphisms in $\mathfrak{F}(\mathcal{A}_n)$, Corollary 5 implies that $\Omega(\mathcal{A}_n) = \sigma$. For every n, Lemma 2 shows that there is a state $\rho_{c,n,m}$ in $\mathcal{A}_n$ such that $H\bigl(\rho_{c,n,m} \,\|\, \Omega(\mathcal{A}_n)\bigr) = H(\rho_{c,n,m} \,\|\, \sigma) = c$. By Theorem 1 of [11] it follows that $\overline{C}(\mathcal{A}_n) \geqslant c$. By assertion 3) of Theorem 1 we have $\overline{C}(\mathcal{A}_n) \leqslant \overline{C}(\mathcal{V}_{\sigma,c}) = c$ and, therefore, $\overline{C}(\mathcal{A}_n) = c$ for all n. On the other hand, $\overline{C}\bigl(\bigcap_n \mathcal{A}_n\bigr) = 0$ because $\bigcap_n \mathcal{A}_n = \{\sigma\}$.

3.4. The set $\mathcal{A} \otimes \mathcal{B}$. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Given arbitrary subsets $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathfrak{S}(\mathcal{K})$, we consider the set

$$\mathcal{A} \otimes \mathcal{B} = \{\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \mid \omega^{\mathcal{H}} \in \mathcal{A},\ \omega^{\mathcal{K}} \in \mathcal{B}\},$$

where $\omega^{\mathcal{H}} = \operatorname{Tr}_{\mathcal{K}} \omega$ and $\omega^{\mathcal{K}} = \operatorname{Tr}_{\mathcal{H}} \omega$.

The following lemma is proved in [12].

Lemma 4. *The set $\mathcal{A} \otimes \mathcal{B}$ is a convex subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are convex subsets of $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{K})$ respectively.*

The set $\mathcal{A} \otimes \mathcal{B}$ is a compact subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are compact subsets of $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{K})$ respectively.

The properties of the restriction of the entropy to $\mathcal{A} \otimes \mathcal{B}$ are also determined by the properties of the restrictions of the entropy to $\mathcal{A}$ and $\mathcal{B}$.

Proposition 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary subsets of $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{K})$ respectively.*

1) *The entropy is bounded on $\mathcal{A} \otimes \mathcal{B}$ if and only if it is bounded on $\mathcal{A}$ and $\mathcal{B}$.*

2) *The entropy is continuous on $\mathcal{A} \otimes \mathcal{B}$ if and only if it is continuous on $\mathcal{A}$ and $\mathcal{B}$.*

Proof. If the entropy is bounded (continuous) on $\mathcal{A} \otimes \mathcal{B}$, then it is bounded (continuous) on $\mathcal{A}$ and $\mathcal{B}$ since the state $\rho \otimes \sigma$ lies in $\mathcal{A} \otimes \mathcal{B}$ for every state ρ in $\mathcal{A}$ and every state σ in $\mathcal{B}$ and we have $H(\rho \otimes \sigma) = H(\rho) + H(\sigma)$.

If the entropy is bounded on $\mathcal{A}$ and $\mathcal{B}$, then it is bounded on $\mathcal{A} \otimes \mathcal{B}$ by the subadditivity property.

Suppose that the entropy is continuous on $\mathcal{A}$ and $\mathcal{B}$. Let ω_0 be a state in $\mathcal{A} \otimes \mathcal{B}$ and $\{\omega_n\}$ a sequence of states in $\mathcal{A} \otimes \mathcal{B}$ converging to ω_0. Since

$$H(\omega_n) = H\bigl(\omega_n^{\mathcal{H}}\bigr) + H\bigl(\omega_n^{\mathcal{K}}\bigr) - H\bigl(\omega_n \,\|\, \omega_n^{\mathcal{H}} \otimes \omega_n^{\mathcal{K}}\bigr), \qquad n = 0, 1, 2, \dots,$$

the assumption of continuity and the lower semicontinuity of the relative entropy yield that

$$\limsup_{n\to+\infty} H(\omega_n) = \lim_{n\to+\infty} H\bigl(\omega_n^{\mathcal{H}}\bigr) + \lim_{n\to+\infty} H\bigl(\omega_n^{\mathcal{K}}\bigr) - \liminf_{n\to+\infty} H\bigl(\omega_n \,\|\, \omega_n^{\mathcal{H}} \otimes \omega_n^{\mathcal{K}}\bigr)$$
$$\leqslant H\bigl(\omega_0^{\mathcal{H}}\bigr) + H\bigl(\omega_0^{\mathcal{K}}\bigr) - H\bigl(\omega_0 \,\|\, \omega_0^{\mathcal{H}} \otimes \omega_0^{\mathcal{K}}\bigr) = H(\omega_0).$$

Using this inequality and the lower semicontinuity of the entropy, we see that $\lim_{n\to+\infty} H(\omega_n) = H(\omega_0)$.

An important example of the set $\mathcal{A}\otimes\mathcal{B}$ is the set of all states ω in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ with given partial traces $\omega^{\mathcal{H}} = \rho$ and $\omega^{\mathcal{K}} = \sigma$. Following [8], we denote this set by $\mathcal{C}(\rho,\sigma)$. Lemma 4 shows that $\mathcal{C}(\rho,\sigma)$ is convex and compact for all ρ and σ. It follows from the subadditivity of the entropy that $\sup_{\omega\in\mathcal{C}(\rho,\sigma)} H(\omega) = H(\rho)+H(\sigma)$. As in the case of the set $\mathcal{L}(\sigma)$, the finiteness of the entropy on $\mathcal{C}(\rho,\sigma)$ guarantees that it is continuous.

