

On Superactivation of Zero-Error Capacities and Reversibility of a Quantum Channel

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Abstract: We propose examples of low dimensional quantum channels demonstrating different forms of superactivation of one-shot zero-error capacities, in particular, the extreme superactivation (this complements the recent result of Cubitt and Smith). We also describe classes of quantum channels whose zero-error classical and quantum capacities cannot be superactivated. We consider implications of the superactivation of one-shot zero-error capacities to analysis of reversibility of a tensor-product channel with respect to families of pure states. Our approach based on the notions of complementary channel and of transitive subspace of operators makes it possible to study the superactivation effects for infinite-dimensional channels as well.

1. Introduction

The effect of superactivation of quantum channel capacities is one of the main recent discoveries in quantum information theory. It means that the particular capacity of a tensor product of two quantum channels may be positive despite the fact that same capacity of each of these channels is zero.

This effect was originally observed by Smith and Yard in [24], who gave examples of two channels Φ and Ψ with zero quantum capacity such that the channel $\Phi \otimes \Psi$ has positive quantum capacity.

The same phenomenon for the (one shot and asymptotic) zero-error classical capacities was established by Cubitt et al. in [3]. Simultaneously and independently, Duan presented an example of low dimensional channels demonstrating superactivation of the one-shot zero-error classical capacity [8].

The extreme form of superactivation of zero-error capacities was observed by Cubitt and Smith in [4], who proved the existence of two channels Φ and Ψ with zero (asymptotic) zero-error classical capacity such that the channel $\Phi \otimes \Psi$ has positive zero-error quantum capacity.

In this paper we present examples of low dimensional quantum channels, which demonstrate different forms of superactivation of one-shot zero-error capacities. In particular, in Corollary 2 we give a *symmetric* example of superactivation of one-shot zero-error classical capacity with the minimal possible input dimension $\dim \mathcal{H}_A = 4$ and the minimal Choi rank $\dim \mathcal{H}_E = 3$ so that $\dim \mathcal{H}_B \leq 12$ (this answers the question stated after Theorem 1 in [8]). As to the extreme form of superactivation of one-shot zero-error capacities, the existence of such channels in high dimensions follows from the results in [4]. However, nothing was known about their minimal dimensions. Here (Corollary 3) we give an explicit example with $\dim \mathcal{H}_A = 8$, $\dim \mathcal{H}_E = 5$ and $\dim \mathcal{H}_B \leq 40$.

The aim of this paper is also to point out the relation between the superactivation of one-shot zero-error capacities and results on transitive and reflexive subspaces of operators [6, 20]. In fact, the notion of transitive subspace is very close to the notion of unextendible subspace traditionally used in analysis of the superactivation (one can easily show that in finite dimensions they are related by the natural isomorphism between the tensor product $\mathcal{H} \otimes \mathcal{K}$ of two Hilbert spaces and the space of all operators from \mathcal{H} to \mathcal{K}). Nevertheless, the recent results concerning transitive subspaces of operators (presented in [6]) seem to be unknown for specialists in quantum information theory. It is also essential that these results can be used for analysis of superactivation effects for infinite dimensional quantum channels.

Some results concerning transitive and reflexive subspaces of operators can also be applied for showing that channels of certain type cannot be superactivated by any other channels. A result in this direction was obtained recently by Park and Lee in [22]. They showed that superactivation of one-shot zero-error classical capacity is not possible if one of two channels is a qubit channel. Our approach gives a very simple proof of this result and also allows us to prove similar statements for some other important classes of channels (Proposition 3, Corollary 6). We also describe classes of channels for which the superactivation of one-shot and asymptotic zero-error quantum capacities does not hold (Proposition 4, Corollary 8).

In this paper we also consider the relations between positivity of one-shot classical and quantum zero-error capacities of a quantum channel and reversibility properties of this channel with respect to families of pure states. These relations show that the superactivation of one-shot classical (correspondingly, quantum) zero-error capacities is equivalent to “superactivation” of reversibility of a channel with respect to orthogonal (correspondingly, non-orthogonal) families of pure states. It is observed that such superactivation of reversibility with respect to *complete* families of pure states is not possible (Proposition 5).

2. On Positivity of Classical and Quantum Zero-Error Capacities of a Quantum Channel

Let \mathcal{H} be a separable¹ Hilbert space, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ —the Banach spaces of all bounded operators in \mathcal{H} and of all trace-class operators in \mathcal{H} correspondingly, $\mathfrak{S}(\mathcal{H})$ – the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of positive operators with unit trace called *states* [12, 21]. If $\dim \mathcal{H} = n < +\infty$ we may identify $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ with the space \mathfrak{M}_n of all $n \times n$ matrices (equipped with the appropriate norm).

¹ In the main part of the paper we may assume that these spaces are finite-dimensional, although all the results are valid in infinite dimensions if we accept the value “ $+\infty$ ” for $\bar{C}_0(\Phi)$, $\bar{Q}_0(\Phi)$, etc. The case of infinite-dimensional quantum channels is included because of our intension to study reversibility properties of a tensor product channel (Sect. 5).

Let $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel, i.e. a completely positive trace-preserving linear map [12, 21]. The *dual* channel $\Phi^* : \mathfrak{B}(\mathcal{H}_B) \rightarrow \mathfrak{B}(\mathcal{H}_A)$ (defined by the relation $\text{Tr} \Phi(\rho)B = \text{Tr} \rho \Phi^*(B)$, $\rho \in \mathfrak{T}(\mathcal{H}_A)$, $B \in \mathfrak{B}(\mathcal{H}_B)$) is a completely positive map such that $\Phi^*(I_{\mathcal{H}_B}) = I_{\mathcal{H}_A}$.

The Stinespring theorem implies the existence of a Hilbert space \mathcal{H}_E and of an isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \text{Tr}_{\mathcal{H}_E} V \rho V^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A). \tag{1}$$

The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni \rho \mapsto \widehat{\Phi}(\rho) = \text{Tr}_{\mathcal{H}_B} V \rho V^* \in \mathfrak{T}(\mathcal{H}_E) \tag{2}$$

is called *complementary* to the channel Φ [12, 13]. The complementary channel is defined uniquely up to isometrical equivalence [13, the Appendix].

