Commun. Math. Phys. 262, 137–159 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-005-1457-8

Communications in
**Mathematical
Physics**

# The Holevo Capacity of Infinite Dimensional Channels and the Additivity Problem

**M.E. Shirokov**

Steklov Mathematical Institute, 119991 Moscow, Russia.
E-mail: msh@mi.ras.ru

Received: 28 January 2005 / Accepted: 2 June 2005
Published online: 29 October 2005 – 

**Abstract:** The Holevo capacity of an arbitrarily constrained infinite dimensional quantum channel is considered and its properties are discussed. The notion of optimal average state is introduced. The continuity properties of the Holevo capacity with respect to constraint and to the channel are explored.

The main result of this paper is the statement that additivity of the Holevo capacity for all finite dimensional channels implies its additivity for all infinite dimensional channels with arbitrary constraints.

## 1. Introduction

The Holevo capacity (in what follows, $\chi$-capacity) of a quantum channel is an important characteristic defining the amount of classical information which can be transmitted by this channel using nonentangled encoding and entangled decoding, see e.g. [8, 10, 21]. For additive channels the $\chi$-capacity coincides with the full classical capacity of a quantum channel. At present the main interest is focused on quantum channels between finite dimensional quantum systems. But having in mind possible applications, it is necessary to deal with infinite dimensional quantum channels, in particular, Gaussian channels.

In this paper the $\chi$-capacity for an arbitrarily constrained infinite dimensional quantum channel is considered. It is shown that despite nonexistence of an optimal ensemble in this case it is possible to define the notion of the optimal average state for such a channel, inheriting important properties of the optimal average state for finite dimensional channels (Proposition 1). A "minimax" expression for the $\chi$-capacity is obtained and an alternative characterization of the image of the optimal average state as the minimum point of a lower semicontinuous function on a compact set is given (Proposition 2).

The notion of the $\chi$-function of an infinite dimensional quantum channel is introduced. It is shown that the $\chi$-function of an arbitrary channel is a concave lower semicontinuous function with natural chain properties, having continuous restriction to any set of continuity of the output entropy (Propositions 3-4). This and the result in [12] imply continuity of the $\chi$-function for Gaussian channels with power constraint (Example 1). For the $\chi$-function the analog of Simon's dominated convergence theorem for quantum entropy is also obtained (Corollary 3).

The question of continuity of the $\chi$-capacity as a function of channel is considered. It is shown that the $\chi$-capacity is a continuous function of channel in the finite dimensional case while in general it is only lower semicontinuous (Theorem 1, Example 2).

The above results make it possible to obtain the infinite dimensional version of Theorem 1 in [11], which shows equivalence of several formulations of the additivity conjecture (Theorem 2).

The main result of this paper is the statement that additivity of the $\chi$-capacity for all finite dimensional channels implies its additivity for all infinite dimensional channels with arbitrary constraints (Theorem 3). This is done in two steps by using several results (Lemma 5, Propositions 5 and 6). These results are also applicable to analysis of individual pairs of channels as it is demonstrated in the proof of additivity of the $\chi$-capacity for two arbitrarily constrained infinite dimensional channels with one of them noiseless or with entanglement breaking (Proposition 7).

## 2. Basic Quantities

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$ with the cone $\mathfrak{B}_+(\mathcal{H})$ of all positive operators, $\mathfrak{T}(\mathcal{H})$ be the Banach space of all trace-class operators with the trace norm $\|\cdot\|_1$ and $\mathfrak{S}(\mathcal{H})$ be the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of all density operators on $\mathcal{H}$, which is a complete separable metric space with the metric defined by the trace norm. Each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$ [1], so in what follows we will also for brevity use the term "state". Note that convergence of a sequence of states to a *state* in the weak operator topology is equivalent to convergence of this sequence to this state in the trace norm [3].

In what follows log denotes the function on $[0, +\infty)$, which coincides with the logarithm on $(0, +\infty)$ and vanishes at zero. Let $A$ and $B$ be positive trace class operators. Let $\{|i\rangle\}$ be a complete orthonormal set of eigenvectors of $A$. The entropy is defined by $H(A) = -\sum_i \langle i|\, A \log A\, |i\rangle$ while the relative entropy — as $H(A\,\|B) = \sum_i \langle i|\,(A\log A - A\log B + B - A)\,|i\rangle$, provided $\operatorname{ran}A \subseteq \operatorname{ran}B$,[1] and $H(A\,\|B) = +\infty$ otherwise (see [16, 17] for a more detailed definition). The entropy and the relative entropy are nonnegative lower semicontinuous (in the trace-norm topology) concave and convex functions of their arguments correspondingly [29].

Arbitrary finite collection $\{\rho_i\}$ of states in $\mathfrak{S}(\mathcal{H})$ with corresponding set of probabilities $\{\pi_i\}$ is called *ensemble* and is denoted by $\Sigma = \{\pi_i, \rho_i\}$. The state $\bar{\rho} = \sum_i \pi_i\rho_i$ is called *the average state* of the above ensemble. Following [12] we treat an arbitrary Borel probability measure $\pi$ on $\mathfrak{S}(\mathcal{H})$ as a *generalized ensemble* and the *barycenter* of the measure $\pi$ defined by the Pettis integral

$$\bar{\rho}(\pi) = \int\limits_{\mathfrak{S}(\mathcal{H})} \rho\pi(d\rho)$$

as the average state of this ensemble. In these notations the conventional ensembles correspond to measures with finite support. For an arbitrary subset $\mathcal{A}$ of $\mathfrak{S}(\mathcal{H})$ we denote by $\mathcal{P}_{\mathcal{A}}$ the set of all probability measures with barycenters contained in $\mathcal{A}$.

[1] ran denotes the closure of the range of an operator in $\mathcal{H}$.

In analysis of the $\chi$-capacity we shall use Donald's identity [6, 19]

$$\sum_{i=1}^{n} \pi_i H(\rho_i \| \hat{\rho}) = \sum_{i=1}^{n} \pi_i H(\rho_i \| \bar{\rho}) + H(\bar{\rho} \| \hat{\rho}), \tag{1}$$

which holds for an arbitrary ensemble $\{\pi_i, \rho_i\}$ of $n$ states with the average state $\bar{\rho}$ and arbitrary state $\hat{\rho}$.

Let $\mathcal{H}, \mathcal{H}'$ be a pair of separable Hilbert spaces which we shall call correspondingly input and output space. A channel $\Phi$ is a linear positive trace preserving map from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ such that the dual map $\Phi^* : \mathfrak{B}(\mathcal{H}') \mapsto \mathfrak{B}(\mathcal{H})$ (which exists since $\Phi$ is bounded [5]) is completely positive. Let $\mathcal{A}$ be an arbitrary closed subset of $\mathfrak{S}(\mathcal{H})$. We consider a constraint on the input ensemble $\{\pi_i, \rho_i\}$, defined by the requirement $\bar{\rho} \in \mathcal{A}$. The channel $\Phi$ with this constraint is called the $\mathcal{A}$ *-constrained* channel. We define the $\chi$*-capacity* of the $\mathcal{A}$-constrained channel $\Phi$ as (cf.[9–11])

$$\bar{C}(\Phi; \mathcal{A}) = \sup_{\bar{\rho} \in \mathcal{A}} \chi_\Phi(\{\pi_i, \rho_i\}), \tag{2}$$

where

$$\chi_\Phi(\{\pi_i, \rho_i\}) = \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})).$$

In [12] it is shown that the $\chi$-capacity of the $\mathcal{A}$-constrained channel $\Phi$ can be also defined by

$$\bar{C}(\Phi; \mathcal{A}) = \sup_{\pi \in \mathcal{P}_\mathcal{A}} \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho) \| \Phi(\bar{\rho}(\pi)))\pi(d\rho), \tag{3}$$

which means coincidence of the above supremum over all measures in $\mathcal{P}_\mathcal{A}$ with the supremum over all measures in $\mathcal{P}_\mathcal{A}$ with finite support.

The $\chi$-capacity $\bar{C}(\Phi; \mathfrak{S}(\mathcal{H}))$ of the unconstrained channel $\Phi$ is also denoted by $\bar{C}(\Phi)$.

The $\chi$-function of the channel $\Phi$ is defined by

$$\chi_\Phi(\rho) = \bar{C}(\Phi; \{\rho\}) = \sup_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\rho)). \tag{4}$$

The $\chi$-function of the finite dimensional channel $\Phi$ is a continuous concave function on $\mathfrak{S}(\mathcal{H})$ [11]. The properties of the $\chi$-function of an arbitrary infinite dimensional channel $\Phi$ are considered in Sect. 4.

## 3. The Optimal Average State

It is a known fact that for an arbitrary finite dimensional channel $\Phi$ and an arbitrary closed set $\mathcal{A}$ there exists an optimal ensemble $\{\pi_i, \rho_i\}$ on which the supremum in definition (2) of the $\chi$-capacity is achieved [4, 22]. The image of the average state of this optimal ensemble plays an important role in the analysis of finite dimensional channels [11].

For a general infinite dimensional constrained channel there are no reasons for existence of an optimal ensemble (with a finite number of states). In this case it is natural to introduce the notion of optimal generalized ensemble (optimal measure) on which the supremum in the definition (3) of the $\chi$-capacity is achieved. In [12] a sufficient condition for existence of an optimal measure for an infinite dimensional constrained channel is obtained and the example of the channel with no optimal measure is given.

The aim of this section is to show that even in the case of nonexistence of an optimal generalized ensemble we can define the notion of "optimal average state", inheriting the basic properties of the average state of the optimal ensemble for the finite dimensional constrained channel. Using this notion we can generalize some results of [11] to the infinite dimensional case.

**Definition 1.** *A sequence of ensembles $\{\pi_i^k, \rho_i^k\}$ with the average $\bar{\rho}^k \in \mathcal{A}$ such that*

$$\lim_{k\to+\infty} \chi_\Phi(\{\pi_i^k, \rho_i^k\}) = \bar{C}(\Phi; \mathcal{A})$$

*is called the approximating sequence for the $\mathcal{A}$-constrained channel $\Phi$.*

*A state $\bar{\rho}$ is called an optimal average state for the $\mathcal{A}$-constrained channel $\Phi$ if this state $\bar{\rho}$ is a limit of the sequence of the average states of some approximating sequence of ensembles for the $\mathcal{A}$-constrained channel $\Phi$.*

This definition admits that an optimal average state may not exist or may not be unique. If $\mathcal{A}$ is a compact set then the set of optimal average states for the $\mathcal{A}$-constrained channel $\Phi$ is nonempty. It turns out that in the case of the convex compact set $\mathcal{A}$ all optimal average states have the same image.

**Proposition 1.** *Let $\mathcal{A}$ be a convex compact subset of $\mathfrak{S}(\mathcal{H})$ such that $\bar{C}(\Phi; \mathcal{A}) < +\infty$. Then there exists the unique state $\Omega(\Phi, \mathcal{A})$ in $\mathfrak{S}(\mathcal{H}')$ such that $\Phi(\bar{\rho}) = \Omega(\Phi, \mathcal{A})$ for the arbitrary optimal average state $\bar{\rho}$ for the $\mathcal{A}$-constrained channel $\Phi$.*

*The state $\Omega(\Phi, \mathcal{A})$ is the unique state in $\mathfrak{S}(\mathcal{H}')$ such that*

$$\sum_j \mu_j H(\Phi(\sigma_j) \| \Omega(\Phi, \mathcal{A})) \leq \bar{C}(\Phi; \mathcal{A})$$

*for any ensemble $\{\mu_j, \sigma_j\}$ with the average state $\bar{\sigma} \in \mathcal{A}$.*

Note that the second part of this proposition is a generalization of Proposition 1 in [11], where the version of the "maximal distance property" of an optimal ensemble [22] adapted to the case of a constrained channel is presented. Since the assumed compactness of the set $\mathcal{A}$ implies existence of at least one optimal average state, Proposition 1 is proved by combining the two following lemmas.