Corollary 6. *The entropy is continuous on the set $\mathcal{C}(\rho,\sigma)$ if and only if $H(\rho)<+\infty$ and $H(\sigma)<+\infty$.*

Let $\{\pi_i,\rho_i\}$ and $\{\lambda_j,\sigma_j\}$ be arbitrary ensembles of states in $\mathcal{A}$ and $\mathcal{B}$ respectively. Their *tensor product* is the ensemble $\{\pi_i\lambda_j,\ \rho_i\otimes\sigma_j\}$ of states in $\mathcal{A}\otimes\mathcal{B}$. Considering the tensor products of all possible ensembles of states in $\mathcal{A}$ and $\mathcal{B}$, we easily see that

$$\overline{C}(\mathcal{A}\otimes\mathcal{B}) \geqslant \overline{C}(\mathcal{A}) + \overline{C}(\mathcal{B}). \tag{30}$$

There are examples of sets $\mathcal{A}$ and $\mathcal{B}$ for which equality holds in (30). For instance, if $\mathcal{A}$ and $\mathcal{B}$ are the sets considered in § 3.2, then (30) is an equation by the subadditivity of the entropy. There are also examples of sets for which strict inequality holds in (30). Moreover, if $\mathcal{A}=\{\rho\}$ and $\mathcal{B}=\{\sigma\}$, where ρ and σ are isomorphic states with infinite entropy in $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{K})$ respectively, then the left-hand side of (30) equals $+\infty$ by Proposition 9 below while the right-hand side is obviously equal to zero.[16]

We note that equality in (30) yields

$$\Omega(\mathcal{A}\otimes\mathcal{B}) = \Omega(\mathcal{A})\otimes\Omega(\mathcal{B}). \tag{31}$$

Indeed, if $\{\{\pi_i^n,\rho_i^n\}\}_n$ and $\{\{\lambda_j^n,\sigma_j^n\}\}_n$ are approximating sequences of ensembles for $\mathcal{A}$ and $\mathcal{B}$ respectively, then equality in (30) implies that the sequence of ensembles $\{\{\pi_i^n\lambda_j^n,\ \rho_i^n\otimes\sigma_j^n\}\}_n$ is an approximating sequence for $\mathcal{A}\otimes\mathcal{B}$. By Theorem 1 of [11], the sequences $\{\overline{\rho}_n\}$ and $\{\overline{\sigma}_n\}$ converge to the optimal average states $\Omega(\mathcal{A})$ and $\Omega(\mathcal{B})$ respectively. Hence the sequence $\{\overline{\rho}_n\otimes\overline{\sigma}_n\}$ converges to the state $\Omega(\mathcal{A})\otimes\Omega(\mathcal{B})$ and, therefore, this state is the optimal average state $\Omega(\mathcal{A}\otimes\mathcal{B})$ for $\mathcal{A}\otimes\mathcal{B}$ by Theorem 1 of [11]. Proposition 9 below shows that (31) does not guarantee equality in (30).

[16] Strict inequality in (30) does not contradict the additivity conjecture for the χ-capacity of quantum channels. Indeed, if $\mathcal{A}$ and $\mathcal{B}$ are the output sets of channels Φ and Ψ respectively, then the output set of $\Phi\otimes\Psi$ is a proper subset of $\mathcal{A}\otimes\mathcal{B}$.

We consider the set $\mathcal{C}(\rho,\sigma)$. Write $\rho=\sum_i \pi_i|e_i\rangle\langle e_i|$ and $\sigma=\sum_j \lambda_j|f_j\rangle\langle f_j|$, where $\{|e_i\rangle\}$ and $\{|f_j\rangle\}$ are orthonormal systems of vectors in $\mathcal{H}$ and $\mathcal{K}$ respectively. Let $E_{ij}=|e_i\rangle\langle e_j|$ and $F_{kl}=|f_k\rangle\langle f_l|$ be the operators of rank 1 in $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{B}(\mathcal{K})$ respectively. Given any probability distributions $\{\pi_i\}$ and $\{\lambda_j\}$, let $\mathcal{C}(\{\pi_i\},\{\lambda_j\})$ be the set of all probability distributions $\{\omega_{ij}\}$ such that $\sum_j \omega_{ij}=\pi_i$ and $\sum_i \omega_{ij}=\lambda_j$. Hence $\mathcal{C}(\{\pi_i\},\{\lambda_j\})$ is a classical analogue of the set $\mathcal{C}(\rho,\sigma)$. Let $\mathcal{C}_s(\rho,\sigma)$ be the closed convex subset of $\mathcal{C}(\rho,\sigma)$ consisting of all states of the form $\sum_{i,j}\omega_{ij}E_{ii}\otimes F_{jj}$, where $\{\omega_{ij}\}\in\mathcal{C}(\{\pi_i\},\{\lambda_j\})$. The set $\mathcal{C}_s(\rho,\sigma)$ can be identified with the classical analogue $\mathcal{C}(\{\pi_i\},\{\lambda_j\})$ of the set $\mathcal{C}(\rho,\sigma)$.

Let G be the group of all unitary operators on $\mathcal{H}\otimes\mathcal{K}$ that are diagonal in the basis $\{|e_i\otimes f_j\rangle\}$. We shall use the following simple observation.

Lemma 5. *Let $\rho=\sum_i \pi_i|e_i\rangle\langle e_i|$ and $\sigma=\sum_j \lambda_j|f_j\rangle\langle f_j|$ be states in $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{K})$ respectively.*

1) *Any state ω in $\mathcal{C}(\rho,\sigma)$ may be represented as*

$$\omega=\sum_{i,j}\omega_{ij}E_{ii}\otimes F_{jj}+\sum_{i\neq j,\,k\neq l}\eta_{ijkl}E_{ij}\otimes F_{kl},$$

where $\{\omega_{ij}\}\in\mathcal{C}(\{\pi_i\},\{\lambda_j\})$.

2) *The set $\mathcal{C}(\rho,\sigma)$ is invariant under the automorphism group $\{U(\,\cdot\,)U^*\}_{U\in G}$. The set $\mathcal{C}_s(\rho,\sigma)$ consists of all invariant states for this group that are contained in $\mathcal{C}(\rho,\sigma)$.*

Proof. Any state ω in $\mathcal{C}(\rho,\sigma)$ may be represented as

$$\omega=\sum_{i,j,k,l}\eta_{ijkl}E_{ij}\otimes F_{kl}.$$

The requirements $\mathrm{Tr}_{\mathcal{K}}\,\omega=\rho=\sum_i \pi_i E_{ii}$ and $\mathrm{Tr}_{\mathcal{H}}\,\omega=\sigma=\sum_j \lambda_j F_{jj}$ yield assertion 1) of the lemma.