The Stinespring representation (1) generates the Kraus representation

$$\Phi(\rho) = \sum_{k=1}^{\dim \mathcal{H}_E} V_k \rho V_k^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A), \tag{3}$$

in which $\{V_k\}$ is a set of linear operators from \mathcal{H}_A into \mathcal{H}_B such that $\sum_k V_k^* V_k = I_{\mathcal{H}_A}$. The operators V_k are defined by the relation

$$\langle \varphi | V_k \psi \rangle = \langle \varphi \otimes k | V \psi \rangle, \quad \varphi \in \mathcal{H}_B, \psi \in \mathcal{H}_A,$$

where $\{|k\rangle\}$ is an orthonormal basis in the space \mathcal{H}_E . The complementary channel (2) can be expressed via these operators as follows

$$\widehat{\Phi}(\rho) = \sum_{k,l=1}^{\dim \mathcal{H}_E} \text{Tr} [V_k \rho V_l^*] |k\rangle \langle l|, \quad \rho \in \mathfrak{T}(\mathcal{H}_A). \tag{4}$$

Among different Stinespring representations (1) of a given channel Φ there are representations with the environment space \mathcal{H}_E of minimal dimension (such representations are called *minimal* [13]). They generate Kraus representations (3) with the minimal number of nonzero summands called *Choi rank* of the channel Φ [12, 21]. We assume in what follows that (1) is a minimal Stinespring representation, so that $\dim \mathcal{H}_E$ is the Choi rank of Φ .

The one-shot zero-error classical capacity $\bar{C}_0(\Phi)$ of a channel Φ can be defined as $\sup_{\mathfrak{S} \in c_0(\Phi)} \log \sharp(\mathfrak{S})$, where $c_0(\Phi)$ is the set of all families $\{\rho_i\}$ of input states such that $\text{supp} \Phi(\rho_i) \perp \text{supp} \Phi(\rho_j)$ for all $i \neq j$.² The (asymptotic) zero-error classical capacity is defined by regularization: $C_0(\Phi) = \sup_n n^{-1} \bar{C}_0(\Phi^{\otimes n})$ [3, 4, 8, 10, 19, 22].

Let $\varphi, \psi \in \mathcal{H}_A$. It follows from (1), (2) and the Schmidt decomposition of the vectors $V\varphi$ and $V\psi$ in $\mathcal{H}_B \otimes \mathcal{H}_E$ that

$$\text{supp} \Phi(|\varphi\rangle \langle \varphi|) \perp \text{supp} \Phi(|\psi\rangle \langle \psi|) \Leftrightarrow \widehat{\Phi}(|\varphi\rangle \langle \psi|) = 0. \tag{5}$$

This observation implies the following lemma.

² The support $\text{supp} \rho$ of a state ρ is the orthogonal complement to its kernel.

Lemma 1. *A channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ has positive one-shot zero-error classical capacity if and only if $\ker \widehat{\Phi}$ contains a 1-rank operator.*

The assertion of Lemma 1 agrees with Lemma 1 in [8], since representation (4) shows that the subspace $\widehat{\Phi}^*(\mathfrak{B}(\mathcal{H}_E))$ is precisely the *noncommutative graph* $\mathcal{G}(\Phi)$ of the channel Φ which is defined as the subspace of $\mathfrak{B}(\mathcal{H}_A)$ spanned by the family of operators $\{V_j^* V_k\}_{kj}$, where $\{V_k\}_k$ is the family of operators from the Kraus representation (3) of the channel Φ [10, Lemma 1].

Definition 1. [6] A subspace $\mathfrak{L} \subseteq \mathfrak{B}(\mathcal{H})$ is (topologically) *transitive* if for any vector $\varphi \in \mathcal{H}$ the set $\mathfrak{L}|\varphi \doteq \{A|\varphi \mid A \in \mathfrak{L}\}$ is dense in \mathcal{H} .

If \mathcal{H} is a finite-dimensional space then “is dense in” in the above definition may be replaced by “coincides with”.

The following lemma is our basic tool for studying the one-shot zero-error classical capacity.

Lemma 2. *A channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ has positive one-shot zero-error classical capacity if and only if the noncommutative graph $\mathcal{G}(\Phi) \doteq \widehat{\Phi}^*(\mathfrak{B}(\mathcal{H}_E))$ is not transitive.*

Proof. It is easy to check that a subspace \mathfrak{L} of $\mathfrak{B}(\mathcal{H})$ is transitive if and only if the subspace $\mathfrak{L}^\perp = \{A \in \mathfrak{T}(\mathcal{H}) \mid \text{Tr} AB = 0 \ \forall B \in \mathfrak{L}\}$ does not contain any 1-rank operator (this was first noticed by Azoff [1], see also [6]). Now the statement follows from Lemma 1. \square

The one-shot zero-error quantum capacity $\bar{Q}_0(\Phi)$ of a channel Φ can be defined as $\sup_{\mathcal{H} \in q_0(\Phi)} \log \dim \mathcal{H}$, where $q_0(\Phi)$ is the set of all subspaces \mathcal{H}_0 of \mathcal{H}_A on which the channel Φ is perfectly reversible (in the sense that there is a channel Ψ such that $\Psi(\Phi(\rho)) = \rho$ for all states ρ supported by \mathcal{H}_0 , see [12, Chap. 10]). The (asymptotic) zero-error quantum capacity is defined by regularization: $Q_0(\Phi) = \sup_n n^{-1} \bar{Q}_0(\Phi^{\otimes n})$ [3,4,8,10,19,22].

Hence the one-shot zero-error quantum capacity $\bar{Q}_0(\Phi)$ is positive if and only if there exists a nontrivial subspace \mathcal{H}_0 of \mathcal{H}_A such that the restriction of the channel $\widehat{\Phi}$ to the subset $\mathfrak{S}(\mathcal{H}_0)$ is completely depolarizing [12, Chap. 10], i.e. $\widehat{\Phi}(\rho_1) = \widehat{\Phi}(\rho_2)$ for all states ρ_1 and ρ_2 supported by \mathcal{H}_0 .

These arguments imply the following modification of Lemma 1 in [4].

Lemma 3. *A channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ has positive one-shot zero-error quantum capacity if and only if there are unit vectors φ and ψ in \mathcal{H}_A such that*

$$\widehat{\Phi}(|\varphi\rangle\langle\psi|) = 0 \quad \text{and} \quad \widehat{\Phi}(|\varphi\rangle\langle\varphi|) = \widehat{\Phi}(|\psi\rangle\langle\psi|) \tag{6}$$

or, equivalently,

$$\langle\psi|A|\varphi\rangle = 0 \quad \text{and} \quad \langle\varphi|A|\varphi\rangle = \langle\psi|A|\psi\rangle \quad \forall A \in \mathcal{G}(\Phi) \doteq \widehat{\Phi}^*(\mathfrak{B}(\mathcal{H}_E)). \tag{7}$$

Proof. It is easy to see that $\widehat{\Phi}(|\varphi\rangle\langle\psi|) = 0$ if and only if

$$\widehat{\Phi}(\rho) = \langle\varphi|\rho|\varphi\rangle \widehat{\Phi}(|\varphi\rangle\langle\varphi|) + \langle\psi|\rho|\psi\rangle \widehat{\Phi}(|\psi\rangle\langle\psi|)$$

for all states ρ supported by the subspace $\mathcal{H}_{\varphi,\psi}$ spanned by the vectors φ and ψ . Hence (6) holds if and only if the restriction of the channel $\widehat{\Phi}$ to the subset $\mathfrak{S}(\mathcal{H}_{\varphi,\psi})$ is completely depolarizing. \square

Lemmas 2 and 3 imply the following conditions for positivity of the one-shot classical and quantum zero-error capacities.