**Lemma 1.** *Let $\mathcal{A}$ be a convex set such that $\bar{C}(\Phi; \mathcal{A}) < +\infty$ and $\bar{\rho}$ be an optimal average state for the $\mathcal{A}$-constrained channel $\Phi$. Then*

$$\sum_j \mu_j H(\Phi(\sigma_j) \| \Phi(\bar{\rho})) \leq \bar{C}(\Phi; \mathcal{A})$$

*for any ensemble $\{\mu_j, \sigma_j\}$ with the average state $\bar{\sigma} \in \mathcal{A}$.*

*Proof.* The proof of this lemma is a generalization of the proof of Proposition 1 in [11]. Let $\{\pi_i^k, \rho_i^k\}$ be an approximating sequence of ensembles such that $\bar{\rho} = \lim_{k\to+\infty} \bar{\rho}^k$, and let $\{\mu_j, \sigma_j\}$ be an arbitrary ensemble of $m$ states with the average $\bar{\sigma} \in \mathcal{A}$. Consider the mixture

$$\Sigma_\eta^k = \{(1-\eta)\pi_1^k \rho_1^k, ..., (1-\eta)\pi_{n(k)}^k \rho_{n(k)}^k, \eta\mu_1\sigma_1, ..., \eta\mu_m\sigma_m\}, \quad \eta \in [0, 1]$$

of the ensemble $\{\mu_j, \sigma_j\}$ with an ensemble $\{\pi_i^k, \rho_i^k\}$ of the above approximating sequence, consisting of $n(k)$ states. We obtain the sequence of ensembles with the corresponding sequence of the average states $\bar{\rho}_\eta^k = (1-\eta)\bar{\rho}^k + \eta\bar{\sigma} \in \mathcal{A}$ converging to the state $\bar{\rho}_\eta = (1-\eta)\bar{\rho} + \eta\bar{\sigma} \in \mathcal{A}$ as $k \to +\infty$.

For arbitrary $k$ we have

$$\chi_\Phi\left(\Sigma_\eta^k\right) = (1-\eta)\sum_{i=1}^{n(k)} \pi_i^k H(\Phi(\rho_i^k)\|\Phi(\bar{\rho}_\eta^k)) + \eta\sum_{j=1}^{m} \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho}_\eta^k)). \quad (5)$$

By assumption $\bar{C}(\Phi;\mathcal{A}) < +\infty$ both sums in the right side of the above expression are finite. Applying Donald's identity (1) to the first sum in the right side we obtain

$$\sum_{i=1}^{n(k)} \pi_i^k H(\Phi(\rho_i^k)\|\Phi(\bar{\rho}_\eta^k)) = \chi_\Phi(\Sigma_0^k) + H(\Phi(\bar{\rho}^k)\|\Phi(\bar{\rho}_\eta^k)).$$

Substitution of the above expression into (5) gives

$$\chi_\Phi\left(\Sigma_\eta^k\right) = \chi_\Phi(\Sigma_0^k) + (1-\eta)H(\Phi(\bar{\rho}^k)\|\Phi(\bar{\rho}_\eta^k))$$
$$+\eta\left[\sum_{j=1}^{m} \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho}_\eta^k)) - \chi_\Phi(\Sigma_0^k)\right].$$

Due to nonnegativity of the relative entropy it follows that

$$\sum_{j=1}^{m} \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho}_\eta^k)) \le \eta^{-1}\left[\chi_\Phi\left(\Sigma_\eta^k\right) - \chi_\Phi\left(\Sigma_0^k\right)\right] + \chi_\Phi\left(\Sigma_0^k\right), \quad \eta \neq 0. \quad (6)$$

By definition of the approximating sequence we have

$$\lim_{k\to+\infty} \chi_\Phi\left(\Sigma_0^k\right) = \bar{C}(\Phi;\mathcal{A}) \ge \chi_\Phi\left(\Sigma_\eta^k\right) \quad (7)$$

for all $k$. It follows that

$$\liminf_{\eta\to+0}\ \liminf_{k\to+\infty}\ \eta^{-1}\left[\chi_\Phi\left(\Sigma_\eta^k\right) - \chi_\Phi\left(\Sigma_0^k\right)\right] \le 0. \quad (8)$$

Lower semicontinuity of the relative entropy [29] with (6),(7) and (8) imply

$$\sum_{j=1}^{m} \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho})) \le \liminf_{\eta\to+0}\ \liminf_{k\to+\infty}\sum_{j=1}^{m} \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho}_\eta^k)) \le \bar{C}(\Phi;\mathcal{A}).$$

□

**Lemma 2.** *Let $\mathcal{A}$ be a set such that $\bar{C}(\Phi;\mathcal{A}) < +\infty$ and $\rho'$ be a state in $\mathfrak{S}(\mathcal{H}')$ such that*

$$\sum_j \mu_j H(\Phi(\sigma_j)\|\,\rho') \le \bar{C}(\Phi;\mathcal{A})$$

*for any ensemble $\{\mu_j, \sigma_j\}$ with the average $\bar{\sigma} \in \mathcal{A}$. Then for the arbitrary approximating sequence $\{\pi_i^k, \rho_i^k\}$ of ensembles $\rho' = \lim_{k\to+\infty}\Phi(\bar{\rho}^k)$.*

*Proof.* Let $\{\pi_i^k, \rho_i^k\}$ an approximating sequence of ensembles with the corresponding sequence of the average states $\bar{\rho}^k$. By assumption we have

$$\sum_i \pi_i^k H(\Phi(\rho_i^k)\| \rho') \leq \bar{C}(\Phi; \mathcal{A}).$$

Applying Donald's identity (1) to the left side we obtain

$$\sum_i \pi_i^k H(\Phi(\rho_i^k)\| \rho') = \sum_i \pi_i^k H(\Phi(\rho_i^k)\|\Phi(\bar{\rho}^k)) + H(\Phi(\bar{\rho}^k)\| \rho').$$

From the two above expressions we have

$$0 \leq H(\Phi(\bar{\rho}^k)\| \rho') \leq \bar{C}(\Phi; \mathcal{A}) - \sum_i \pi_i^k H(\Phi(\rho_i^k)\|\Phi(\bar{\rho}^k)).$$

But the right side tends to zero as $k$ tends to infinity due to the approximating property of the sequence $\{\pi_i^k, \rho_i^k\}$. □

Proposition 1 provides the following generalization of Corollary 1 in [11].

**Corollary 1.** *Let $\mathcal{A}$ be a convex compact set such that $\bar{C}(\Phi; \mathcal{A}) < +\infty$ then*

$$\bar{C}(\Phi; \mathcal{A}) \geq \chi_{\Phi}(\rho) + H(\Phi(\rho)\|\Omega(\Phi, \mathcal{A})) \quad \textit{for arbitrary state } \rho \textit{ in } \mathcal{A}.$$

*Proof.* Let $\{\pi_i, \rho_i\}$ be an arbitrary ensemble such that $\sum_i \pi_i \rho_i = \rho \in \mathcal{A}$. By Proposition 1,

$$\sum_i \pi_i H(\Phi(\rho_i)\|\Omega(\Phi, \mathcal{A})) \leq \bar{C}(\Phi; \mathcal{A}).$$

This inequality and Donald's identity

$$\sum_i \pi_i H(\Phi(\rho_i)\|\Omega(\Phi, \mathcal{A})) = \chi_{\Phi}(\{\pi_i, \rho_i\}) + H(\Phi(\rho)\|\Omega(\Phi, \mathcal{A}))$$

complete the proof. □

**Corollary 2.** *Let $\mathcal{A}$ be a convex compact set such that $\bar{C}(\Phi; \mathcal{A}) < +\infty$. Then*

$$H(\Phi(\rho)\|\Omega(\Phi, \mathcal{A})) \leq \bar{C}(\Phi; \mathcal{A}) \quad \textit{for arbitrary state } \rho \textit{ in } \mathcal{A}.$$

*For the arbitrary approximating sequence $\{\pi_i^k, \rho_i^k\}$ of ensembles*

$$\lim_{k \to +\infty} \Phi(\bar{\rho}^k) = \Omega(\Phi, \mathcal{A}).$$

The first assertion of the corollary directly follows from Proposition 1 while the second is proved by using Lemma 2.

There exists another approach to the definition of the state $\Omega(\Phi, \mathcal{A})$. For the arbitrary ensemble $\{\mu_j, \sigma_j\}$ with the average $\bar{\sigma} \in \mathcal{A}$ consider the lower semicontinuous function $F_{\{\mu_j,\sigma_j\}}(\rho') = \sum_j \mu_j H(\Phi(\sigma_j)\| \rho')$ on the set $\Phi(\mathcal{A})$. The function $F(\rho') = \sup_{\sum_j \mu_j\sigma_j \in \mathcal{A}} F_{\{\mu_j,\sigma_j\}}(\rho')$ is also lower semicontinuous on the compact set $\Phi(\mathcal{A})$ and, hence, achieves its minimum on this set. The following proposition asserts, in particular, that the state $\Omega(\Phi, \mathcal{A})$ can be defined as the unique minimal point of the function $F(\rho')$.

**Proposition 2.** *Let $\mathcal{A}$ be a convex compact set such that $\bar{C}(\Phi;\mathcal{A}) < +\infty$. The $\chi$-capacity of the $\mathcal{A}$-constrained channel $\Phi$ can be expressed as*

$$\bar{C}(\Phi;\mathcal{A}) = \min_{\rho'\in\Phi(\mathcal{A})}\left[\sup_{\sum_j \mu_j\sigma_j\in\mathcal{A}}\sum_j \mu_j H(\Phi(\sigma_j)\|\,\rho')\right],$$

*and $\Omega(\Phi,\mathcal{A})$ is the only state on which the minimum in the right side is achieved.*

*Proof.* We will show first that

$$\sup_{\sum_j \mu_j\sigma_j\in\mathcal{A}}\sum_j \mu_j H(\Phi(\sigma_j)\|\Omega(\Phi,\mathcal{A})) = \bar{C}(\Phi;\mathcal{A}). \tag{9}$$

It follows from Proposition 1 that "$\leq$" takes place in (9).

Let $\{\pi_i^k,\rho_i^k\}$ be an approximating sequence. By Donald's identity (1) we have

$$\sum_i \pi_i^k H(\Phi(\rho_i^k)\|\Omega(\Phi,\mathcal{A})) = \sum_i \pi_i^k H(\Phi(\rho_i^k)\|\Phi(\bar{\rho}^k)) + H(\Phi(\bar{\rho}^k)\|\Omega(\Phi,\mathcal{A})).$$

The first term in the right side tends to $\bar{C}(\Phi;\mathcal{A})$ as $k$ tends to infinity due to the approximating property of the sequence $\{\pi_i^k,\rho_i^k\}$, while the second one is nonnegative. This implies "$\geq$" and, hence, "$=$" in (9).