Any operator U in G is determined by the set $\{\varphi_{ij}(U)\}_{i,j}$ of numbers in $[0,2\pi)$ via the formula

$$U=\sum_{i,j}\exp\bigl(\mathrm{i}\,\varphi_{ij}(U)\bigr)E_{ii}\otimes F_{jj}.$$

Therefore we have $UE_{ii}\otimes F_{jj}U^*=E_{ii}\otimes F_{jj}$ and $UE_{ij}\otimes F_{kl}U^*=\exp\bigl(\mathrm{i}(\varphi_{ik}-\varphi_{jl})\bigr)E_{ij}\otimes F_{kl}$. Thus, given $\omega\in\mathcal{C}(\rho,\sigma)$ and U as above, we obtain

$$U\omega U^*=\sum_{i,j}\omega_{ij}E_{ii}\otimes F_{jj}+\sum_{i\neq j,\,k\neq l}\eta_{ijkl}\exp\bigl(\mathrm{i}(\varphi_{ik}-\varphi_{jl})\bigr)E_{ij}\otimes F_{kl}.$$

This proves assertion 2) of the lemma.

The following proposition shows that the problems of calculating the χ-capacity and finding the optimal average state of $\mathcal{C}(\rho,\sigma)$ are non-trivial even in the symmetric case $\rho\cong\sigma$.

Proposition 9. *Let $\rho=\sum_i \lambda_i|e_i\rangle\langle e_i|$ and $\sigma=\sum_j \lambda_j|f_j\rangle\langle f_j|$ be isomorphic states with $H(\rho)=H(\sigma)=-\sum_i \lambda_i\log\lambda_i=h\leqslant+\infty$. Then the following assertions hold.*

1) *We have* $h \leqslant \overline{C}\big(\mathcal{C}(\rho,\sigma)\big) \leqslant 2h$. *Equality on the left holds if and only if* ρ *and* σ *are pure states.*

2) *In the case* $h < +\infty$, *there is an optimal measure* $\mu_*(\rho,\sigma)$ *with barycentre* $\Omega\big(\mathcal{C}(\rho,\sigma)\big)$ *in* $\mathcal{C}_s(\rho,\sigma)$ *such that* $\operatorname{supp}\Omega\big(\mathcal{C}(\rho,\sigma)\big) = \operatorname{supp}\rho\otimes\operatorname{supp}\sigma$ *and the following statements are equivalent.*

(i) $\overline{C}\big(\mathcal{C}(\rho,\sigma)\big) = 2h$.

(ii) $\Omega\big(\mathcal{C}(\rho,\sigma)\big) = \rho\otimes\sigma$.

(iii) *The states* ρ *and* σ *are multiples of projectors of the same finite rank.*

(iv) *The measure* $\mu_*(\rho,\sigma)$ *is supported by pure states.*

Proof. Since the entropy is subadditive, we have $H(\omega) \leqslant H(\rho) + H(\sigma) = 2h$ for all ω in $\mathcal{C}(\rho,\sigma)$. This yields the upper bound for $\overline{C}\big(\mathcal{C}(\rho,\sigma)\big)$.

Suppose that $\overline{C}\big(\mathcal{C}(\rho,\sigma)\big) < +\infty$. By Theorem 1 of [11], there is a unique state $\Omega\big(\mathcal{C}(\rho,\sigma)\big)$ in $\mathcal{C}(\rho,\sigma)$ such that

$$H\big(\omega \,\|\, \Omega(\mathcal{C}(\rho,\sigma))\big) \leqslant \overline{C}\big(\mathcal{C}(\rho,\sigma)\big) \qquad \forall\,\omega\in\mathcal{C}(\rho,\sigma). \tag{32}$$

By Corollary 4, $\Omega\big(\mathcal{C}(\rho,\sigma)\big)$ is invariant under the automorphism $U(\,\cdot\,)U^*$ for every U in G. Using Lemma 5, we obtain

$$\Omega\big(\mathcal{C}(\rho,\sigma)\big) = \sum_{i,j}\omega_{ij}E_{ii}\otimes F_{jj} \tag{33}$$

for some probability distribution $\{\omega_{ij}\}$ in $\mathcal{C}(\{\lambda_i\},\{\lambda_j\})$. All the probabilities ω_{ij} in this distribution must be positive since otherwise one can easily find a state ω in $\mathcal{C}(\rho,\sigma)$ such that $H\big(\omega \,\|\, \Omega(\mathcal{C}(\rho,\sigma))\big) = +\infty$, contrary to (32).

Let $\omega = \sum_{i,j}\sqrt{\lambda_i}\,\sqrt{\lambda_j}\,E_{ij}\otimes F_{ij}$ be a pure state in $\mathcal{C}(\rho,\sigma)$. Using (32) and (33), we obtain

$$\begin{aligned} \overline{C}\big(\mathcal{C}(\rho,\sigma)\big) &\geqslant H\big(\omega \,\|\, \Omega(\mathcal{C}(\rho,\sigma))\big) = -\operatorname{Tr}\omega\log\big(\Omega(\mathcal{C}(\rho,\sigma))\big) \\ &= -\operatorname{Tr}\sum_{i,j}\sqrt{\lambda_i}\,\sqrt{\lambda_j}\,(\log\omega_{jj})E_{ij}\otimes F_{ij} = -\sum_i\lambda_i\log\omega_{ii}. \end{aligned} \tag{34}$$

If the states ρ and σ are not pure, then the right-hand side of (34) is greater than $-\sum_i\lambda_i\log\lambda_i = h$ since $\omega_{ii} + \sum_{j\neq i}\omega_{ij} = \lambda_i$ and $\omega_{ij} > 0$ for all i and j.

The existence of an optimal measure in the case $h < +\infty$ follows from Theorem 2 in [11] and Corollary 6.