Proposition 1. *Let $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel and $\mathcal{G}(\Phi) \doteq \widehat{\Phi}^*(\mathfrak{B}(\mathcal{H}_E))$ its noncommutative graph. Then*

$$[\mathcal{G}(\Phi)]' \text{ is non-trivial } (\neq \{\lambda I\}) \implies \bar{C}_0(\Phi) > 0, \tag{8}$$

$$[\mathcal{G}(\Phi)]' \text{ is noncommutative} \implies \bar{Q}_0(\Phi) > 0. \tag{9}$$

If $\mathcal{G}(\Phi)$ is an algebra then “ \Leftrightarrow ” holds in the above implications.

Remark 1. In general “ \Leftrightarrow ” does not hold in (8) and (9). There exists a channel Φ with $\bar{Q}_0(\Phi) > 0$ for which $[\mathcal{G}(\Phi)]' = \{\lambda I\}$. Indeed, since the subspace of \mathfrak{M}_4 consisting of the matrices

$$\begin{bmatrix} \lambda I_2 & A \\ B & C \end{bmatrix}, \quad A, B, C \in \mathfrak{M}_2,$$

is symmetric and contains the unit matrix I_4 , Proposition 2 below (or Lemma 2 in [8]) shows that this subspace is the noncommutative graph of some channel Φ . It follows from Lemma 3 that $\bar{Q}_0(\Phi) > 0$, but it is easy to see that the commutant of this subspace is trivial.

Proof. If the algebra $[\mathcal{G}(\Phi)]'$ is non-trivial, then it contains a non-trivial projection P . Then $\mathcal{G}(\Phi)P(\mathcal{H}_A) \subseteq P(\mathcal{H}_A)$ and hence $\mathcal{G}(\Phi)$ is not transitive. The first implication follows now from Lemma 2.

If the algebra $[\mathcal{G}(\Phi)]'$ is noncommutative, then, by Lemma 4 below, there exists a partial isometry $W \in [\mathcal{G}(\Phi)]'$ such that the projections $P = W^*W$ and $Q = WW^*$ are orthogonal. Let $|\varphi\rangle$ be an arbitrary vector in $P(\mathcal{H}_A)$ and $|\psi\rangle = W|\varphi\rangle \in Q(\mathcal{H}_A)$. Then it is easy to see that (7) holds and by Lemma 3 the second implication follows.

By Lemma 2 $\bar{C}_0(\Phi) > 0$ implies the existence of a non-zero vector φ such that $\mathcal{H}_\varphi = \overline{\{A|\varphi\rangle, A \in \mathcal{G}(\Phi)\}} \neq \mathcal{H}_A$. If $\mathcal{G}(\Phi)$ is an algebra then \mathcal{H}_φ is an invariant subspace for $\mathcal{G}(\Phi)$. Since the algebra $\mathcal{G}(\Phi)$ is symmetric, it implies that the orthogonal projection onto \mathcal{H}_φ commutes with $\mathcal{G}(\Phi)$.

Suppose $\mathcal{G}(\Phi)$ is an algebra and $\bar{Q}_0(\Phi) > 0$. We will show that $[\mathcal{G}(\Phi)]'$ contains two orthogonal equivalent projections and hence is noncommutative. By Lemma 3 there are vectors φ and ψ in \mathcal{H}_A such that (7) holds. Let $\mathcal{H}_\varphi = \overline{\{A|\varphi\rangle \mid A \in \mathcal{G}(\Phi)\}}$ and $\mathcal{H}_\psi = \overline{\{A|\psi\rangle \mid A \in \mathcal{G}(\Phi)\}}$. It follows from (7) that $\mathcal{H}_\varphi \perp \mathcal{H}_\psi$ and that $\|A|\varphi\rangle\| = \|A|\psi\rangle\|$ for all $A \in \mathcal{G}(\Phi)$. Hence the operator W defined by the relations

$$WA|\varphi\rangle = A|\psi\rangle \quad \forall A \in \mathcal{G}(\Phi) \quad \text{and} \quad W|\phi\rangle = 0 \quad \forall \phi \in \mathcal{H}_\varphi^\perp$$

is a partial isometry for which \mathcal{H}_φ and \mathcal{H}_ψ are initial and final subspaces. Since these subspaces are invariant for all operators in $\mathcal{G}(\Phi)$, it is easy to see that $W \in [\mathcal{G}(\Phi)]'$. Thus, the algebra $[\mathcal{G}(\Phi)]'$ contains the orthogonal equivalent projections W^*W and WW^* (onto \mathcal{H}_φ and \mathcal{H}_ψ respectively). \square

Lemma 4. ³ *A von Neumann algebra \mathfrak{M} is noncommutative if and only if it contains two orthogonal equivalent projections.*⁴

³ We are grateful to Victor Shulman for this observation.

⁴ Two projections P and Q are said to be equivalent relative to a von Neumann algebra \mathfrak{M} when $P = W^*W$ and $Q = WW^*$ for some $W \in \mathfrak{M}$ [18, Definition 6.1.4].

Proof. If \mathfrak{M} is noncommutative then it contains a noncentral projection P . Let $\bar{P} = I - P$. By the Comparison Theorem [18, Theorem 6.2.7.] there exists a central projection E such that $PE \preceq \bar{P}E$ and $\bar{P}\bar{E} \preceq P\bar{E}$, where $\bar{E} = I - E$ and “ \preceq ” denotes the projection ordering (relative to \mathfrak{M}) [18]. Since P is noncentral, either $PE \neq 0$ or $\bar{P}\bar{E} \neq 0$ (otherwise $P = \bar{E}$).

If $PE \neq 0$ then PE is equivalent to some projection $Q \leq \bar{P}E$. It is clear that the projections PE and Q are orthogonal.

If $\bar{P}\bar{E} \neq 0$ then the similar arguments shows the existence of a projection $Q' \leq P\bar{E}$ equivalent to $\bar{P}\bar{E}$. \square

Example 1. An important class of channels for which “ \Leftrightarrow ” holds in (8) and in (9) consists of Bosonic Gaussian channels defined as follows.