Let $\varrho'$ be a minimal point of the function $F(\rho')$ introduced before Proposition 2. By (9) it follows that

$$\sup_{\sum_j \mu_j\sigma_j\in\mathcal{A}}\sum_j \mu_j H(\Phi(\sigma_j)\|\,\varrho') = F(\varrho') \leq F(\Omega(\Phi,\mathcal{A})) = \bar{C}(\Phi;\mathcal{A}).$$

By Proposition 1 this implies that $\varrho' = \Omega(\Phi,\mathcal{A})$. $\square$

Note that the expression for the $\chi$-capacity in the above proposition can be considered as a generalization of the "mini-max formula for $\chi^*$" in [22] to the case of an infinite dimensional constrained channel.

*Remark 1.* Propositions 1-2 and Corollaries 1-2 do not hold without assumption of convexity of the set $\mathcal{A}$. To show this it is sufficient to consider the noiseless channel $\Phi = \mathrm{Id}$ and the compact set $\mathcal{A}$, consisting of two states $\rho_1$ and $\rho_2$ such that $H(\rho_1) = H(\rho_2) < +\infty$ and $H(\rho_1\|\rho_2) = +\infty$. In this case $\bar{C}(\Phi;\mathcal{A}) = H(\rho_1) = H(\rho_2)$, the states $\rho_1$ and $\rho_2$ are optimal average states in the sense of Definition 1 with the different images $\Phi(\rho_1) = \rho_1$ and $\Phi(\rho_2) = \rho_2$. Moreover the first assertion of Corollary 2 is false in the following extreme form: $\bar{C}(\Phi;\mathcal{A}) < H(\Phi(\rho_1)\|\Phi(\rho_2)) = H(\rho_1\|\rho_2) = +\infty$.

## 4. The $\chi$-Function

The function $\chi_\Phi(\rho)$ on $\mathfrak{S}(\mathcal{H})$ is defined by (4). It is shown in [12] that

$$\chi_\Phi(\rho) = \sup_{\pi\in\mathcal{P}_{\{\rho\}}}\int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\sigma)\|\Phi(\rho))\pi(d\sigma), \tag{10}$$

where $\mathcal{P}_{\{\rho\}}$ is the set of all probability measures on $\mathfrak{S}(\mathcal{H})$ with the barycenter $\rho$, and that under the condition $H(\Phi(\rho)) < +\infty$ the supremum in (10) is achieved on some measure supported by pure states.

Note that $H(\Phi(\rho)) = +\infty$ does not imply $\chi_\Phi(\rho) = +\infty$. Indeed, it is easy to construct a channel $\Phi$ from a finite dimensional system into an infinite dimensional one such that $H(\Phi(\rho)) = +\infty$ for any $\rho \in \mathfrak{S}(\mathcal{H})$.[2] On the other hand, by the monotonicity property of the relative entropy [18]

$$\sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho)) \leq \sum_i \pi_i H(\rho_i\|\rho) \leq \log\dim\mathcal{H} < +\infty$$

for the arbitrary ensemble $\{\pi_i, \rho_i\}$, and hence $\chi_\Phi(\rho) \leq \log\dim\mathcal{H} < +\infty$ for any $\rho \in \mathfrak{S}(\mathcal{H})$.

For the arbitrary state $\rho$ such that $H(\Phi(\rho)) < +\infty$ the $\chi$-function has the following representation

$$\chi_\Phi(\rho) = H(\Phi(\rho)) - \hat{H}_\Phi(\rho), \tag{11}$$

where

$$\hat{H}_\Phi(\rho) = \inf_{\pi\in\mathcal{P}_{\{\rho\}}} \int\limits_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho))\pi(d\rho) = \inf_{\sum_i \pi_i\rho_i=\rho} \sum_i \pi_i H(\Phi(\rho_i)) \tag{12}$$

is a convex closure of the output entropy $H(\Phi(\rho))$ (this is proved in [24]).[3]

Note that the notion of the convex closure of the output entropy is widely used in the quantum information theory in connection with the notion of the entanglement of formation (EoF). Namely, in the finite dimensional case EoF was defined in [2] as the convex hull (=convex closure) of the output entropy of a partial trace channel from the state space of a bipartite system onto the state space of its single subsystem. In the infinite dimensional case the definition of EoF as the $\sigma$-convex hull of the output entropy of a partial trace channel is proposed in [7] while some advantages of the definition of EoF as the convex closure of the output entropy of a partial trace channel are considered in [24]. It is shown that the two above definitions coincide on the set of states with finite entropy of partial trace, but their coincidence for an arbitrary state remains an open problem.

In the finite dimensional case the output entropy $H(\Phi(\rho))$ and its convex closure (=convex hull) $\hat{H}_\Phi(\rho)$ are continuous concave and convex functions on $\mathfrak{S}(\mathcal{H})$ correspondingly and representation (11) is valid for all states. It follows that in this case the function $\chi_\Phi(\rho)$ is continuous and concave on $\mathfrak{S}(\mathcal{H})$.

In the infinite dimensional case the output entropy $H(\Phi(\rho))$ is only lower semicontinuous and, hence, the function $\chi_\Phi(\rho)$ is not continuous even in the case of the noiseless channel $\Phi$, for which $\chi_\Phi(\rho) = H(\Phi(\rho))$. But it turns out that the function $\chi_\Phi(\rho)$ for the arbitrary channel $\Phi$ has properties similar to the properties of the output entropy $H(\Phi(\rho))$.

**Proposition 3.** *The function $\chi_\Phi(\rho)$ is a nonnegative concave and lower semicontinuous function on $\mathfrak{S}(\mathcal{H})$.*

*If the restriction of the output entropy $H(\Phi(\rho))$ to a particular subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ is continuous then the restriction of the function $\chi_\Phi(\rho)$ to this subset $\mathcal{A}$ is continuous as well.*

---

2 For example, the channel $\Phi : \rho \mapsto \frac{1}{2}\rho \oplus \frac{1}{2}\mathrm{Tr}(\rho)\tau$, where $\tau$ is a fixed state with infinite entropy.

3 Note that the second equality in (12) holds under the condition $H(\Phi(\rho)) < +\infty$, it is not valid in general (see Lemma 2 in [24] and the notes below).

The proof of this proposition is based on the following lemma.

**Lemma 3.** *Let $\{\pi_i, \rho_i\}$ be an arbitrary ensemble of m states with the average state $\rho$ and let $\{\rho_n\}$ be an arbitrary sequence of states converging to the state $\rho$. There exists the sequence $\{\pi_i^n, \rho_i^n\}$ of ensembles of m states such that*

$$\lim_{n\to+\infty} \pi_i^n = \pi_i, \quad \lim_{n\to+\infty} \rho_i^n = \rho_i, \quad \textit{and} \quad \rho_n = \sum_{i=1}^{m} \pi_i^n \rho_i^n.$$

*Proof.* Without loss of generality we may assume that $\pi_i > 0$ for all $i$. Let $\mathcal{D} \subseteq \mathcal{H}$ be the support of $\rho = \sum_{i=1}^m \pi_i \rho_i$ and $P$ be the projector onto $\mathcal{D}$. Since $\rho_i \leq \pi_i^{-1} \rho$ we have

$$0 \leq A_i \equiv \rho^{-1/2} \rho_i \rho^{-1/2} \leq \pi_i^{-1} I,$$

where we denote by $\rho^{-1/2}$ the generalized (Moore-Penrose) inverse of the operator $\rho^{1/2}$ (equal 0 on the orthogonal complement to $\mathcal{D}$).

Consider the sequence $B_i^n = \rho_n^{1/2} A_i \rho_n^{1/2} + \rho_n^{1/2}(I_{\mathcal{H}} - P)\rho_n^{1/2}$ of operators in $\mathfrak{B}(\mathcal{H})$. Since $\lim_{n\to+\infty} \rho_n = \rho = P\rho$ in the trace norm, we have

$$\lim_{n\to+\infty} B_i^n = \rho^{1/2} A_i \rho^{1/2} = \rho_i$$

in the weak operator topology. The last equality implies $A_i \neq 0$. Note that $\mathrm{Tr} B_i^n = \mathrm{Tr} A_i \rho_n + \mathrm{Tr}(I_{\mathcal{H}} - P)\rho_n < +\infty$ and hence

$$\lim_{n\to+\infty} \mathrm{Tr} B_i^n = \mathrm{Tr} A_i \rho = \mathrm{Tr} \rho_i = 1.$$

Denote by $\rho_i^n = (\mathrm{Tr} B_i^n)^{-1} B_i^n$ a state and by $\pi_i^n = \pi_i \mathrm{Tr} B_i^n$ a positive number for each $i$, then $\lim_{n\to+\infty} \pi_i^n = \pi_i$ and $\lim_{n\to+\infty} \rho_i^n = \rho_i$ in the weak operator topology and hence, by the result in [3], in the trace norm. Moreover,

$$\sum_{i=1}^{m} \pi_i^n \rho_i^n = \sum_{i=1}^{m} \pi_i B_i^n = \rho_n^{1/2} \rho^{-1/2} \sum_{i=1}^{m} \pi_i \rho_i \rho^{-1/2} \rho_n^{1/2} + \rho_n^{1/2}(I_{\mathcal{H}} - P)\rho_n^{1/2} = \rho_n.$$

□

*Proof of Proposition 3.* Nonnegativity of the $\chi$-function is obvious. Let us show first the concavity property of the $\chi$-function. Note that for a convex set of states with finite output entropy this concavity easily follows from (11). But to prove concavity on the whole state space we will use a different approach.

Let $\rho$ and $\sigma$ be arbitrary states. By definition for arbitrary $\varepsilon > 0$ there exist ensembles $\{\pi_i, \rho_i\}_{i=1}^n$ and $\{\mu_j, \sigma_j\}_{j=1}^m$ with the average states $\rho$ and $\sigma$ correspondingly such that $\sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho)) > \chi_\Phi(\rho) - \varepsilon$ and $\sum_j \mu_j H(\Phi(\sigma_j)\|\Phi(\sigma)) > \chi_\Phi(\sigma) - \varepsilon$.

Taking the mixture

$$\{(1-\eta)\pi_1\rho_1, ..., (1-\eta)\pi_n\rho_n, \eta\mu_1\sigma_1, ..., \eta\mu_m\sigma_m\}, \quad \eta \in [0, 1]$$

of the above two ensembles we obtain the ensemble with the average state $(1-\eta)\rho+\eta\sigma$. By using Donald's identity (1) we have

$$\chi_{\Phi}((1-\eta)\rho+\eta\sigma) \geq (1-\eta)\sum_{i}\pi_i H(\Phi(\rho_i)\|\Phi((1-\eta)\rho+\eta\sigma))$$
$$+\eta\sum_{j}\mu_j H(\Phi(\sigma_j)\|\Phi((1-\eta)\rho+\eta\sigma)) = (1-\eta)\sum_{i}\pi_i H(\Phi(\rho_i)\|\Phi(\rho))$$
$$+(1-\eta)H(\Phi(\rho)\|\Phi((1-\eta)\rho+\eta\sigma)) + \eta\sum_{j}\mu_j H(\Phi(\sigma_j)\|\Phi(\sigma))$$
$$+\eta H(\Phi(\sigma)\|\Phi((1-\eta)\rho+\eta\sigma) \geq (1-\eta)\sum_{i}\pi_i H(\Phi(\rho_i)\|\Phi(\rho))$$
$$+\eta\sum_{j}\mu_j H(\Phi(\sigma_j)\|\Phi(\sigma)) \geq (1-\eta)\chi_{\Phi}(\rho)+\eta\chi_{\Phi}(\sigma)-\varepsilon,$$

where nonnegativity of the relative entropy was used. Since $\varepsilon$ can be arbitrary small the concavity of the $\chi$-function is established.