The equivalence of statements (i)–(iv) will be proved in the following order: (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i). Suppose that

$$\Omega\big(\mathcal{C}(\rho,\sigma)\big) = \rho\otimes\sigma = \sum_{i,j}\lambda_i\lambda_jE_{ii}\otimes F_{jj}.$$

Let ω be the pure state introduced above. Using (34) with $\omega_{ij} = \lambda_i\lambda_j$, we obtain

$$\overline{C}\big(\mathcal{C}(\rho,\sigma)\big) \geqslant H\big(\omega \,\|\, \Omega(\mathcal{C}(\rho,\sigma))\big) = -\sum_i\lambda_i\log\lambda_i^2 = 2h.$$

Since the reverse inequality has already been proved, we have $\overline{C}\big(\mathcal{C}(\rho,\sigma)\big) = 2h$.

(i) $\Longrightarrow$ (iv). Suppose that $\overline{C}\big(\mathcal{C}(\rho,\sigma)\big)=2h=H(\rho\otimes\sigma)$. Let μ_* be an arbitrary optimal measure for $\mathcal{C}(\rho,\sigma)$. Since $\chi(\mu_*)=2h$ is the maximal value of the entropy on $\mathcal{C}(\rho,\sigma)$, we see from formula (2) of [11] that

$$\widehat{H}(\mu_*)=\int H(\omega)\mu_*(d\omega)=0.$$

Hence the measure μ_* is supported by pure states.

(iv) $\Longrightarrow$ (iii). Suppose that μ_* is an optimal measure for $\mathcal{C}(\rho,\sigma)$ and μ_* is supported by pure states. This means that the barycentre $\Omega\big(\mathcal{C}(\rho,\sigma)\big)$ of this measure belongs to the convex closure of the set of pure states in $\mathcal{C}(\rho,\sigma)$. By the observation above, $\Omega\big(\mathcal{C}(\rho,\sigma)\big)$ is a state in $\mathcal{C}_s(\rho,\sigma)$ supported by $\operatorname{supp}\rho\otimes\operatorname{supp}\sigma$. Therefore Lemma 6 (see below) yields that ρ and σ are multiples of projectors of the same finite rank.

(iii) $\Longrightarrow$ (ii). Suppose that ρ and σ are multiples of projectors of finite rank. By Lemma 6 below, there is an ensemble of pure states in $\mathcal{C}(\rho,\sigma)$ with average state $\rho\otimes\sigma$. This ensemble is clearly optimal for $\mathcal{C}(\rho,\sigma)$. Hence its average state coincides with $\Omega\big(\mathcal{C}(\rho,\sigma)\big)$.

Lemma 6. *Let ρ and σ be states in $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{K})$ respectively. Then the following statements are equivalent.*

(i) *The set $\mathcal{C}_s(\rho,\sigma)$ contains a state which is supported by* $\operatorname{supp}\rho\otimes\operatorname{supp}\sigma$ *and belongs to the convex closure of the set of all pure states in $\mathcal{C}(\rho,\sigma)$.*

(ii) *The states ρ and σ are multiples of projectors of the same finite rank.*

(iii) *The state $\rho\otimes\sigma$ is a finite convex combination of pure states in $\mathcal{C}(\rho,\sigma)$.*

Proof. Each of the statements in the lemma implies that the states ρ and σ are isomorphic, for otherwise the set $\mathcal{C}(\rho,\sigma)$ contains no pure states.

It suffices to show that (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii).

(i) $\Longrightarrow$ (ii). Let $\widehat{\omega}=\sum_{i,j}\omega_{ij}E_{ii}\otimes F_{jj}$ be a state in $\mathcal{C}_s(\rho,\sigma)$ that belongs to the convex closure of the set of pure states in $\mathcal{C}(\rho,\sigma)$. By Lemma 1 of [1], there is a measure μ supported by pure states in $\mathcal{C}(\rho,\sigma)$ such that

$$\widehat{\omega}=\int_{\mathcal{C}(\rho,\sigma)}\omega\mu(d\omega).$$

It suffices to prove that the state ρ has no distinct positive eigenvalues. Suppose that λ_i and λ_j are such eigenvalues. Using the Schmidt decomposition for any pure state ω in $\mathcal{C}(\rho,\sigma)$, it is easy to see that $E_{ii}\otimes F_{jj}\omega=0$. Thus,

$$\omega_{ij}E_{ii}\otimes F_{jj}=E_{ii}\otimes F_{jj}\widehat{\omega}=\int_{\mathcal{C}(\rho,\sigma)}E_{ii}\otimes F_{jj}\omega\mu(d\omega)=0.$$

Hence the support of $\widehat{\omega}$ does not coincide with $\operatorname{supp}\rho\otimes\operatorname{supp}\sigma$.

(ii) $\Longrightarrow$ (iii). Suppose that $\rho=d^{-1}P$ and $\sigma=d^{-1}Q$, where P and Q are d-dimensional projectors in $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{B}(\mathcal{K})$ respectively. Let $\{|\varphi_i\rangle\}$ be a basis of maximally entangled vectors in $P(\mathcal{H})\otimes Q(\mathcal{K})$. Then $\rho\otimes\sigma=d^{-2}\sum_i|\varphi_i\rangle\langle\varphi_i|$.

Remark 5. It is interesting to compare the χ-capacity of $\mathcal{C}(\rho,\sigma)$ with that of the set $\mathcal{C}_s(\rho,\sigma)$, which can be identified with the classical analogue $\mathcal{C}(\{\pi_i\},\{\lambda_j\})$ of the set $\mathcal{C}(\rho,\sigma)$. Let ρ and σ be multiples of d-dimensional projectors. Then the set $\mathcal{C}(\{\pi_i\},\{\lambda_j\})$ consists of all probability distributions $\{\omega_{ij}\}_{i,j=1}^{d}$ such that

$$\sum_{i=1}^{d}\omega_{ij}=d^{-1}=\sum_{j=1}^{d}\omega_{ij}.$$

The optimal ensemble for the set $\mathcal{C}_s(\rho,\sigma)\cong\mathcal{C}(\{\pi_i\},\{\lambda_j\})$ consists of d states having one non-zero element d^{-1} in each row and each column, with equal probabilities d^{-1}. Hence its average state is the uniform distribution $\{\omega_{ij}=d^{-2}\}$. Thus,

$$\overline{C}\big(\mathcal{C}_s(\rho,\sigma)\big)=\log d^2-\log d=\log d=h=\frac{1}{2}\,\overline{C}\big(\mathcal{C}(\rho,\sigma)\big),$$

where the last equality follows from Proposition 9. Thus the existence of entangled states in $\mathcal{C}(\rho,\sigma)$ doubles the χ-capacity.