Let \mathcal{H}_X ($X = A, B$) be the space of irreducible representation of the Canonical Commutation Relations (CCR)

$$W_X(z)W_X(z') = \exp\left(-\frac{i}{2} \Delta_X(z, z')\right) W_X(z' + z), \quad z, z' \in Z_X,$$

where (Z_X, Δ_X) is a symplectic space and $W_X(z)$ are the Weyl operators [2, 11],[12, Chap. 12]. Denote by s_X the number of modes of the system X , i.e. $2s_X = \dim Z_X$. A Bosonic Gaussian channel $\Phi_{K,l,\alpha} : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is defined via the action of its dual $\Phi_{K,l,\alpha}^* : \mathfrak{B}(\mathcal{H}_B) \rightarrow \mathfrak{B}(\mathcal{H}_A)$ on the Weyl operators:

$$\Phi_{K,l,\alpha}^*(W_B(z)) = W_A(Kz) \exp\left[ilz - \frac{1}{2}z^\top \alpha z\right], \quad z \in Z_B,$$

where $K : Z_B \rightarrow Z_A$ is a linear operator, l is a $2s_B$ -dimensional real row and α is a real symmetric $(2s_B) \times (2s_B)$ matrix satisfying the inequality $\alpha \geq \pm \frac{1}{2} [\Delta_B - K^\top \Delta_A K]$ [2, 11, 12].

Any Bosonic Gaussian channel $\Phi_{K,l,\alpha}$ is unitary equivalent to the channel $\Phi_{K,0,\alpha}$ for which Bosonic unitary dilation always exists [2, 12]. So, Lemma 2 in [23] shows that the noncommutative graph of the channel $\Phi_{K,0,\alpha}$ coincides with the algebra generated by the family $\{W_A(z)\}_{z \in [K(\ker \alpha)]^\complement}$ of Weyl operators in \mathcal{H}_A , where $[K(\ker \alpha)]^\complement$ is the skew-orthogonal complement to the subspace $K(\ker \alpha) \subseteq Z_A$. It follows that $[\mathcal{G}(\Phi_{K,0,\alpha})]' = [\{W_A(z)\}_{z \in K(\ker \alpha)}]''$.

Since $\ker K \cap \ker \alpha = \{0\}$ and $\Delta_A(Kz_1, Kz_2) = \Delta_B(z_1, z_2)$ for all z_1, z_2 in $\ker \alpha$ (see [12, Chap. 12] or [23, Lemma 2]), the algebra $[\{W_A(z)\}_{z \in K(\ker \alpha)}]''$ is nontrivial if and only if $\ker \alpha \neq \{0\}$ and it is noncommutative if and only if $\Delta_B|_{\ker \alpha} \neq 0$. Thus, Proposition 1 shows that

$$\begin{aligned} \{\bar{C}_0(\Phi_{K,l,\alpha}) > 0\} &\Leftrightarrow \{\ker \alpha \neq \{0\}\}, \\ \{\bar{Q}_0(\Phi_{K,l,\alpha}) > 0\} &\Leftrightarrow \{\exists z_1, z_2 \in \ker \alpha \text{ such that } \Delta_B(z_1, z_2) \neq 0\}. \end{aligned} \tag{10}$$

In fact, positivity of these capacities means that they are equal to $+\infty$.⁵

Since the tensor product of two Gaussian channels Φ_{K_1,l_1,α_1} and Φ_{K_2,l_2,α_2} is a Gaussian channel $\Phi_{K,l,\alpha}$ with $\alpha = \alpha_1 \oplus \alpha_2$, it is easy to see that equivalence relations (10) are valid for the asymptotic zero-error capacities as well, i.e. for $C_0(\Phi_{K,l,\alpha})$ and $Q_0(\Phi_{K,l,\alpha})$ instead of $\bar{C}_0(\Phi_{K,l,\alpha})$ and $\bar{Q}_0(\Phi_{K,l,\alpha})$.

⁵ This follows from the observations in [23, Section 4C].

3. Superactivation of One-Shot Zero-Error Capacities

3.1. *The case of zero-error classical capacities.* The superactivation of one-shot zero-error classical capacity means that

$$\bar{C}_0(\Phi_1) = \bar{C}_0(\Phi_2) = 0, \quad \text{but} \quad \bar{C}_0(\Phi_1 \otimes \Phi_2) > 0. \tag{11}$$

for some channels Φ_1 and Φ_2 . The existence of such channels was shown independently in [3, 8]. In particular, in [8] an example of two channels $\Phi_1 \neq \Phi_2$ having input dimension $\dim \mathcal{H}_A = 4$ such that (11) holds was constructed and it was mentioned that this is the minimal input dimension for which superactivation (11) may take place. Then by using these two channels and a direct sum construction a *symmetric* example of superactivation (i.e. (11) with $\Phi_1 = \Phi_2$) with input dimension $\dim \mathcal{H}_A = 8$ was obtained [8, Theorem 1]. In this section we will construct a symmetric example of superactivation (11) with the minimal input dimension $\dim \mathcal{H}_A = 4$ and the minimal Choi rank $\dim \mathcal{H}_E = 3$.

Since a subspace \mathcal{L} of the algebra \mathfrak{M}_n of $n \times n$ -matrices is a noncommutative graph of a particular channel if and only if

$$\mathcal{L} \text{ is symmetric } (\mathcal{L} = \mathcal{L}^*) \text{ and contains the unit matrix} \tag{12}$$

(see Lemma 2 in [8] and Proposition 2 below), Lemma 2 reduces the problem of finding channels for which (11) holds to the problem of finding transitive subspaces \mathcal{L}_1 and \mathcal{L}_2 satisfying (12) such that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is not transitive. It is this way that was used in [8] to construct the channels Φ_1 and Φ_2 mentioned above.

It is interesting that the non-preserving of transitivity under tensor product was known in the theory of operator subspaces: a transitive subspace $\mathcal{L}_0 \subset \mathfrak{M}_4$ such that $\mathcal{L}_0 \otimes \mathcal{L}_0$ is not transitive was constructed in [6, Example 3.10]. Moreover, the subspace $\mathcal{L}_0^\perp \doteq \{A \mid \text{Tr}AB = 0 \ \forall B \in \mathcal{L}_0\}$ in this example also has the same property. The above subspaces \mathcal{L}_0 and \mathcal{L}_0^\perp consist respectively of the matrices

$$\begin{bmatrix} a & b & h & 2g \\ c & d & f & e \\ e & f & a & b \\ g & h & c & d \end{bmatrix}, \quad \begin{bmatrix} a & b & -h & -g \\ c & d & -f & -e \\ e & f & -a & -b \\ g/2 & h & -c & -d \end{bmatrix}, \quad a, b, c, d, e, f, g, h \in \mathbb{C}.$$

This example does not give an example of superactivation of one-shot zero-error classical capacity, since the subspaces \mathcal{L}_0 and \mathcal{L}_0^\perp are not symmetric. Nevertheless, using a similar approach one can construct a symmetric example.