To prove lower semicontinuity of the $\chi$-function we have to show

$$\liminf_{n\to+\infty}\chi_{\Phi}(\rho_n) \geq \chi_{\Phi}(\rho_0) \tag{13}$$

for arbitrary state $\rho_0$ and arbitrary sequence $\rho_n$ converging to this state $\rho_0$.

For arbitrary $\varepsilon > 0$ let $\{\pi_i, \rho_i\}$ be an ensemble with the average $\rho_0$ such that

$$\sum_{i}\pi_i H(\Phi(\rho_i)\|\Phi(\rho_0)) \geq \chi_{\Phi}(\rho_0)-\varepsilon.$$

By Lemma 3 there exists the sequence of ensembles $\{\pi_i^n, \rho_i^n\}$ of fixed size such that

$$\lim_{n\to+\infty}\pi_i^n = \pi_i, \qquad \lim_{n\to+\infty}\rho_i^n = \rho_i, \qquad \text{and} \qquad \rho_n = \sum_{i=1}^{m}\pi_i^n\rho_i^n.$$

By definition we have

$$\liminf_{n\to+\infty}\chi_{\Phi}(\rho_n) \geq \liminf_{n\to+\infty}\textstyle\sum_i \pi_i^n H(\Phi(\rho_i^n)\|\Phi(\rho_n))$$
$$\geq \textstyle\sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho_0)) \geq \chi_{\Phi}(\rho_0)-\varepsilon,$$

where lower semicontinuity of the relative entropy [29] was used. This implies (13) (due to the freedom of the choice of $\varepsilon$).

The last assertion of Proposition 3 follows from the representation (11) and from lower semicontinuity of the function $\hat{H}_{\Phi}(\rho)$ established in [24]. □

The similarity of the properties of the functions $\chi_{\Phi}(\rho)$ and $H(\Phi(\rho))$ is stressed by the following analog of Simon's dominated convergence theorem for quantum entropy [28], which will be used later.

**Corollary 3.** *Let $\rho_n$ be a sequence of states in $\mathfrak{S}(\mathcal{H})$, converging to the state $\rho$ and such that $\lambda_n\rho_n \leq \rho$ for some sequence $\lambda_n$ of positive numbers, converging to* 1. *Then*

$$\lim_{n\to+\infty}\chi_{\Phi}(\rho_n) = \chi_{\Phi}(\rho).$$

*Proof.* The condition $\lambda_n \rho_n \leq \rho$ implies decomposition $\rho = \lambda_n \rho_n + (1-\lambda_n)\rho'_n$, where $\rho'_n = (1-\lambda_n)^{-1}(\rho - \lambda_n \rho_n)$ is a state. By concavity of the $\chi$-function we have

$$\chi_\Phi(\rho) \geq \lambda_n \chi_\Phi(\rho_n) + (1-\lambda_n)\chi_\Phi(\rho'_n) \geq \lambda_n \chi_\Phi(\rho_n),$$

which implies $\limsup_{n\to+\infty} \chi_\Phi(\rho_n) \leq \chi_\Phi(\rho)$. This and lower semicontinuity of the $\chi$-function completes the proof. □

*Example 1.* Let $H'$ be a positive unbounded operator on the space $\mathcal{H}'$ such that $\mathrm{Tr}\exp(-\beta H') < +\infty$ for all $\beta > 0$ and $h'$ be a positive number. In the proof of Proposition 3 in [12] continuity of the restriction of the output entropy $H(\Phi(\rho))$ to the subset $\mathcal{A}_{h'} = \{\rho \in \mathfrak{S}(\mathcal{H}) \mid \mathrm{Tr}\,\Phi(\rho)H' \leq h'\}$ was established.[4] By Proposition 3 the restriction of the $\chi$-function to the set $\mathcal{A}_{h'}$ is continuous. As it is mentioned in [12], the above continuity condition is fulfilled for Gaussian channels with the power constraint of the form $\mathrm{Tr}\rho H \leq h$, where $H = R^T \epsilon R$ is the many-mode oscillator Hamiltonian with nondegenerate energy matrix $\epsilon$ and $R$ are the canonical variables of the system.

We shall use the following chain properties of the $\chi$-function.

**Proposition 4.** *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi : \mathfrak{S}(\mathcal{H}') \mapsto \mathfrak{S}(\mathcal{H}'')$ be two channels. Then*

$$\chi_{\Psi\circ\Phi}(\rho) \leq \chi_\Phi(\rho) \quad \textit{and} \quad \chi_{\Psi\circ\Phi}(\rho) \leq \chi_\Psi(\Phi(\rho)) \quad \textit{for arbitrary } \rho \textit{ in } \mathfrak{S}(\mathcal{H}).$$

*Proof.* The first inequality follows from the monotonicity property of the relative entropy [18] and (4), while the second one is a direct corollary of the definition (4) of the $\chi$-function. □

## 5. On Continuity of the $\chi$-Capacity

In this section the question of continuity of the $\chi$-capacity as a function of channel is considered. Dealing with this question we must choose a topology on the set $\mathcal{C}(\mathcal{H},\mathcal{H}')$ of all quantum channels from $\mathfrak{S}(\mathcal{H})$ into $\mathfrak{S}(\mathcal{H}')$. This choice is essential only in the infinite dimensional case because all locally convex Hausdorff topologies on a finite dimensional space are equivalent.

Let $\mathcal{L}(\mathcal{H},\mathcal{H}')$ be the linear space of all continuous linear mapping from $\mathfrak{T}(\mathcal{H})$ into $\mathfrak{T}(\mathcal{H}')$. We will use the topology on $\mathcal{C}(\mathcal{H},\mathcal{H}') \subset \mathcal{L}(\mathcal{H},\mathcal{H}')$ generated by the topology of strong convergence on $\mathcal{L}(\mathcal{H},\mathcal{H}')$.

**Definition 2.** *The topology on the linear space $\mathcal{L}(\mathcal{H},\mathcal{H}')$ defined by the family of seminorms $\{\|\Phi\|_\rho = \|\Phi(\rho)\|_1\}_{\rho\in\mathfrak{T}(\mathcal{H})}$ is called the topology of strong convergence.*

Since an arbitrary operator in $\mathfrak{T}(\mathcal{H})$ can be represented as a linear combination of operators in $\mathfrak{S}(\mathcal{H})$ it is possible to consider only seminorms $\|\cdot\|_\rho$ corresponding to $\rho \in \mathfrak{S}(\mathcal{H})$ in the above definition.

Note that a sequence $\Phi_n$ of channels in $\mathcal{C}(\mathcal{H},\mathcal{H}')$ strongly converges to a channel $\Phi \in \mathcal{C}(\mathcal{H},\mathcal{H}')$ if and only if $\lim_{n\to+\infty}\Phi_n(\rho) = \Phi(\rho)$ for all $\rho \in \mathfrak{S}(\mathcal{H})$. Due to the result in [3] the above limit may be in the weak operator topology.

[4] The value $\mathrm{Tr}\,\Phi(\rho)H'$ is defined as a limit of nondecreasing sequence $\mathrm{Tr}\,\Phi(\rho)Q'_n H'$, where $Q'_n$ is the spectral projector of $H'$ corresponding to the lowest $n$ eigenvalues [10].

**Theorem 1.** *Let $\mathcal{A}$ be an arbitrary closed and convex subset of $\mathfrak{S}(\mathcal{H})$.*[5]

*In the case of finite dimensional spaces $\mathcal{H}$ and $\mathcal{H}'$ the $\chi$-capacity $\bar{C}(\Phi, \mathcal{A})$ is a continuous function on the set $\mathcal{C}(\mathcal{H}, \mathcal{H}')$. If $\Phi_n$ is an arbitrary sequence of channels in $\mathcal{C}(\mathcal{H}, \mathcal{H}')$, converging to some channel $\Phi$ in $\mathcal{C}(\mathcal{H}, \mathcal{H}')$, then there exists*

$$\lim_{n\to\infty} \Omega(\Phi_n, \mathcal{A}) = \Omega(\Phi, \mathcal{A}). \tag{14}$$

*In general the $\chi$-capacity $\bar{C}(\Phi, \mathcal{A})$ is a lower semicontinuous function on the set $\mathcal{C}(\mathcal{H}, \mathcal{H}')$ equipped with the topology of strong convergence.*

*Proof.* Let us first show lower semicontinuity of the $\chi$-capacity. Let $\varepsilon > 0$ and $\Phi_\lambda$ be an arbitrary net of channels, strongly converging to the channel $\Phi$, and $\{\pi_i, \rho_i\}$ be an ensemble with the average $\bar{\rho}$ such that $\chi_\Phi(\{\pi_i, \rho_i\}) > \bar{C}(\Phi, \mathcal{A}) - \varepsilon$. By lower semicontinuity of the relative entropy [29],

$$\liminf_{\lambda} \sum_i \pi_i H(\Phi_\lambda(\rho_i) \| \Phi_\lambda(\bar{\rho})) \geq \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})) > \bar{C}(\Phi, \mathcal{A}) - \varepsilon.$$

This implies

$$\liminf_{\lambda} \bar{C}(\Phi_\lambda, \mathcal{A}) \geq \bar{C}(\Phi, \mathcal{A}).$$

It follows that

$$\liminf_{n\to+\infty} \bar{C}(\Phi_n, \mathcal{A}) \geq \bar{C}(\Phi, \mathcal{A}) \tag{15}$$

for an arbitrary sequence $\Phi_n$ of channels strongly converging to a channel $\Phi$.

Now to prove the continuity of the $\chi$-capacity in the finite dimensional case it is sufficient to show that for the above sequence of channels

$$\limsup_{n\to+\infty} \bar{C}(\Phi_n, \mathcal{A}) \leq \bar{C}(\Phi, \mathcal{A}). \tag{16}$$

For an arbitrary $\mathcal{A}$-constrained channel from $\mathcal{C}(\mathcal{H}, \mathcal{H}')$ there exists an optimal ensemble consisting of $m = (\dim \mathcal{H})^2$ states (probably, some states with zero weights) [4, 22]. Let $\mathfrak{P}$ be the compact space of all probability distributions with $m$ outcomes. Consider the compact space[6]

$$\mathfrak{P}\mathfrak{C}^m = \mathfrak{P} \times \underbrace{\mathfrak{S}(\mathcal{H}) \times \cdots \times \mathfrak{S}(\mathcal{H})}_{m},$$

consisting of sequences $(\{\pi_i\}_{i=1}^m, \rho_1, ..., \rho_m)$, corresponding to an arbitrary input ensemble $\{\pi_i, \rho_i\}_{i=1}^m$ of $m$ states.