3.5. Orbits of compact automorphism groups. Let G be a compact group, $\{U_g\}_{g\in G}$ a unitary (projective) representation of G on the Hilbert space $\mathcal{H}$ and σ an arbitrary state in $\mathfrak{S}(\mathcal{H})$. We consider the set $\mathcal{O}_{G,U_g,\sigma}=\{U_g\sigma U_g^*\}_{g\in G}$. Being the image of the compact set G under the continuous map $g\mapsto U_g\sigma U_g^*$, it is compact. Its convex closure $\overline{\mathrm{co}}(\mathcal{O}_{G,U_g,\sigma})$ is also compact. Let $\omega(G,U_g,\sigma)=\int_G U_g\sigma U_g^*\mu_H(dg)$ be a state in $\overline{\mathrm{co}}(\mathcal{O}_{G,U_g,\sigma})$, where μ_H is the Haar measure on G.

Proposition 10. *The entropy is bounded on the set $\overline{\mathrm{co}}(\mathcal{O}_{G,U_g,\sigma})$ if and only if $H\big(\omega(G,U_g,\sigma)\big)<+\infty$. In this case the entropy is continuous on the set $\overline{\mathrm{co}}(\mathcal{O}_{G,U_g,\sigma})$ and attains its maximum at the state*

$$\Gamma\big(\overline{\mathrm{co}}(\mathcal{O}_{G,U_g,\sigma})\big)=\omega(G,U_g,\sigma).$$

The χ-capacity $\overline{C}(\mathcal{O}_{G,U_g,\sigma})$ of the set $\mathcal{O}_{G,U_g,\sigma}$ is equal to $H\big(\sigma\,\|\,\omega(G,U_g,\sigma)\big)$. If $\overline{C}(\mathcal{O}_{G,U_g,\sigma})<+\infty$, then the set $\mathcal{O}_{G,U_g,\sigma}$ is regular and the image of the Haar measure μ_H under the map $g\mapsto U_g\sigma U_g^*$ is an optimal measure for the set $\mathcal{O}_{G,U_g,\sigma}$ with barycentre

$$\Omega(\mathcal{O}_{G,U_g,\sigma})=\omega(G,U_g,\sigma).$$

Proof. Since $\int_G U_g\rho U_g^*\mu_H(dg)=\omega(G,U_g,\sigma)$ for every state ρ in $\overline{\mathrm{co}}(\mathcal{O}_{G,U_g,\sigma})$, the boundedness assertion follows from the concavity of the entropy and Jensen's inequality.[17] The continuity assertion follows from Corollary 3 of [11] since

$$\mathrm{Tr}\,\sigma\big(-\log\omega(G,U_g,\sigma)\big)=H\big(\omega(G,U_g,\sigma)\big).$$

The set $\mathcal{O}_{G,U_g,\sigma}$ is invariant under the action of the automorphism group $\{U_g(\,\cdot\,)U_g^*\}_{g\in G}$, and $\omega(G,U_g,\sigma)$ is the unique invariant state in $\overline{\mathrm{co}}(\mathcal{O}_{G,U_g,\sigma})$ for

[17] Jensen's inequality is applicable in this case since the entropy can be represented as the pointwise limit of a monotone increasing sequence of concave continuous functions [4].

this group. It follows from Corollary 5 that $\overline{C}(\mathcal{O}_{G,U_g,\sigma}) = H\big(\sigma \,\|\, \omega(G,U_g,\sigma)\big)$ and $\Omega(\mathcal{O}_{G,U_g,\sigma}) = \omega(G,U_g,\sigma)$.

The assertion on the existence of an optimal measure for $\mathcal{O}_{G,U_g,\sigma}$ is obvious.

The regularity assertion follows from the above observations since we have $H\big(\rho \,\|\, \omega(G,U_g,\sigma)\big) = H\big(\sigma \,\|\, \omega(G,U_g,\sigma)\big)$ for all ρ in $\mathcal{O}_{G,U_g,\sigma}$.

Example 4 (a closed set that has an optimal measure but no atomic optimal measure). Let $G = \mathbb{T}$ be the one-dimensional rotation group represented as the interval $[-\pi,\pi)$. Then the Haar measure is the normalized Lebesgue measure $\frac{dx}{2\pi}$. Put $\mathcal{H} = \mathcal{L}_2\big([-\pi,\pi)\big)$. The elements of $\mathcal{L}_2\big([-\pi,\pi)\big)$ may be regarded as 2π-periodic functions on $\mathbb{R}$. We define a unitary representation $\{U_\lambda\}_{\lambda\in\mathbb{T}}$ of the group $\mathbb{T}$ by

$$U_\lambda\big(\psi(x)\big) = \psi(x-\lambda), \qquad \psi(x) \in \mathcal{L}_2\big([-\pi,\pi)\big).$$