Theorem 1. *There exists a symmetric transitive subspace $\mathcal{L} \subseteq \mathfrak{M}_4$ with $\dim \mathcal{L} = 8$ containing the unit matrix such that $\mathcal{L} \otimes \mathcal{L}$ is not transitive.*

We will need two lemmas. The first one is similar to Lemma 2.1 in [6].

Lemma 5. *Let $\Phi : \mathfrak{M}_n \rightarrow \mathfrak{M}_n$ be a linear isomorphism with n^2 different eigenvalues and such that all eigenvectors of Φ^* have rank more than or equal to 2. Then the subspace*

$$\mathcal{L} = \left\{ \begin{bmatrix} A & \Phi(B) \\ B & A \end{bmatrix} \mid A, B \in \mathfrak{M}_n \right\}$$

is transitive.

Proof. Given $z_1, z_2, x, y \in \mathbb{C}^n$ with $\|x\|^2 + \|y\|^2 \neq 0$, we need to find A and B in \mathfrak{M}_n such that

$$\begin{bmatrix} A & \Phi(B) \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Case 1: $x, y \neq 0, x \neq \lambda y$. Take $B = 0, A$ such that $Ax = z_1, Ay = z_2$.

Case 2: $x = 0, y \neq 0$. Take A such that $Ay = z_2$ and B such that $\Phi(B)y = z_1$ (this is possible, since Φ is an isomorphism).

Case 3: $x \neq 0, y = 0$. It is similar to the case 2.

Case 4: $x, y \neq 0, x = \lambda y$. We need to find A, B such that

$$\lambda Ay + \Phi(B)y = z_1, \quad \lambda By + Ay = z_2.$$

Expressing Ay from the second equation and substituting into the first one, we get:

$$Ay = z_2 - \lambda By, \tag{13}$$

and $\lambda z_2 - \lambda^2 By + \Phi(B)y = z_1$, whence $(\Phi(B) - \lambda^2 B)y = z_1 - \lambda z_2$. It has a solution if $\text{Ran}(\Phi - \lambda^2)$ is transitive or, equivalently, $\text{Ker}(\Phi^* - \bar{\lambda}^2)$ does not contain a 1-rank operator. If λ^2 is not an eigenvalue of Φ then it holds. If it is an eigenvalue, then this kernel is a 1-dimensional subspace generated by a matrix of rank ≥ 2 , so it again holds. And now one finds A from (13). \square

Lemma 6. *Let \mathcal{L} be a subspace of \mathfrak{M}_n . The subspace $\mathcal{L} \otimes \mathcal{L}$ is transitive if and only if the subspace $\mathcal{L}A\mathcal{L}^\top \doteq \{\sum_i X_i AY_i^\top \mid X_i, Y_i \in \mathcal{L}\}$ coincides with \mathfrak{M}_n for each $A \in \mathfrak{M}_n$.*

Proof. We may identify $\mathbb{C}^n \otimes \mathbb{C}^n$ with \mathfrak{M}_n by the linear isomorphism $U : x \otimes y \mapsto x \cdot y^\top$ (we assume that x, y are columns).

There exists a linear isomorphism $\Lambda : \mathfrak{B}(\mathbb{C}^n \otimes \mathbb{C}^n) \rightarrow \mathfrak{B}(\mathfrak{M}_n)$ given by $\Lambda(T \otimes S) = L_T R_{S^\top}$ (left multiplication by T and right multiplication by S^\top), which agrees with U in the sense that

$$U[T \otimes S]z = \Lambda(T \otimes S)Uz \quad \forall z \in \mathbb{C}^n \otimes \mathbb{C}^n.$$

This implies the assertion of the lemma. \square

Proof of Theorem 1. Let

$$C_1 = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These matrices form an orthogonal basis in \mathfrak{M}_2 . Let $\lambda_1 = i, \lambda_2 = -i, \lambda_3 = 1, \lambda_4 = -1$. We define an unitary map $\Phi : \mathfrak{M}_2 \rightarrow \mathfrak{M}_2$ by $\Phi(C_i) = \lambda_i C_i$.

Let $\mathcal{L} = \left\{ \begin{bmatrix} A & \Phi(B) \\ B & A \end{bmatrix} \mid A, B \in \mathfrak{M}_2 \right\}$ be a subspace of \mathfrak{M}_4 . Since $\Phi \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} d & -c \\ b & a \end{bmatrix}$, the subspace \mathcal{L} consists of the matrices

$$\begin{bmatrix} a & b & h & -g \\ c & d & f & e \\ e & f & a & b \\ g & h & c & d \end{bmatrix}, \quad a, b, c, d, e, f, g, h \in \mathbb{C}.$$

It is clear that $\dim \mathcal{L} = \dim \mathfrak{M}_2 + \dim \mathfrak{M}_2 = 8$ and that the subspace \mathcal{L} is symmetric. Transitivity of \mathcal{L} follows from Lemma 5.

To prove that $\mathcal{L} \otimes \mathcal{L}$ is not transitive it suffices, by Lemma 6, to show that $\mathcal{L} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathcal{L}^\top \neq \mathfrak{M}_4$. We have

$$\begin{aligned} & \mathcal{L} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathcal{L}^\top \\ &= \left\{ \sum_i \begin{bmatrix} A_i^1 & \Phi(B_i^1) \\ B_i^1 & A_i^1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} A_i^2 & B_i^{2\top} \\ \Phi(B_i^2)^\top & A_i^2 \end{bmatrix} \mid A_i^{1,2}, B_i^{1,2} \in \mathfrak{M}_2 \right\} \\ &= \left\{ \sum_i \begin{bmatrix} A_i^1 & -\Phi(B_i^1) \\ B_i^1 & -A_i^1 \end{bmatrix} \begin{bmatrix} A_i^2 & B_i^{2\top} \\ \Phi(B_i^2)^\top & A_i^2 \end{bmatrix} \mid A_i^{1,2}, B_i^{1,2} \in \mathfrak{M}_2 \right\} \\ &= \left\{ \begin{bmatrix} \sum_i (A_i^1 A_i^2 - \Phi(B_i^1) \Phi(B_i^2)^\top) & \dots \\ \dots & \sum_i (B_i^1 B_i^{2\top} - A_i^1 A_i^2) \end{bmatrix} \mid A_i^{1,2}, B_i^{1,2} \in \mathfrak{M}_2 \right\} \end{aligned} \tag{14}$$