Suppose (16) is not true. Without loss of generality we may assume that

$$\lim_{n\to+\infty} \bar{C}(\Phi_n, \mathcal{A}) > \bar{C}(\Phi, \mathcal{A}). \tag{17}$$

Let $\{\pi_i^n, \rho_i^n\}_{i=1}^m$ be an optimal ensemble for the $\mathcal{A}$-constrained channel $\Phi_n$. By compactness of $\mathfrak{P}\mathfrak{C}^m$ we can choose a subsequence $(\{\pi_i^{n_k}\}_{i=1}^m, \rho_1^{n_k}, ..., \rho_m^{n_k})$ converging to some

[5] Convexity of $\mathcal{A}$ is used only in the proof of (14).

[6] with product topology.

element $(\{\pi_i^*\}_{i=1}^m, \rho_1^*, ..., \rho_m^*)$ of the space $\mathfrak{P}\mathfrak{C}^m$. By definition of the product topology on $\mathfrak{P}\mathfrak{C}^m$ it means that

$$\lim_{k\to\infty} \pi_i^{n_k} = \pi_i^*, \quad \lim_{k\to\infty} \rho_i^{n_k} = \rho_i^*.$$

The average state of the ensemble $\{\pi_i^*, \rho_i^*\}_{i=1}^m$ is a limit of the sequence of average states of the ensembles $\{\pi_i^{n_k}, \rho_i^{n_k}\}_{i=1}^m$ and hence lies in $\mathcal{A}$ (which is closed by the assumption).

By continuity of the quantum entropy in finite dimensional case we have

$$\lim_{k\to+\infty} \bar{C}(\Phi_{n_k}, \mathcal{A}) = \lim_{k\to+\infty} \chi_{\Phi_{n_k}}(\{\pi_i^{n_k}, \rho_i^{n_k}\}) = \chi_\Phi(\{\pi_i^*, \rho_i^*\}) \leq \bar{C}(\Phi, \mathcal{A}),$$

which contradicts (17).

Comparing (15) and (16) we see that

$$\lim_{n\to+\infty} \bar{C}(\Phi_n, \mathcal{A}) = \bar{C}(\Phi, \mathcal{A}).$$

It follows that the above ensemble $\{\pi_i^*, \rho_i^*\}_{i=1}^m$ is optimal for the $\mathcal{A}$-constrained channel $\Phi$. Hence, there exists the optimal average state $\bar{\rho}^*$ for the $\mathcal{A}$-constrained channel $\Phi$ which is a partial limit of the sequence $\{\bar{\rho}^n\}$ of the optimal average states for the $\mathcal{A}$-constrained channels $\Phi_n$.

Suppose (14) is not true. Without loss of generality we may (by compactness argument) assume that there exists $\lim_{n\to\infty} \Omega(\Phi_n, \mathcal{A}) \neq \Omega(\Phi, \mathcal{A})$ . By Proposition 1 this contradicts the previous observation. □

The assumption of finite dimensionality in the first part of Theorem 1 is essential. The following example shows that generally the $\chi$-capacity is not a continuous function of a channel even in the stronger trace norm topology on the space of all channels. The example is a purely classical channel which has a standard extension to a quantum one.

*Example 2.* Consider the Abelian von Neumann algebra $l_\infty$ and its predual $l_1$. Let $\{\Phi_n^q; n = 1, 2, ...; q \in (0, 1)\}$ be the family of classical unconstrained channels defined by the formula

$$\Phi_n^q(\{x_1, x_2, ..., x_n, ...\}) = \{(1-q)\textstyle\sum_{i=1}^\infty x_i, q\sum_{i=n+1}^\infty x_i, qx_1, ..., qx_n, 0, 0, ...\}$$

for $\{x_1, x_2, ..., x_n, ...\} \in l_1$. Defining $\Phi^0(\{x_1, x_2, ..., x_n, ...\}) = \{\sum_{i=1}^\infty x_i, 0, 0, ...\}$ we have

$$\|(\Phi_n^q - \Phi^0)(\{x_i\}_{i=1}^\infty)\|_1 = q\|\{-\textstyle\sum_{i=1}^\infty x_i, \sum_{i=n+1}^\infty x_i, x_1, ..., x_n, 0, 0, ...\}\|_1$$

$$= q(|\textstyle\sum_{i=1}^\infty x_i| + |\sum_{i=n+1}^\infty x_i| + |x_1| + \cdots + |x_n|) \leq 3q\|\{x_i\}_{i=1}^\infty\|_1,$$

hence $\|\Phi_n^q - \Phi^0\| \to 0$ as $q \to 0$ uniformly in $n$.

To evaluate the $\chi$-capacity of the channel $\Phi_n^q$ it is sufficient to note that

$$H(\Phi_n^q \text{ (any pure state)}) = h_2(q) = -q\log q - (1-q)\log(1-q) \text{ and}$$

$$H(\Phi_n^q \text{ (any state)}) \leq H(\Phi_n^q(\{\underbrace{\tfrac{1}{n+1}, \tfrac{1}{n+1}, ..., \tfrac{1}{n+1}}_{n+1}, 0, 0...\})) = q\log(n+1) + h_2(q).$$

It follows by definition that $\bar{C}(\Phi_n^q) = q\log(n+1),\ q \in (0, 1),\ n \in \mathbb{N}$.

Take arbitrary $C$ such that $0 < C \leq +\infty$ and choose a sequence $q(n)$ such that $\lim_{n\to\infty} q(n) = 0$ while $\lim_{n\to\infty} q(n)\log(n+1) = C$ . Then we have $\lim_{n\to\infty} \|\Phi_n^{q(n)} - \Phi^0\| = 0$ but $\lim_{n\to\infty} \bar{C}(\Phi_n^{q(n)}) = C > 0 = \bar{C}(\Phi^0)$. □

*Remark 2.* The above example demonstrates harsh discontinuity of the $\chi$-capacity in the infinite dimensional case. One can see that a similar discontinuity underlies Shor's construction [27] allowing to prove equivalence of different additivity properties by using channel extension and a limiting procedure.

## 6. Additivity for Constrained Channels

Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi : \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ be two channels with the constraints, defined by closed subsets $\mathcal{A} \subset \mathfrak{S}(\mathcal{H})$ and $\mathcal{B} \subset \mathfrak{S}(\mathcal{K})$ correspondingly. For the channel $\Phi \otimes \Psi$ we consider the constraint defined by the requirements $\bar{\omega}^{\mathcal{H}} := \mathrm{Tr}_{\mathcal{K}}\bar{\omega} \in \mathcal{A}$ and $\bar{\omega}^{\mathcal{K}} := \mathrm{Tr}_{\mathcal{H}}\bar{\omega} \in \mathcal{B}$, where $\bar{\omega}$ is the average state of an input ensemble $\{\mu_i, \omega_i\}$. The closed subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ consisting of states $\omega$ such that $\mathrm{Tr}_{\mathcal{K}}\omega \in \mathcal{A}$ and $\mathrm{Tr}_{\mathcal{H}}\omega \in \mathcal{B}$ will be denoted $\mathcal{A}\otimes\mathcal{B}$. The application of the results of Sect. 3 to the $\mathcal{A}\otimes\mathcal{B}$-constrained channel $\Phi \otimes \Psi$ is based on the following lemma.

**Lemma 4.** *The set $\mathcal{A} \otimes \mathcal{B}$ is a convex subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ if and only if the sets $\mathcal{A}$ and $\mathcal{B}$ are convex subsets of $\mathfrak{S}(\mathcal{H})$ and of $\mathfrak{S}(\mathcal{K})$ correspondingly.*

*The set $\mathcal{A} \otimes \mathcal{B}$ is a compact subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ if and only if the sets $\mathcal{A}$ and $\mathcal{B}$ are compact subsets of $\mathfrak{S}(\mathcal{H})$ and of $\mathfrak{S}(\mathcal{K})$ correspondingly.*

*Proof.* The first statement of this lemma is trivial. To prove the second, note that compactness of the set $\mathcal{A} \otimes \mathcal{B}$ implies compactness of the sets $\mathcal{A}$ and $\mathcal{B}$ due to continuity of the partial trace.

The proof of the converse implication is based on the following characterization of a compact set of states: *a closed subset $\mathcal{A}$ of $\mathfrak{S}(\mathcal{H})$ is compact if and only if for any $\varepsilon > 0$ there exists a finite dimensional projector $P_\varepsilon$ such that* $\mathrm{Tr}P_\varepsilon\rho > 1 - \varepsilon$ *for all $\rho \in \mathcal{A}$.* This characterization can be deduced by combining results of [20] and [3] (see the proof of the lemma in [10]). Its proof is also presented in the Appendix of [12].

Let $\mathcal{A}$ and $\mathcal{B}$ be compact. By the above characterization for arbitrary $\varepsilon > 0$ there exist finite rank projectors $P_\varepsilon$ and $Q_\varepsilon$ such that

$$\mathrm{Tr}P_\varepsilon\rho > 1 - \varepsilon,\ \forall\rho \in \mathcal{A} \quad \text{and} \quad \mathrm{Tr}Q_\varepsilon\sigma > 1 - \varepsilon,\ \forall\sigma \in \mathcal{B}.$$

Since $\omega^{\mathcal{H}} \in \mathcal{A}$ and $\omega^{\mathcal{K}} \in \mathcal{B}$ for arbitrary $\omega \in \mathcal{A} \otimes \mathcal{B}$ we have

$$\mathrm{Tr}((P_\varepsilon \otimes Q_\varepsilon) \cdot \omega) = \mathrm{Tr}((P_\varepsilon \otimes I_{\mathcal{K}}) \cdot \omega) - \mathrm{Tr}(P_\varepsilon \otimes (I_{\mathcal{K}} - Q_\varepsilon)) \cdot \omega)$$

$$\geq \mathrm{Tr}P_\varepsilon\omega^{\mathcal{H}} - \mathrm{Tr}(I_{\mathcal{K}} - Q_\varepsilon)\omega^{\mathcal{K}} > 1 - 2\varepsilon.$$

The above characterization implies compactness of the set $\mathcal{A} \otimes \mathcal{B}$. □

The conjecture of additivity of the $\chi$-capacity for the $\mathcal{A}$-constrained channel $\Phi$ and the $\mathcal{B}$-constrained channel $\Psi$ is [11, 12]

$$\bar{C}\,(\Phi \otimes \Psi; \mathcal{A} \otimes \mathcal{B}) = \bar{C}(\Phi; \mathcal{A}) + \bar{C}(\Psi; \mathcal{B}). \tag{18}$$

*Remark 3.* Let $\bar{\rho}$ and $\bar{\sigma}$ be the optimal average states for the $\mathcal{A}$-constrained channel $\Phi$ and the $\mathcal{B}$-constrained channel $\Psi$ correspondingly. The additivity (18) implies that the state $\bar{\rho}\otimes\bar{\sigma}$ is an optimal average state for the $\mathcal{A}\otimes\mathcal{B}$-constrained channel $\Phi\otimes\Psi$. Indeed, the tensor product of ensembles of the approximating sequence for the $\mathcal{A}$-constrained channel $\Phi$ with ensembles of the approximating sequence for the $\mathcal{B}$-constrained channel $\Psi$ provides (due to (18)) an approximating sequence of ensembles for the $\mathcal{A} \otimes \mathcal{B}$-constrained channel $\Phi \otimes \Psi$.