Given an element $|\varphi_0\rangle$ in $\mathcal{L}_2\big([\pi,\pi)\big)$, we consider the set $\mathcal{O}_{\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|}$. In this case,

$$\omega(\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} |\varphi_\lambda\rangle\langle\varphi_\lambda| d\lambda,$$

where $|\varphi_\lambda\rangle = U_\lambda|\varphi_0\rangle$. Note that $\overline{\mathrm{co}}(\mathcal{O}_{\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|})$ is the closure of the output set of the channel Φ considered in [1]. It is shown in [1] that

$$\overline{C}(\mathcal{O}_{\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|}) = H\big(\omega(\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|)\big) = -\sum_{n=-\infty}^{+\infty} c_n^2(\varphi_0)\log c_n^2(\varphi_0), \tag{35}$$

where $\{c_n(\varphi_0)\}_{n\in\mathbb{Z}}$ is the set of Fourier coefficients of the function $\varphi_0(x)$ with respect to the trigonometric orthonormal system $\{\exp(\mathrm{i}\,nx)\}_{n\in\mathbb{Z}}$. By Proposition 10, the finiteness of this series means that the entropy is continuous on $\overline{\mathrm{co}}(\mathcal{O}_{\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|})$. Proposition 10 also implies that the image of the normalized Lebesgue measure $\frac{dx}{2\pi}$ under the map $\lambda \mapsto U_\lambda|\varphi_0\rangle\langle\varphi_0|U_\lambda^*$ is an optimal measure for the set $\mathcal{O}_{\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|}$. This measure is non-atomic. However, its existence does not automatically mean that there is no purely atomic optimal measure in this case. We now show that there is a function $\varphi_0(x)$ such that the set $\mathcal{O}_{\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|}$ has no purely atomic optimal measure.

Put

$$\varphi_0(x) = \begin{cases} 0, & x \in [-\pi,0), \\ \sqrt{2}\,, & x \in [0,+\pi). \end{cases}$$

Then $c_n(\varphi_0) \sim n^{-1}$ and, therefore, the sum in (35) is finite.

To prove the absence of an atomic optimal measure, it suffices to show that the state $\Omega(\mathcal{O}_{\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|}) = \omega(\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|)$ cannot be represented as a countable convex combination of states in $\mathcal{O}_{\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|}$.

Suppose that

$$\omega(\mathbb{T},U_\lambda,|\varphi_0\rangle\langle\varphi_0|) = \sum_{i=1}^{+\infty} \pi_i |\varphi_{\lambda_i}\rangle\langle\varphi_{\lambda_i}|.$$

We may assume without loss of generality that $\pi_1 \geqslant \pi_i$ for all $i > 1$ and that $\lambda_1 = 0$. For arbitrary η we have

$$\sum_{i=1}^{+\infty} \pi_i \langle \varphi_\eta | \varphi_{\lambda_i} \rangle^2 = \langle \varphi_\eta | \omega(\mathbb{T}, U_\lambda, |\varphi_0\rangle\langle\varphi_0|) | \varphi_\eta \rangle$$
$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \langle \varphi_\eta | \varphi_\lambda \rangle^2 \, d\lambda = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \langle \varphi_0 | \varphi_\lambda \rangle^2 \, d\lambda. \tag{36}$$

Let $\theta(x)$ be the 2π-periodic function that equals $(1-\pi^{-1}|x|)^2$ on $[-\pi, +\pi]$. Then $\langle \varphi_\eta | \varphi_\lambda \rangle^2 = \theta(\eta - \lambda)$ for all λ and η. Since the function $\theta_\lambda(x) = \theta_0(x - \lambda)$ is locally integrable for each λ, it generates an element $\widetilde{\theta}_\lambda$ of the space $\mathfrak{D}'$ of generalized functions. (Here $\mathfrak{D}'$ is the linear space of all continuous linear functionals on the space $\mathfrak{D}$ of smooth compactly supported functions [3].) Let $\widetilde{\theta}'_\lambda \in \mathfrak{D}'$ be the derivative of the generalized function $\widetilde{\theta}_\lambda \in \mathfrak{D}'$. Then (36) implies that

$$\mathfrak{D}' - \lim_{n \to +\infty} \sum_{i=1}^{n} \pi_i \widetilde{\theta}'_{\lambda_i} = 0. \tag{37}$$

The function

$$\omega_\delta(x) = \begin{cases} \exp\bigl(-(1-(x/\delta)^2)^{-1}\bigr), & x \in [-\delta, +\delta], \\ 0, & x \in \mathbb{R} \setminus [-\delta, +\delta], \end{cases}$$

belongs to $\mathfrak{D}$ for every $\delta > 0$. Direct integration shows that

$$\widetilde{\theta}'_\lambda(\omega'_\delta) = \int_{-\infty}^{+\infty} \theta'_\lambda(x) \omega'_\delta(x) \, dx = \frac{2}{\pi} \int_{-\delta}^{\lambda} \left(1 + \frac{x-\lambda}{\pi}\right) \omega'_\delta(x) \, dx$$
$$+ \frac{2}{\pi} \int_{\lambda}^{+\delta} \left(\frac{x-\lambda}{\pi} - 1\right) \omega'_\delta(x) \, dx = \frac{4\omega_\delta(\lambda)}{\pi} - \frac{2\delta I}{\pi^2}\,,$$

if $\lambda \in [-\delta, +\delta]$, and

$$\widetilde{\theta}'_\lambda(\omega'_\delta) = \int_{-\infty}^{+\infty} \theta'_\lambda(x) \omega'_\delta(x) \, dx = \frac{2}{\pi^2} \int_{-\delta}^{+\delta} x \omega'_\delta(x) \, dx = -\frac{2\delta I}{\pi^2}$$

if $\lambda \in \mathbb{R} \setminus [-\delta, +\delta]$, where

$$I = \delta^{-1} \int_{-\delta}^{+\delta} \omega_\delta(x) \, dx = \int_{-1}^{+1} \exp\bigl(-(1-x^2)^{-1}\bigr) \, dx$$

is a positive number.