Let $B^1, B^2 \in \mathfrak{M}_2$. We can write them as $B^1 = \sum_i t_i C_i, B^2 = \sum_i s_i C_i$. Since $\text{Tr} C_i C_j^\top \neq 0$ only in the cases: a) $i = 1, j = 2$, b) $i = 2, j = 1$, c) $i = j = 3$, d) $i = j = 4$, we obtain

$$\begin{aligned} & \text{Tr}(B^1 B^{2\top} - \Phi(B^1) \Phi(B^2)^\top) \\ &= \text{Tr} \left(\sum_{i,j} t_i s_j C_i C_j^\top - \sum_{i,j} \lambda_i t_i \lambda_j s_j C_i C_j^\top \right) \\ &= \sum_{i,j} \text{Tr}(1 - \lambda_i \lambda_j) t_i s_j C_i C_j^\top \\ &= \text{Tr}(1 - \lambda_1 \lambda_2) t_1 s_2 C_1 C_2^\top + \text{Tr}(1 - \lambda_2 \lambda_1) t_2 s_1 C_2 C_1^\top \\ &\quad + \text{Tr}(1 - \lambda_3^2) t_3 s_3 C_3 C_3^\top + \text{Tr}(1 - \lambda_4^2) t_4 s_4 C_4 C_4^\top = 0. \end{aligned} \tag{15}$$

It follows from (14) and (15) that for any $T \in \mathcal{L} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathcal{L}^\top$ we have

$$\text{Tr}(T_{11} + T_{22}) = 0.$$

Thus $\mathcal{L} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathcal{L}^\top \neq \mathfrak{M}_4$. \square

To derive from Theorem 1 an example of superactivation of one-shot zero-error classical capacity with smallest possible dimension we need the following observation (which is a strengthened version of Lemma 2 in [8]).

Proposition 2. *Let \mathcal{L} be a subspace of $\mathfrak{M}_n, n \geq 2$, and m the minimal natural number such that $\dim \mathcal{L} \leq m^2$. The following statements are equivalent:*

- (i) \mathfrak{L} is symmetric ($\mathfrak{L}^* = \mathfrak{L}$) and contains the unit matrix;
- (ii) there exists an entanglement-breaking channel $\Psi : \mathfrak{M}_n \rightarrow \mathfrak{M}_m$ such that $\mathfrak{L} = \Psi^*(\mathfrak{M}_m)$ ($\Psi^* : \mathfrak{M}_m \rightarrow \mathfrak{M}_n$ is a dual map to the channel Ψ).
- (iii) there exists a pseudo-diagonal⁶ channel $\Phi : \mathfrak{M}_n \rightarrow \mathfrak{M}_{nm}$ with the Choi rank m such that $\mathfrak{L} = \mathcal{G}(\Phi)$ (the noncommutative graph of Φ).

Proof. (ii) \Rightarrow (i) is obvious.

(i) \Rightarrow (ii). We will show first that there is a basis $\{A_i\}_{i=1}^d$ of \mathfrak{L} with all A_i 's being positive such that $\sum_{i=1}^d A_i = I_n$ (the unit matrix in \mathfrak{M}_n). It is sufficient to show that such a basis exists in the real space $\mathfrak{L}_{sa} = \{A \in \mathfrak{L} \mid A = A^*\}$, since it will also be a basis for \mathfrak{L} over \mathbb{C} (by symmetricity of \mathfrak{L}). Since any ball generates the whole space, we can find a basis $I_n, \tilde{A}_2, \dots, \tilde{A}_d$ with all \tilde{A}_i belonging to a ball in \mathfrak{L}_{sa} with centrum I_n and of radius, say, $1/2$. Since for any $A = A^* \in \mathfrak{M}_n$, $\|I_n - A\| < 1$ implies that $A \geq 0$, we conclude that $\tilde{A}_i \geq 0$. Now let M be a sufficiently large number such that $I_n - \sum_{i=2}^d \tilde{A}_i/M \geq 0$. Let $A_1 = I_n - \sum_{i=2}^d \tilde{A}_i/M$. It is easy to see that $A_1, \tilde{A}_2, \dots, \tilde{A}_n$ form a basis and

$$I_n = A_1 + \sum_{i=2}^d \tilde{A}_i/M.$$

Now take $A_i = \tilde{A}_i/M, i = 2, \dots, d$.

Let $\{B_i\}_{i=1}^d, d = \dim \mathfrak{L}$, be a set of positive linearly independent matrices in \mathfrak{M}_m with unit trace. Consider the unital completely positive map

$$\mathfrak{M}_m \ni X \mapsto \Psi^*(X) = \sum_{i=1}^d [\text{Tr} B_i X] A_i \in \mathfrak{M}_n$$

Apparently $\text{Ran} \Psi^* \subseteq \mathfrak{L}$. To see that it is exactly \mathfrak{L} , we will show that each A_i is in the range. For that we just take any $X \in \mathfrak{M}_m$ such that $\text{Tr} B_j X = 0$ for all $j \neq i$ and $\text{Tr} B_i X \neq 0$, which exists since B_i 's are linearly independent.

Since the map Ψ^* has the Kraus representation consisting of 1-rank operators, the predual map $\Psi : \mathfrak{M}_n \rightarrow \mathfrak{M}_m$ is an entanglement-breaking quantum channel.

(ii) \Leftrightarrow (iii) It suffices to note that a pseudo-diagonal channel is complementary to an entanglement-breaking channel and vice versa [5]. \square

The proof of Proposition 2 can be used to obtain an explicit formula for a channel Φ with given noncommutative graph.

Corollary 1. *Let \mathfrak{L} be a subspace of $\mathfrak{M}_n, n \geq 2$, satisfying (12) and m the minimal natural number such that $d = \dim \mathfrak{L} \leq m^2$. There is a pseudo-diagonal channel Φ with*

⁶ A channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called pseudo-diagonal if it has the representation

$$\Phi(\rho) = \sum_{i,j} c_{ij} \langle \psi_i | \rho | \psi_j \rangle |i\rangle \langle j|, \quad \rho \in \mathfrak{T}(\mathcal{H}_A),$$

where $\{c_{ij}\}$ is a Gram matrix of a collection of unit vectors, $\{|\psi_i\rangle\}$ is a collection of vectors in \mathcal{H}_A such that $\sum_i |\psi_i\rangle \langle \psi_i| = I_{\mathcal{H}_A}$ and $\{|i\rangle\}$ is an orthonormal basis in \mathcal{H}_B [5].