The results of the previous sections make it possible to obtain the following infinite dimensional version of Theorem 1 in [11].

**Theorem 2.** *Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi : \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ be arbitrary channels. The following properties are equivalent:*

*(i) Eq. (18) holds for arbitrary subsets $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathfrak{S}(\mathcal{K})$ such that $H(\Phi(\rho)) < +\infty$ for all $\rho \in \mathcal{A}$ and $H(\Psi(\sigma)) < +\infty$ for all $\sigma \in \mathcal{B}$;*
*(ii) inequality*

$$\chi_{\Phi\otimes\Psi}(\omega) \leq \chi_{\Phi}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) \tag{19}$$

*holds for an arbitrary state $\omega$ such that $H(\Phi(\omega^{\mathcal{H}})) < +\infty$ and $H(\Psi(\omega^{\mathcal{K}})) < +\infty$;*
*(iii) inequality*

$$\hat{H}_{\Phi\otimes\Psi}(\omega) \geq \hat{H}_{\Phi}(\omega^{\mathcal{H}}) + \hat{H}_{\Psi}(\omega^{\mathcal{K}}) \tag{20}$$

*holds for an arbitrary state $\omega$ such that $H(\Phi(\omega^{\mathcal{H}})) < +\infty$ and $H(\Psi(\omega^{\mathcal{K}})) < +\infty$.*

*Proof.* $(i) \Rightarrow (iii)$. Let $\omega$ be an arbitrary state with finite $H(\Phi(\omega^{\mathcal{H}}))$ and $H(\Psi(\omega^{\mathcal{K}}))$. The validity of $(i)$ implies

$$\bar{C}\left(\Phi \otimes \Psi; \{\omega^{\mathcal{H}}\} \otimes \{\omega^{\mathcal{K}}\}\right) = \bar{C}(\Phi; \{\omega^{\mathcal{H}}\}) + \bar{C}(\Psi; \{\omega^{\mathcal{K}}\}).$$

By Remark 3 the state $\omega^{\mathcal{H}} \otimes \omega^{\mathcal{K}}$ is the optimal average state for the $\{\omega^{\mathcal{H}}\} \otimes \{\omega^{\mathcal{K}}\}$ -constrained channel $\Phi \otimes \Psi$. By Lemma 4 the set $\{\omega^{\mathcal{H}}\} \otimes \{\omega^{\mathcal{K}}\}$ is a convex compact subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. Noting that $\omega \in \{\omega^{\mathcal{H}}\} \otimes \{\omega^{\mathcal{K}}\}$ and applying Corollary 1 we obtain

$$\begin{aligned} \chi_{\Phi}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) &= \bar{C}(\Phi; \{\omega^{\mathcal{H}}\}) + \bar{C}(\Psi; \{\omega^{\mathcal{K}}\}) \\ &= \bar{C}\left(\Phi \otimes \Psi; \{\omega^{\mathcal{H}}\} \otimes \{\omega^{\mathcal{K}}\}\right) \\ &\geq \chi_{\Phi\otimes\Psi}(\omega) + H((\Phi \otimes \Psi)(\omega) \| \Phi(\omega^{\mathcal{H}}) \otimes \Psi(\omega^{\mathcal{K}})). \end{aligned} \tag{21}$$

Due to

$$H((\Phi \otimes \Psi)(\omega) \| \Phi(\omega^{\mathcal{H}}) \otimes \Psi(\omega^{\mathcal{K}})) = H(\Phi(\omega^{\mathcal{H}})) + H(\Psi(\omega^{\mathcal{K}})) - H((\Phi \otimes \Psi)(\omega))$$

the inequality (21) together with (11) implies (20).

$(iii) \Rightarrow (ii)$. It can be derived from expression (11) for the $\chi$-function and subadditivity of the (output) entropy.

$(ii) \Rightarrow (i)$. It follows from the definition of the $\chi$-capacity (2) and inequality (19) that

$$\bar{C}\,(\Phi \otimes \Psi; \mathcal{A} \otimes \mathcal{B}) \leq \bar{C}(\Phi; \mathcal{A}) + \bar{C}(\Psi; \mathcal{B}).$$

Since the converse inequality is obvious, there is equality here. $\square$

The validity of inequality (19) for arbitrary $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ seems to be substantially stronger than the equivalent properties in Theorem 2. This property is called *subadditivity* of the $\chi$-function for the channels $\Phi$ and $\Psi$. By using arguments from the proof of Theorem 2 it is easy to see that subadditivity of the $\chi$-function for the channels $\Phi$ and $\Psi$ is equivalent to validity of Eq. (18) for *arbitrary* subsets $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathfrak{S}(\mathcal{K})$.

By using Proposition 6 below it is possible to show that properties (*i*)–(*iii*) in the above theorem are equivalent to subadditivity of the $\chi$-function for the channels $\Phi$ and $\Psi$ having the following property: $H(\Phi(\rho)) < +\infty$ and $H(\Psi(\sigma)) < +\infty$ for arbitrary finite rank states $\rho \in \mathfrak{S}(\mathcal{H})$ and $\sigma \in \mathfrak{S}(\mathcal{K})$.

We see later (Proposition 7) that the set of quantum infinite dimensional channels for which the subadditivity of the $\chi$-function holds is nontrivial.

*Remark 4.* By Theorem 1 in [11] the subadditivity of the $\chi$-function for the arbitrary finite dimensional channels $\Phi$ and $\Psi$ is equivalent to validity of inequality (20) for the arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, which implies additivity of the minimal output entropy

$$\inf_{\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})} H(\Phi \otimes \Psi(\omega)) = \inf_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi(\rho)) + \inf_{\sigma \in \mathfrak{S}(\mathcal{K})} H(\Psi(\sigma)) \tag{22}$$

for these channels. This follows from the inequality

$$\begin{aligned} H(\Phi \otimes \Psi(\omega)) &\geq \hat{H}_{\Phi \otimes \Psi}(\omega) \geq \hat{H}_{\Phi}(\omega^{\mathcal{H}}) + \hat{H}_{\Psi}(\omega^{\mathcal{K}}) \\ &\geq \inf_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi(\rho)) + \inf_{\sigma \in \mathfrak{S}(\mathcal{K})} H(\Psi(\sigma)), \end{aligned} \tag{23}$$

valid for the arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ for which inequality (20) holds.

In contrast to this in the infinite dimensional case we can not prove the above implication (without some additional assumptions). The problem consists in the existence of pure states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ with infinite entropies of partial traces, which can be called *superentangled*. To show this note first that the monotonicity property of the relative entropy [18] provides the following inequality

$$\begin{aligned} &H(\omega^{\mathcal{H}}) + H(\omega^{\mathcal{K}}) - H(\omega) \\ &\quad = H\left(\omega \,\|\, \omega^{\mathcal{H}} \otimes \omega^{\mathcal{K}}\right) \geq H\left(\Phi \otimes \Psi(\omega) \| \, \Phi\left(\omega^{\mathcal{H}}\right) \otimes \Psi\left(\omega^{\mathcal{K}}\right)\right) \\ &\quad = H\left(\Phi\left(\omega^{\mathcal{H}}\right)\right) + H\left(\Psi\left(\omega^{\mathcal{K}}\right)\right) - H(\Phi \otimes \Psi(\omega)), \end{aligned}$$

which shows that $H(\omega^{\mathcal{H}}) = H(\omega^{\mathcal{K}}) < +\infty$ implies $H(\Phi(\omega^{\mathcal{H}})) < +\infty$ and $H(\Psi(\omega^{\mathcal{K}})) < +\infty$ for the arbitrary pure state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ with finite output entropy $H(\Phi \otimes \Psi(\omega))$. By this and Theorem 2 the subadditivity of the $\chi$-function for arbitrary infinite dimensional channels $\Phi$ and $\Psi$ implies validity of inequality (20) and hence validity of inequality (23) for all pure states $\omega$ such that $H(\omega^{\mathcal{H}}) = H(\omega^{\mathcal{K}}) < +\infty$ and $H(\Phi \otimes \Psi(\omega)) < +\infty$. So if we considered only such states $\omega$ in the calculation of the minimal output entropy for the channel $\Phi \otimes \Psi$, we would obtain that it is equal to the sum of $\inf_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi(\rho))$ and $\inf_{\sigma \in \mathfrak{S}(\mathcal{K})} H(\Psi(\sigma))$, but this additivity can be (probably) broken by taking into account superentangled states.

## 7. Generalization of the Additivity Conjecture

The main aim of this section is to show that the conjecture of additivity of the $\chi$-capacity for arbitrary finite dimensional channels implies the additivity of the $\chi$-capacity for arbitrary infinite dimensional channels with arbitrary constraints.

It is convenient to introduce the following notation. The channel $\Phi$ is

- FF-channel if $\dim\mathcal{H} < +\infty$ and $\dim\mathcal{H}' < +\infty$;
- FI-channel if $\dim\mathcal{H} < +\infty$ and $\dim\mathcal{H}' \leq +\infty$.

Speaking about the quantum channel $\Phi$ without reference to FF or FI we will assume that $\dim\mathcal{H} \leq +\infty$ and $\dim\mathcal{H}' \leq +\infty$.

Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be an arbitrary channel such that $\dim\mathcal{H}' = +\infty$ and $P'_n$ be a sequence of finite rank projectors in $\mathcal{H}'$ increasing to $I_{\mathcal{H}'}$ and $\mathcal{H}'_n = P'_n(\mathcal{H}')$. Consider the channel

$$\Phi_n(\rho) = P'_n\Phi(\rho)P'_n + \left(\mathrm{Tr}(I_{\mathcal{H}'} - P'_n)\Phi(\rho)\right)\tau_n \tag{24}$$

from $\mathfrak{S}(\mathcal{H})$ into $\mathfrak{S}(\mathcal{H}'_n \oplus \mathcal{H}''_n) \subset \mathfrak{S}(\mathcal{H}')$, where $\tau_n$ is a pure state in some finite dimensional subspace $\mathcal{H}''_n$ of $\mathcal{H}' \ominus \mathcal{H}'_n$. If $\dim\mathcal{H}' < +\infty$ we will assume that $\Phi_n = \Phi$ for all $n$. Note that for the arbitrary FI-channel $\Phi$ the corresponding channel $\Phi_n$ is a FF-channel for all $n$.

For arbitrary channel $\Psi : \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ we will consider the sequences $\Phi_n$ and $\Phi_n \otimes \Psi$ of channels as approximations for the channels $\Phi$ and $\Phi \otimes \Psi$ correspondingly. Despite the discontinuity of the $\chi$-capacity as a function of a channel in the infinite dimensional case the following result is valid.