Put $\mathcal{N}(\delta) = \{i \in \mathbb{N} \mid \lambda_i \in [-\delta, +\delta]\}$ and $\mathcal{K}_n = \{2, 3, \dots, n\}$. Using the previous formula, we get

$$\sum_{i=1}^{n} \pi_i \widetilde{\theta}'_{\lambda_i}(\omega'_\delta) = \pi_1 \widetilde{\theta}'_0(\omega'_\delta) + \sum_{i \in \mathcal{N}(\delta) \cap \mathcal{K}_n} \pi_i \widetilde{\theta}'_{\lambda_i}(\omega'_\delta) + \sum_{i \in (\mathbb{N} \setminus \mathcal{N}(\delta)) \cap \mathcal{K}_n} \pi_i \widetilde{\theta}'_{\lambda_i}(\omega'_\delta)$$
$$\geqslant \pi_1 \left(\frac{4}{e\pi} - \frac{2\delta I}{\pi^2}\right) - \left(\frac{4}{e\pi} - \frac{2\delta I}{\pi^2}\right) \sum_{i \in \mathcal{N}(\delta),\, i>1} \pi_i - \frac{2\delta I}{\pi^2} \quad \forall\, n.$$

Since $\sum_{i\in\mathcal{N}(\delta),\, i>1}\pi_i$ tends to zero with δ, this inequality implies that

$$\liminf_{n\to+\infty}\sum_{i=1}^{n}\pi_i\widetilde{\theta}'_{\lambda_i}(\omega'_\delta)>0$$

for all sufficiently small δ. This contradicts (37).

§ 4. Another definition of $\overline{C}(\mathcal{A})$ and $\Omega(\mathcal{A})$

It is known that the entropy and relative entropy of general quantum states can be introduced via their definitions for states of finite rank and a limiting procedure. To show this, we consider the non-linear map

$$\Theta_P(\rho)=(\operatorname{Tr}P\rho)^{-1}P\rho P$$

determined by an arbitrary projector P and having the domain $\mathfrak{D}(\Theta_P)=\{\rho\in\mathfrak{S}(\mathcal{H})\mid P\rho\neq 0\}$. By Lemma 4 in [4], the entropy $H(\rho)$ of an arbitrary state ρ can be defined by

$$H(\rho)=\lim_{n\to+\infty}H\bigl(\Theta_{P_n}(\rho)\bigr)$$

and the relative entropy $H(\rho\,\|\,\sigma)$ can be defined for any states ρ and σ by

$$H(\rho\,\|\,\sigma)=\lim_{n\to+\infty}H\bigl(\Theta_{P_n}(\rho)\,\|\,\Theta_{P_n}(\sigma)\bigr),$$

where $\{P_n\}$ is an arbitrary increasing sequence of finite-rank projectors strongly converging to the identity operator $I_{\mathcal{H}}$. (Here we assume that n is so large that ρ and σ lie in $\mathfrak{D}(\Theta_{P_n})$.) This means that both limits (finite or infinite) exist and are independent of the choice of the sequence $\{P_n\}$. Since the states $\Theta_{P_n}(\rho)$ and $\Theta_{P_n}(\sigma)$ are supported by the finite-dimensional subspace $P_n(\mathcal{H})$ for each n, this observation reduces the definition of the entropy and relative entropy to the finite-dimensional case.

In this section we obtain analogous results for the χ-capacity and optimal average of any set of states. Given any closed set of states in the d-dimensional Hilbert space, one can take the supremum in the definition of χ-capacity over all ensembles of d^2 states. Hence the χ-capacity and optimal average of this set can be determined by the methods of linear programming [13]. Thus the results of this section provide a constructive definition of the χ-capacity and optimal average state for an arbitrary set of infinite-dimensional states. In principle, one can use this for the numerical approximation of these quantities.

For any projector P it is clear that the corresponding map Θ_P is continuous at every point of its domain. Although this map is non-linear, we have the following result.

Lemma 7. 1) *For any convex set $\mathcal{A}\subseteq\mathfrak{D}(\Theta_P)$, its image $\Theta_P(\mathcal{A})$ under Θ_P is a convex subset of $\mathfrak{S}(\mathcal{H})$.*

2) *For every ensemble $\{\pi_i,\rho_i\}_{i=1}^m$ of states in $\Theta_P(\mathcal{A})$, there is an ensemble $\{\lambda_i,\sigma_i\}_{i=1}^m$ of states in $\mathcal{A}$ such that*

$$\Theta_P(\sigma_i)=\rho_i,\qquad \lambda_i\operatorname{Tr}P\sigma_i=\pi_i\sum_{j=1}^{m}\lambda_j\operatorname{Tr}P\sigma_j,\qquad i=1,\dots,m.$$

Proof. It suffices to prove assertion 2) of the lemma since it implies that

$$\Theta_P\Big(\sum_i \lambda_i\sigma_i\Big) = \sum_i \pi_i\rho_i.$$

For every i, the state ρ_i in $\Theta_P(\mathcal{A})$ is the image of some state σ_i in $\mathcal{A}$. Let $\eta_i = \pi_i(\operatorname{Tr} P\sigma_i)^{-1}$ be a positive number for each $i = 1, \dots, m$, and let $\big\{\lambda_i = \eta_i\big(\sum_{j=1}^m \eta_j\big)^{-1}\big\}_{i=1}^m$ be a probability distribution. Adding the equations $\lambda_i \operatorname{Tr} P\sigma_i = \pi_i\big(\sum_{j=1}^m \eta_j\big)^{-1}$, we get $\sum_{i=1}^m \lambda_i \operatorname{Tr} P\sigma_i = \big(\sum_{j=1}^m \eta_j\big)^{-1}$.

Lemma 8. *Let $\mathcal{A}$ be a set of finite χ-capacity and P a projector such that $\eta(\mathcal{A}, P) = \inf_{\rho\in\mathcal{A}} \operatorname{Tr} P\rho > 0$. Then*

$$\eta(\mathcal{A}, P)\overline{C}\big(\Theta_P(\mathcal{A})\big) \leqslant \overline{C}(\mathcal{A}).$$

Proof. For any ensemble $\{\pi_i, \rho_i\}$ of states in $\Theta_P(\mathcal{A})$, let $\{\lambda_i, \sigma_i\}$ be the corresponding ensemble of states in $\mathcal{A}$ provided by Lemma 7. It follows that $\eta = \sum_i \lambda_i\eta_i$, where $\eta_i = \operatorname{Tr} P\sigma_i$ and $\eta = \operatorname{Tr} P\,\overline{\sigma}$.