$\dim \mathcal{H}_A = n$, $\dim \mathcal{H}_E = m$ and $\dim \mathcal{H}_B \leq mn$ such that $\mathcal{G}(\Phi) = \mathcal{L}$ represented as follows

$$\mathfrak{M}_n \ni \rho \mapsto \Phi(\rho) = \sum_{i,j=1}^d c_{ij} A_i^{1/2} \rho A_j^{1/2} \otimes |i\rangle\langle j| \in \mathfrak{M}_n \otimes \mathfrak{M}_d, \tag{16}$$

where $\{A_i\}_{i=1}^d$ is a basis of \mathcal{L} such that $\sum_{i=1}^d A_i = I_n$ and $A_i \geq 0$ for all i , $\{c_{ij}\}$ is the Gram matrix of a set $\{|\psi_i\rangle\}_{i=1}^d$ of unit vectors in \mathbb{C}^m such that the set $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^d$ is linearly independent and $\{|i\rangle\}$ is the canonical basis in \mathbb{C}^d .

By representation (16) the channel Φ maps a state $\rho \in \mathfrak{M}_n$ into the $d \times d$ matrix $\left[c_{ij} A_i^{1/2} \rho A_j^{1/2} \right]$ with entries in \mathfrak{M}_n . Its formal output dimension nd may be greater than mn , but the real output dimension is $\leq mn$ (since Φ is complementary to a channel from \mathfrak{M}_n into \mathfrak{M}_m , see the proof). If $d > m$ this means that all the states $\Phi(\rho)$ in (16) are supported by a proper subspace $\mathcal{H}_0 \subset \mathbb{C}^n \otimes \mathbb{C}^d$ such that $\dim \mathcal{H}_0 \leq mn$.

Proof. The proof of Proposition 2 shows that a channel Φ with the stated properties can be constructed as the complementary channel to the channel

$$\Psi(\rho) = \sum_{i=1}^d [\text{Tr} A_i \rho] B_i,$$

where $\{A_i\} \subset \mathfrak{M}_n$ is a basis of \mathcal{L} determined in that proof and $\{B_i\} \subset \mathfrak{M}_m$ is any linearly independent set of positive matrices with unit trace. We may assume that $B_i = |\psi_i\rangle\langle\psi_i|$ for all $i = \overline{1, d}$, where $\{|\psi_i\rangle\}_{i=1}^d$ is a set of unit vectors in \mathbb{C}^m such that the set $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^d$ is linearly independent. Consider the linear operator

$$V : |\varphi\rangle \mapsto \sum_{i=1}^d A_i^{1/2} |\varphi\rangle \otimes |i\rangle \otimes |\psi_i\rangle$$

from \mathbb{C}^n into $\mathbb{C}^n \otimes \mathbb{C}^d \otimes \mathbb{C}^m$, where $\{|i\rangle\}$ is the canonical basis in \mathbb{C}^d .

Since $\sum_{i=1}^d A_i = I_n$ and $\|\psi_i\| = 1$ for all i , V is an isometry. It is easy to see that

$$\text{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^d} V |\varphi\rangle\langle\varphi| V^* = \sum_{i=1}^d [\text{Tr} A_i |\varphi\rangle\langle\varphi|] |\psi_i\rangle\langle\psi_i|, \quad \varphi \in \mathbb{C}^n.$$

So, $\Psi(\rho) = \text{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^d} V \rho V^*$ and hence

$$\Phi(\rho) = \widehat{\Psi}(\rho) = \text{Tr}_{\mathbb{C}^m} V \rho V^* = \sum_{i,j=1}^d \langle\psi_j|\psi_i\rangle A_i^{1/2} \rho A_j^{1/2} \otimes |i\rangle\langle j|, \quad \rho \in \mathfrak{M}_n.$$

□

Using the subspace \mathcal{L} from Theorem 1 and applying Proposition 2 we obtain the following corollary.

Corollary 2. *There is a pseudo-diagonal channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ with $\dim \mathcal{H}_A = 4$, $\dim \mathcal{H}_E = 3$ and $\dim \mathcal{H}_B \leq 12$, for which the following symmetric form of superactivation of one-shot zero-error classical capacity holds:*

$$\bar{C}_0(\Phi) = 0, \quad \text{but} \quad \bar{C}_0(\Phi \otimes \Phi) > 0. \tag{17}$$

By finding a basis $\{A_i\}_{i=1}^8$ of \mathfrak{L} such that $\sum_{i=1}^8 A_i = I_4$ and $A_i \geq 0$ for all i and applying Corollary 1 one can obtain an explicit expression for a channel Φ having the properties stated in Corollary 2.

In [8, Theorem 1] the same statement was established with $\dim \mathcal{H}_A = 8$ and it was mentioned that (17) does not hold for any channel Φ with $\dim \mathcal{H}_A < 4$. So, Corollary 2 gives a symmetric example of superactivation of one-shot zero-error classical capacity with *minimal input dimension* $\dim \mathcal{H}_A$ and *minimal Choi rank* $\dim \mathcal{H}_E$. Minimality of $\dim \mathcal{H}_E = 3$ follows from the fact that any transitive subspace of \mathfrak{M}_4 has dimension ≥ 7 [6].

3.2. The extreme form of superactivation. According to the notations in [4], the extreme form of superactivation of one-shot zero-error capacity means the existence of two channels Φ_1 and Φ_2 such that

$$\bar{C}_0(\Phi_1) = \bar{C}_0(\Phi_2) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi_1 \otimes \Phi_2) > 0. \tag{18}$$

Since \bar{Q}_0 is less than or equal to \bar{C}_0 , the channels Φ_1 and Φ_2 demonstrate superactivation of both classical and quantum one-shot zero-error capacities simultaneously, i.e. (11) and

$$\bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi_1 \otimes \Phi_2) > 0. \tag{19}$$

In [4] a very sophisticated method is used to show the existence of two channels Φ_1 and Φ_2 of sufficiently high dimensions ($\dim \mathcal{H}_A = 48$, $\dim \mathcal{H}_E = 1140$, $\dim \mathcal{H}_B = 54720$) for which the extreme form of superactivation of asymptotic zero-error capacity holds (which means validity of (18) with \bar{C}_0 and \bar{Q}_0 replaced by C_0 and Q_0).

This result directly implies the existence of two channels Φ_1 and Φ_2 for which (18) holds, but it neither gives an explicit form of these channels, nor says anything about their minimal dimensions.

We want to fill this gap and present a low-dimensional example of such channels expressed in terms of their noncommutative graphs.