**Lemma 5.** *Let $\Phi$ and $\Psi$ be arbitrary channels. If subadditivity of the $\chi$-function holds for the channel $\Phi_n$ defined by (24) and the channel $\Psi$ for all $n$, then subadditivity of the $\chi$-function holds for the channels $\Phi$ and $\Psi$.*

*Proof.* The channel $\Phi_n$ can be represented as the composition $\Pi_n \circ \Phi$ of the channel $\Phi$ with the channel $\Pi_n : \mathfrak{S}(\mathcal{H}') \mapsto \mathfrak{S}(\mathcal{H}'_n \oplus \mathcal{H}''_n)$ defined by

$$\Pi_n(\rho') = P'_n\rho' P'_n + \left(\mathrm{Tr}(I_{\mathcal{H}'} - P'_n)\rho'\right)\tau_n.$$

Proposition 4 implies

$$\chi_{\Phi_n}(\rho) = \chi_{\Pi_n\circ\Phi}(\rho) \leq \chi_{\Phi}(\rho), \quad \forall\rho \in \mathfrak{S}(\mathcal{H}), \quad \forall n \in \mathbb{N}.$$

Since

$$\lim_{n\mapsto+\infty} \Phi_n(\rho) = \Phi(\rho), \quad \forall\rho \in \mathfrak{S}(\mathcal{H}),$$

it follows from Theorem 1 that

$$\liminf_{n\mapsto+\infty} \chi_{\Phi_n}(\rho) \geq \chi_{\Phi}(\rho), \quad \forall\rho \in \mathfrak{S}(\mathcal{H}).$$

The two above inequalities imply

$$\lim_{n\mapsto+\infty} \chi_{\Phi_n}(\rho) = \chi_{\Phi}(\rho), \quad \forall\rho \in \mathfrak{S}(\mathcal{H}). \tag{25}$$

It is easy to see that

$$\Phi_n\otimes\Psi(\omega)=(P'_n\otimes I_{\mathcal{K}'})\cdot(\Phi\otimes\Psi(\omega))\cdot(P'_n\otimes I_{\mathcal{K}'})$$
$$+\tau_n\otimes\mathrm{Tr}_{\mathcal{H}'}\left(((I_{\mathcal{H}'}-P'_n)\otimes I_{\mathcal{K}'})\cdot(\Phi\otimes\Psi(\omega))\right),\quad \forall\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}).$$

Hence

$$\lim_{n\mapsto+\infty}\Phi_n\otimes\Psi(\omega)=\Phi\otimes\Psi(\omega),\quad \forall\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}),$$

and by Theorem 1 we have

$$\liminf_{n\mapsto+\infty}\chi_{\Phi_n\otimes\Psi}(\omega)\geq\chi_{\Phi\otimes\Psi}(\omega),\quad \forall\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}).\tag{26}$$

By the assumption

$$\chi_{\Phi_n\otimes\Psi}(\omega)\leq\chi_{\Phi_n}(\omega^{\mathcal{H}})+\chi_{\Psi}(\omega^{\mathcal{K}}),\quad \forall\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}),\quad \forall n\in\mathbb{N}.$$

This, (25) and (26) imply

$$\chi_{\Phi\otimes\Psi}(\omega)\leq\chi_{\Phi}(\omega^{\mathcal{H}})+\chi_{\Psi}(\omega^{\mathcal{K}}),\quad \forall\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}).$$

□

**Proposition 5.** *Subadditivity of the $\chi$-function for all FF-channels implies subadditivity of the $\chi$-function for all FI-channels.*

*Proof.* This can be proved by double application of Lemma 5. First, we prove the subadditivity of the $\chi$-function for any two channels, when one of them is of FI-type while another is of FF-type. Second, we remove the FF restriction from the last channel. □

Now we will turn to channels with an infinite dimensional input quantum system. We will use the following notion of subchannel.

**Definition 3.** *The restriction of a channel $\Phi:\mathfrak{S}(\mathcal{H})\mapsto\mathfrak{S}(\mathcal{H}')$ to the set of states with support contained in a subspace $\mathcal{H}_0$ of the space $\mathcal{H}$ is called the subchannel $\Phi_0$ of the channel $\Phi$, corresponding to the subspace $\mathcal{H}_0$.*

It is easy to see that subadditivity of the $\chi$-function for the channels $\Phi$ and $\Psi$ implies subadditivity of the $\chi$-function for arbitrary subchannels $\Phi_0$ and $\Psi_0$ of the channels $\Phi$ and $\Psi$. The properties of the $\chi$-function established in Sect. 4 make it possible to prove the following important result.

**Proposition 6.** *Let $\Phi:\mathfrak{S}(\mathcal{H})\mapsto\mathfrak{S}(\mathcal{H}')$ and $\Psi:\mathfrak{S}(\mathcal{K})\mapsto\mathfrak{S}(\mathcal{K}')$ be arbitrary channels. Subadditivity of the $\chi$-function for any two FI-subchannels of the channels $\Phi$ and $\Psi$ implies subadditivity of the $\chi$-function for the channels $\Phi$ and $\Psi$.*

*Proof.* It is sufficient to consider the case $\dim\mathcal{H}=+\infty$, $\dim\mathcal{K}\leq+\infty$. Let $\omega$ be an arbitrary state in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. Let $\{|\varphi_k\rangle\}_{k=1}^{+\infty}$ and $\{|\psi_k\rangle\}_{k=1}^{\dim\mathcal{K}}$ be ONB of eigenvectors of the compact positive operators $\omega^{\mathcal{H}}$ and $\omega^{\mathcal{K}}$ such that the corresponding sequences of eigenvalues are nonincreasing. Let $P_n=\sum_{k=1}^{n}|\varphi_k\rangle\langle\varphi_k|$ and $Q_n=\sum_{k=1}^{n}|\psi_k\rangle\langle\psi_k|$. In the case $\dim\mathcal{K}<+\infty$ we will assume $Q_n=I_{\mathcal{K}}$ for all $n\geq\dim\mathcal{K}$. The nondecreasing sequences $\{P_n\}$ and $\{Q_n\}$ of finite rank projectors converge to $I_{\mathcal{H}}$ and to $I_{\mathcal{K}}$ correspondingly in the strong operator topology. Let $\mathcal{H}_n=P_n(\mathcal{H})$ and $\mathcal{K}_n=Q_n(\mathcal{K})$.

Consider the sequence of states

$$\omega_n = (\mathrm{Tr}\,((P_n \otimes Q_n) \cdot \omega))^{-1}(P_n \otimes Q_n) \cdot \omega \cdot (P_n \otimes Q_n),$$

which are well defined for all $n$ by the choice of the projectors $P_n$ and $Q_n$. Since obviously

$$\lim_{n\to+\infty} \omega_n = \omega, \tag{27}$$

Proposition 3 implies

$$\liminf_{n\to+\infty} \chi_{\Phi\otimes\Psi}(\omega_n) \geq \chi_{\Phi\otimes\Psi}(\omega). \tag{28}$$

The next part of the proof is based on the following operator inequalities:

$$\lambda_n \omega_n^{\mathcal{H}} \leq \omega^{\mathcal{H}}, \quad \lambda_n \omega_n^{\mathcal{K}} \leq \omega^{\mathcal{K}}, \quad \text{where} \quad \lambda_n = \mathrm{Tr}\,((P_n \otimes Q_n) \cdot \omega)\,. \tag{29}$$

Let us prove the first inequality. By the choice of $P_n$ and due to $\mathrm{supp}\omega_n^{\mathcal{H}} \subseteq \mathcal{H}_n$ it is sufficient to show that $\lambda_n \omega_n^{\mathcal{H}} \leq P_n \omega^{\mathcal{H}}$. Let $\varphi \in \mathcal{H}_n$. By the definition of a partial trace,

$$\begin{aligned}\langle\varphi|\lambda_n \omega_n^{\mathcal{H}}|\varphi\rangle &= \sum_{k=1}^{\dim\mathcal{K}} \langle\varphi\otimes\psi_k|P_n\otimes Q_n\cdot\omega\cdot P_n\otimes Q_n|\varphi\otimes\psi_k\rangle \\ &= \sum_{k=1}^{m}\langle\varphi\otimes\psi_k|\omega|\varphi\otimes\psi_k\rangle \leq \sum_{k=1}^{\dim\mathcal{K}} \langle\varphi\otimes\psi_k|\omega|\varphi\otimes\psi_k\rangle = \langle\varphi|\omega^{\mathcal{H}}|\varphi\rangle,\end{aligned}$$

where $m = \min\{n, \dim\mathcal{K}\}$. The second inequality is proved the same way.

By using (27) and applying Corollary 3 due to (29) we obtain

$$\lim_{n\to+\infty} \chi_\Phi(\omega_n^{\mathcal{H}}) = \chi_\Phi(\omega^{\mathcal{H}}) \quad \text{and} \quad \lim_{n\to+\infty} \chi_\Psi(\omega_n^{\mathcal{K}}) = \chi_\Psi(\omega^{\mathcal{K}}). \tag{30}$$

For each $n$ the $\{\omega_n^{\mathcal{H}}\}$-constrained channel $\Phi$ and the $\{\omega_n^{\mathcal{K}}\}$-constrained channel $\Psi$ can be considered as FI-subchannels of the channels $\Phi$ and $\Psi$ corresponding to the subspaces $\mathcal{H}_n$ and $\mathcal{K}_n$. Hence by the assumption,

$$\chi_{\Phi\otimes\Psi}(\omega_n) \leq \chi_\Phi(\omega_n^{\mathcal{H}}) + \chi_\Psi(\omega_n^{\mathcal{K}}), \quad \forall n \in \mathbb{N}.$$

This, (28) and (30) imply

$$\chi_{\Phi\otimes\Psi}(\omega) \leq \chi_\Phi(\omega^{\mathcal{H}}) + \chi_\Psi(\omega^{\mathcal{K}}).$$

□

It is known that additivity of the $\chi$-capacity for all unconstrained FF-channels is equivalent to subadditivity of the $\chi$-function for all FF-channels [11, 27]. By combining this with Proposition 5 and Proposition 6 we obtain the following extension of the additivity conjecture.

**Theorem 3.** *The additivity of the $\chi$-capacity for all FF-channels implies additivity of the $\chi$-capacity for all channels with arbitrary constraints.*

This theorem and Theorem 2 imply the following result concerning superadditivity of the convex closure of the output entropy for infinite dimensional channels. Note that in the case of a partial trace channel the convex closure of the output entropy coincides with the entanglement of formation (EoF).

**Corollary 4.** *If inequality (20) holds for all FF-channels $\Phi$ and $\Psi$ and all states $\omega$ then inequality (20) holds for all channels $\Phi$ and $\Psi$ and all states $\omega$ such that $H(\Phi(\omega^{\mathcal{H}})) < +\infty$ and $H(\Psi(\omega^{\mathcal{K}})) < +\infty$.*

*Proof.* The validity of inequality (20) for two FF-channels $\Phi$ and $\Psi$ and for all states $\omega$ is equivalent to subadditivity of the $\chi$-function for these channels [11]. Hence the assumption of the corollary and Theorem 3 imply subadditivity of the $\chi$-function for any channels, which, by Theorem 2, implies the validity of inequality (20) for all channels $\Phi$ and $\Psi$ and all states $\omega$ such that $H(\Phi(\omega^{\mathcal{H}})) < +\infty$ and $H(\Psi(\omega^{\mathcal{K}})) < +\infty$. □

*Remark 5.* By combining Shor's theorem in [27] and Theorem 3 we obtain that additivity of the minimal output entropy (22) for all FF-channels implies additivity of the $\chi$-capacity (18) for all channels with arbitrary constraints. But due to existence of superentangled states (see Remark 4) we can not show that it implies additivity of minimal output entropy for all channels. So, in the infinite dimensional case the conjecture of additivity of the minimal output entropy *for all channels* seems to be substantially stronger than the conjecture of additivity of the $\chi$-capacity for all channels with arbitrary constraints.