Consider the channel

$$\Phi(\rho) = P\rho P + \big(\operatorname{Tr}(I - P)\rho\big)\tau,$$

where τ is the pure state corresponding to an arbitrary unit vector in $\mathcal{H} \ominus P(\mathcal{H})$. Using general properties of the relative entropy, we obtain

$$\begin{aligned}\chi\big(\{\lambda_i, \Phi(\sigma_i)\}\big) &= \sum_i \lambda_i H(P\sigma_i P \,\|\, P\,\overline{\sigma}P)\\ &\quad+ \sum_i \lambda_i H\big((\operatorname{Tr}(I - P)\sigma_i)\tau \,\|\, (\operatorname{Tr}(I - P)\overline{\sigma})\tau\big) \geqslant \sum_i \lambda_i H(P\sigma_i P \,\|\, P\,\overline{\sigma}P)\\ &= \sum_i \lambda_i H(\eta_i\rho_i \,\|\, \eta\overline{\rho}) \geqslant \sum_i \lambda_i\eta_i H(\rho_i \,\|\, \overline{\rho}) = \eta\sum_i \pi_i H(\rho_i \,\|\, \overline{\rho}) \geqslant \eta(\mathcal{A}, P)\chi(\{\pi_i, \rho_i\}).\end{aligned}$$

Since the relative entropy is monotone, we have

$$\chi\big(\{\lambda_i, \Phi(\sigma_i)\}\big) \leqslant \chi(\{\lambda_i, \sigma_i\}).$$

The lemma follows from the last two inequalities.

Remark 6. One cannot replace the coefficient $\eta(\mathcal{A}, P)$ by 1 in Lemma 8. (See the example in Remark 7 below.)

Theorem 2. *Let $\mathcal{A}$ be an arbitrary subset of $\mathfrak{S}(\mathcal{H})$.*

1) *If the χ-capacity of $\mathcal{A}$ is finite, then*

$$\lim_{n\to+\infty} \overline{C}\big(\Theta_{P_n}(\mathcal{A})\big) = \overline{C}(\mathcal{A}), \qquad \lim_{n\to+\infty} \Omega\big(\Theta_{P_n}(\mathcal{A})\big) = \Omega(\mathcal{A})$$

for every sequence $\{P_n\}$ of projectors strongly converging to $I_{\mathcal{H}}$.

2) *If there is a sequence $\{P_n\}$ of projectors strongly converging to $I_{\mathcal{H}}$ such that all maps $\{\Theta_{P_n}\}$ are well defined on $\mathcal{A}$ and the sequence $\big\{\overline{C}\big(\Theta_{P_n}(\mathcal{A})\big)\big\}$ is bounded, then the χ-capacity $\overline{C}(\mathcal{A})$ is finite.*

Proof. Suppose that $\overline{C}(\mathcal{A})<+\infty$ and let $\{P_n\}$ be an arbitrary sequence of projectors strongly converging to $I_{\mathcal{H}}$. The set $\mathcal{A}$ is relatively compact by assertion 4) of Theorem 1. The compactness criterion yields that $\lim_{n\to+\infty}\eta(\mathcal{A},P_n)=1$, where $\eta(\mathcal{A},P_n)=\inf_{\rho\in\mathcal{A}}\operatorname{Tr}P_n\rho$. Thus $\mathcal{A}\subseteq\mathfrak{D}(\Theta_{P_n})$ for all sufficiently large n. Using Lemma 8, we obtain

$$\limsup_{n\to+\infty}\overline{C}\bigl(\Theta_{P_n}(\mathcal{A})\bigr)\leqslant\overline{C}(\mathcal{A}).$$

Since $\Theta_{P_n}(\rho)\to\rho$ as $n\to+\infty$, assertion 1) of Lemma 1 implies that

$$\liminf_{n\to+\infty}\overline{C}\bigl(\Theta_{P_n}(\mathcal{A})\bigr)\geqslant\overline{C}(\mathcal{A}).$$

Using the last two inequalities, we obtain the first limit formula of the theorem. The second follows from assertion 2) of Lemma 1.

Suppose that $\overline{C}(\mathcal{A})=+\infty$ and let $\{P_n\}$ be a sequence of finite-rank projectors strongly converging to $I_{\mathcal{H}}$ and such that $\mathcal{A}\subseteq\mathfrak{D}(\Theta_{P_n})$ for all sufficiently large n. Then assertion 1) of Lemma 1 implies that

$$\lim_{n\to+\infty}\overline{C}\bigl(\Theta_{P_n}(\mathcal{A})\bigr)=+\infty.$$

Remark 7. The convergence of the sequence $\bigl\{\overline{C}\bigl(\Theta_{P_n}(\mathcal{A})\bigr)\bigr\}$ to $\overline{C}(\mathcal{A})$ may be of a different nature, depending on the choice of the sequence $\{P_n\}$. It is interesting to note that one can find a set $\mathcal{A}$ and a sequence $\{P_n\}$ such that $\bigl\{\overline{C}\bigl(\Theta_{P_n}(\mathcal{A})\bigr)\bigr\}$ converges to $\overline{C}(\mathcal{A})$ in a strictly decreasing way. Indeed, let $\mathcal{A}$ be the set consisting of two states $\bigl\{\frac{1}{2}\rho+\frac{1}{2}\sigma_i\bigr\}_{i=1,2}$. Here ρ is a state with infinite-dimensional support $\mathcal{H}_\rho$ such that the subspace $\mathcal{H}\ominus\mathcal{H}_\rho$ is two-dimensional, and σ_1, σ_2 are the states corresponding to orthogonal unit vectors in $\mathcal{H}\ominus\mathcal{H}_\rho$. Let $\{P_n\}$ be a sequence of finite-rank projectors such that $P_n(\mathcal{H})\supseteq\mathcal{H}\ominus\mathcal{H}_\rho$ and the sequence $\{\eta_n=\operatorname{Tr}P_n\rho\}$ strictly increases to 1. We easily see that

$$\overline{C}\bigl(\Theta_{P_n}(\mathcal{A})\bigr)=\frac{1}{1+\eta_n}\log 2\searrow\frac{1}{2}\log 2=\overline{C}(\mathcal{A}),\qquad n\to+\infty.$$

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