Note that in contrast to Proposition 5, Proposition 6 relates the subadditivity of the $\chi$-function for the initial channels with the subadditivity of the $\chi$-function for its FI-subchannels (not any FI-channels!). This makes it applicable for analysis of individual channels as it is illustrated in the proof of Proposition 7 below.

We will use the following natural generalization of the notion of entanglement breaking a finite dimensional channel [15].

**Definition 4.** *A channel $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ is called entanglement breaking if for an arbitrary Hilbert space $\mathcal{K}$ and for an arbitrary state $\omega$ in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ the state $\Phi \otimes \mathrm{Id}(\omega)$ lies in the closure of the convex hull of all product states in $\mathfrak{S}(\mathcal{H}' \otimes \mathcal{K})$, where* Id *is the identity channel from $\mathfrak{S}(\mathcal{K})$ onto itself.*

Generalizing the result in [15] it is possible to show that a channel $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ is entanglement-breaking if and only if it admits the representation

$$\Phi(\rho) = \int_X \rho'(x)\mu_\rho(dx),$$

where $X$ is a complete separable metric space, $\rho'(x)$ is a Borel $\mathfrak{S}(\mathcal{H}')$-valued function on $X$ and $\mu_\rho(A) = \mathrm{Tr}(\rho M(A))$ for any Borel $A \subset X$, with $M$ positive operator valued measure on $X$ [14].

The following proposition is a generalization of Proposition 2 in [11].

**Proposition 7.** *Let $\Psi$ be an arbitrary channel. The subadditivity of the $\chi$-function holds in each of the following cases:*

*(i) $\Phi$ is a noiseless channel;*

*(ii)* $\Phi$ *is an entanglement breaking channel;*

*(iii)* $\Phi$ *is a direct sum mixture (cf.[11]) of a noiseless channel and a channel* $\Phi_0$ *such that the subadditivity of the* $\chi$*-function holds for* $\Phi_0$ *and* $\Psi$ *(in particular, an entanglement breaking channel).*

*Proof.* In the proof of each point of this proposition for FF-channels the finite dimensionality of the underlying Hilbert spaces was used (cf.[26, 11]). The idea of this proof consists in using our extension results (Proposition 6 and Lemma 5).

(i) Note that any FI-subchannel of an arbitrary noiseless channel is a noiseless FF-channel. Hence by Proposition 6 it is sufficient to prove the subadditivity of the $\chi$-function for arbitrary noiseless FF-channel $\Phi$ and the arbitrary FI-channel $\Psi$. But this can be done with the help of Lemma 5. Indeed, using this lemma with the noiseless FF-channel in the role of the fixed channel $\Psi$ we can deduce the above assertion from the subadditivity of the $\chi$-function for arbitrary two FF-channels with one of them noiseless (Proposition 2 in [11]).

(ii) Note that any FI-subchannel of an arbitrary entanglement breaking channel is entanglement breaking. Hence by Proposition 6 it is sufficient to prove the subadditivity of the $\chi$-function for an arbitrary entanglement breaking FI-channel $\Phi$ and an arbitrary FI-channel $\Psi$. Similar to the proof of $(i)$ this can be done with the help of Lemma 5, but in this case it is necessary to apply this lemma twice. First we prove the subadditivity of the $\chi$-function for arbitrary entanglement breaking FI-channel $\Phi$ and arbitrary FF-channel $\Psi$ by noting that any FF-channel $\Phi_n$, involved in lemma 5, inherits the entanglement breaking property from the channel $\Phi$ and using the subadditivity of the $\chi$-function for arbitrary two FF-channels with one of them entanglement breaking [26]. Second, by using the result of the first step we remove the FF restriction from another channel $\Psi$.

(iii) Note that any FI-subchannel of the channel $\Phi_q = q\mathrm{Id} \oplus (1-q)\Phi_0$ has the same structure with FF-channel Id and FI-channel $\Phi_0$. By the remark before Proposition 6 subadditivity of the $\chi$-function for the channels $\Phi_0$ and $\Psi$ implies subadditivity of the $\chi$-function for their arbitrary subchannels. Hence by Proposition 6 it is sufficient to prove *(iii)* for the FI-channel $\Phi_q$ and the FI-channel $\Psi$. $\square$

Let $\omega$ be a state in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ with $\dim\mathcal{H} < +\infty$ and $\dim\mathcal{K} < +\infty$. It follows that $\chi_{\mathrm{Id}}(\omega^{\mathcal{H}}) = H(\omega^{\mathcal{H}}) < +\infty$. By the established subadditivity of the $\chi$-function for the FF-channel Id and the FI-channel $\Psi$ and by the assumed subadditivity of the $\chi$-function for the FI-channel $\Phi_0$ and the FI-channel $\Psi$ we have

$$\chi_{\mathrm{Id}\otimes\Psi}(\omega) \le \chi_{\mathrm{Id}}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) \quad \text{and} \quad \chi_{\Phi_0\otimes\Psi}(\omega) \le \chi_{\Phi_0}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}).$$

Using this and Lemma 3 in [11][7] we obtain

$$\begin{aligned}\chi_{\Phi_q\otimes\Psi}(\omega) &\le q\chi_{\mathrm{Id}\otimes\Psi}(\omega) + (1-q)\chi_{\Phi_0\otimes\Psi}(\omega)\\ &\le q\chi_{\mathrm{Id}}(\omega^{\mathcal{H}}) + q\chi_{\Psi}(\omega^{\mathcal{K}}) + (1-q)\chi_{\Phi_0}(\omega^{\mathcal{H}}) + (1-q)\chi_{\Psi}(\omega^{\mathcal{K}})\\ &= qH(\omega^{\mathcal{H}}) + (1-q)\chi_{\Phi_0}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) = \chi_{\Phi_q}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}),\end{aligned}$$

[7] This lemma implies that for arbitrary channels $\Phi_1$ and $\Phi_2$ from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}(\mathcal{H}'_1)$ and to $\mathfrak{S}(\mathcal{H}'_2)$ correspondingly one has

$$\chi_{q\Phi_1\oplus(1-q)\Phi_2}(\{\pi_i,\rho_i\}) = q\chi_{\Phi_1}(\{\pi_i,\rho_i\}) + (1-q)\chi_{\Phi_2}(\{\pi_i,\rho_i\})$$

for the arbitrary ensemble $\{\pi_i,\rho_i\}$ of states in $\mathfrak{S}(\mathcal{H})$ and arbitrary $q\in[0;1]$.

where the last equality follows from the existence of the approximating sequence of *pure* state ensembles for the $\{\omega^{\mathcal{H}}\}$-constrained FI-channel $\Phi_0$. □

*Acknowledgements.* The author is grateful to A. S. Holevo for the idea of this work and for continued help, and to M.B.Ruskai for useful remarks. The author also acknowledges support from the QIS Program, the Newton Institute, Cambridge, where this paper was completed. The work was also partially supported by INTAS grant 00-738.

*Note.* After this paper had been completed it was found that the compactness assumption in Propositions 1 and 2 can be taken off. The generalized versions of these propositions and of Proposition 3 are presented in [25].

## References

1. Bratteli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics. vol.I, New York-Heidelberg-Berlin, Springer Verlag, 1979
2. Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K.: Mixed State Entanglement and Quantum Error Correction. Phys. Rev. A **54**, 3824–3851 (1996)
3. Dell'Antonio, G.F.: On the limits of sequences of normal states. Commun. Pure Appl. Math. **20**, 413–430 (1967)
4. Davies, E.B.: Information and Quantum Measurements. IEEE Trans.Inf.Theory **24**, 596–599 (1978)
5. Davies, E.B.: Quantum theory of open systems. London, Academic Press, 1976
6. Donald, M.J.: Further results on the relative entropy. Math. Proc. Cam. Phil. Soc. **101**, 363–373 (1987)
7. Eisert, J., Simon, C., Plenio, M.B.: The quantification of entanglement in infinite-dimensional quantum systems. J. Phys. A **35**, 3911 (2002)
8. Holevo, A.S.: Quantum coding theorems. Russ. Math. Surv. **53**, N6, 1295–1331 (1998)
9. Holevo, A.S.: On quantum communication channels with constrained inputs. http://arxiv.org/list/quant-ph/9705054, 1997
10. Holevo, A.S.: Classical capacities of quantum channels with constrained inputs. Probability Theory and Applications. **48**, N.2, 359–374 (2003)
11. Holevo, A.S., Shirokov, M.E.: On Shor's channel extension and constrained channels. Commun. Math. Phys. **249**, 417–430 (2004)
12. Holevo, A.S., Shirokov, M.E.: Continuous ensembles and the $\chi$-capacity of infinite dimensional channels. Probability Theory and Applications **50**, N.1, 98–114 (2005)
13. Holevo, A.S., Werner, R.F.: Evaluating capacities of Bosonic Gaussian channels. http://arxiv.org/list/quant-ph/9912067, 1999
14. Holevo, A.S., Shirokov, M.E., Werner, R.F.: On the notion of entanglement in Hilbert space. Russ. Math. Surv. **60**, N.2, 359–360 (2005)
15. Horodecki, M., Shor, P.W., Ruskai, M.B.: General Entanglement Breaking Channels. Rev. Math. Phys. **15**, 629–641 (2003)
16. Lindblad, G.: Entropy, Information and Quantum Measurements. Commun. Math. Phys. **33**, N.4, 305–322 (1973)
17. Lindblad, G.: Expectation and Entropy Inequalities for Finite Quantum Systems. Commun. Math. Phys. **39**, N.2, 111–119 (1974)
18. Lindblad, G.: Completely Positive Maps and Entropy Inequalities. Commun. Math. Phys. 40, N.2, 147–151 (1975)
19. Ohya, M., Petz, D.: Quantum Entropy and Its Use. Texts and Monographs in Physics, Berlin: Springer-Verlag, 1993
20. Sarymsakov, T.A.: Introduction to Quantum Probability Theory. Tashkent, FAN: 1985, (In Russian)
21. Schumacher, B., Westmoreland, M.D.: Sending Classical Information via Noisy Quantum Channels. Phys. Rev. A **56**, 131–138 (1997)
22. Schumacher, B., Westmoreland, M.D.: Optimal signal ensemble. Phys. Rev. A **63**, 022308 (2001)
23. Shirokov, M.E.: On the additivity conjecture for channels with arbitrary constraints. http://arxiv.org/list/quant-ph/0308168, 2003
24. Shirokov, M.E.: On entropic quantities related to the classical capacity of infinite dimensional quantum channels. http://arxiv.org/list/quant-ph/0411091, 2004
25. Shirokov, M.E.: The Holevo capacity of infinite dimensional channels and the additivity problem. http://arxiv.org/list/quant-ph/0408009, 2004

26. Shor, P.W.: Additivity of the classical capacity of entanglement breaking quantum channel. J.Math.Phys, **43**, 4334–4340 (2002)
27. Shor, P.W.: Equivalence of additivity questions in quantum information theory. Commun. Math. Phys. **246**, N.3, 453–472 (2004)
28. Simon, B.: Convergence theorem for entropy. Appendix in Lieb, E.H., Ruskai, M.B.: Proof of the strong subadditivity of quantum mechanical entropy. J.Math.Phys. **14**, 1938–1941 (1973)
29. Wehrl, A.: General properties of entropy. Rev. Mod. Phys. **50**, 221–250 (1978)

Communicated by M.B. Ruskai