By Lemmas 2 and 3 (with Proposition 2) the problem of finding channels for which (18) holds is reduced to the problem of finding transitive subspaces $\mathfrak{L}_1 \subset \mathfrak{M}_{n_1}$ and $\mathfrak{L}_2 \subset \mathfrak{M}_{n_2}$ satisfying (12) such that

$$\langle \psi | A | \varphi \rangle = 0 \quad \text{and} \quad \langle \varphi | A | \varphi \rangle = \langle \psi | A | \psi \rangle \quad \forall A \in \mathfrak{L}_1 \otimes \mathfrak{L}_2 \tag{20}$$

for some unit vectors φ and ψ in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$.

Let $A \mapsto \hat{A}$ be the linear isomorphism of \mathfrak{M}_4 corresponding to the Schur multiplication by the matrix

$$T = [t_{ij}] = \begin{bmatrix} 1 & 1 & -i & -i \\ 1 & 1 & -i & -i \\ +i & +i & 1 & 1 \\ +i & +i & 1 & 1 \end{bmatrix},$$

i.e. $\{\hat{a}_{ij}\} = \{a_{ij}t_{ij}\}$, and \mathcal{L}_0 the subspace of \mathfrak{M}_4 constructed in Example 3.10 in [6] (\mathcal{L}_0 and \mathcal{L}_0^\perp are described in Subsect. 3.1). Consider the subspaces

$$\mathcal{L}_1 = \left\{ M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & \widehat{A}_1 \end{bmatrix}, A_1 \in \mathfrak{M}, B_1, C_1^* \in \mathcal{L}_0^\perp \right\}$$

and

$$\mathcal{L}_2 = \left\{ M_2 = \begin{bmatrix} \widehat{A}_2 & B_2 \\ C_2 & A_2 \end{bmatrix}, A_2 \in \mathfrak{M}, B_2, C_2^* \in \mathcal{L}_0^\perp \right\},$$

of \mathfrak{M}_8 , where \mathfrak{M} is a subspace of \mathfrak{M}_4 having the properties stated in Lemma 7 below. Since $\dim \mathcal{L}_0^\perp = 8$, $\dim \mathcal{L}_1 = \dim \mathcal{L}_2 = 8 + 8 + 7 = 23$.

Since $[\widehat{A}]^* = \widehat{[A]^*}$ and $\widehat{I}_4 = I_4$, the subspaces \mathcal{L}_1 and \mathcal{L}_2 are symmetric and contain the unit matrix I_8 . It is easy to see that they are transitive (since $\mathcal{L}_0^\perp, [\mathcal{L}_0^\perp]^*, \mathfrak{M}$ and $\widehat{\mathfrak{M}} \doteq \{\widehat{A} \mid A \in \mathfrak{M}\}$ are transitive subspaces of \mathfrak{M}_4).

Theorem 2. *There exist unit vectors φ and ψ in $\mathbb{C}^8 \otimes \mathbb{C}^8$ such that (20) holds for the above transitive subspaces \mathcal{L}_1 and \mathcal{L}_2 of \mathfrak{M}_8 .*

Proof. We have to show the existence of two orthogonal unit vectors φ, ψ in $[\mathbb{C}^4 \oplus \mathbb{C}^4] \otimes [\mathbb{C}^4 \oplus \mathbb{C}^4]$ such that

$$\langle \psi \mid M_1 \otimes M_2 \mid \varphi \rangle = 0 \quad \forall M_1 \in \mathcal{L}_1, M_2 \in \mathcal{L}_2 \tag{21}$$

and

$$\langle \psi \mid M_1 \otimes M_2 \mid \psi \rangle = \langle \varphi \mid M_1 \otimes M_2 \mid \varphi \rangle \quad \forall M_1 \in \mathcal{L}_1, M_2 \in \mathcal{L}_2. \tag{22}$$

Let $|u\rangle = \sum_{i=1}^4 |x_i\rangle \otimes |y_i\rangle$ and $|v\rangle = \sum_{i=1}^4 s_i |x_i\rangle \otimes |y_i\rangle$ be the vectors in $\mathbb{C}^4 \otimes \mathbb{C}^4$, where $|x_i\rangle = |e_i\rangle, |y_i\rangle = |e_{5-i}\rangle$ ($\{|e_i\rangle\}$ is the canonical basis in \mathbb{C}^4) and $s_1 = s_2 = 1, s_3 = s_4 = -1$. It is shown in [6] that $|u\rangle\langle v| \in [\mathcal{L}_0^\perp \otimes \mathcal{L}_0^\perp]^\perp$, which means that

$$0 = \langle v \mid B_1 \otimes B_2 \mid u \rangle = \sum_{i,j=1}^4 s_i \langle x_i \otimes y_i \mid B_1 \otimes B_2 \mid x_j \otimes y_j \rangle \quad \forall B_1, B_2 \in \mathcal{L}_0^\perp. \tag{23}$$

Let $|\varphi\rangle = \frac{1}{2} \sum_{i=1}^4 |0, x_i\rangle \otimes |0, y_i\rangle$ and $|\psi\rangle = \frac{1}{2} \sum_{i=1}^4 s_i |x_i, 0\rangle \otimes |y_i, 0\rangle$. Then we have

$$M_1 \otimes M_2 \mid \varphi \rangle = \frac{1}{2} \sum_{i=1}^4 |B_1 x_i, \widehat{A}_1 x_i\rangle \otimes |B_2 y_i, A_2 y_i\rangle \tag{24}$$

and hence

$$\begin{aligned} \langle \psi \mid M_1 \otimes M_2 \mid \varphi \rangle &= \frac{1}{4} \sum_{i,j=1}^4 s_i \langle x_i, 0 \mid \otimes \langle y_i, 0 \mid \cdot |B_1 x_j, \widehat{A}_1 x_j\rangle \otimes |B_2 y_j, A_2 y_j\rangle \\ &= \frac{1}{4} \sum_{i,j=1}^4 s_i \langle x_i \mid B_1 \mid x_j \rangle \langle y_i \mid B_2 \mid y_j \rangle = 0, \end{aligned}$$

where the last equality follows from (23). Thus (21) is valid. It follows from (24) that

$$\begin{aligned} \langle \varphi | M_1 \otimes M_2 | \varphi \rangle &= \frac{1}{4} \sum_{i,j=1}^4 \langle 0, x_i | \otimes \langle 0, y_i | \cdot | B_1 x_j, \widehat{A}_1 x_j \rangle \otimes | B_2 y_j, A_2 y_j \rangle \\ &= \frac{1}{4} \sum_{i,j=1}^4 \langle x_i | \widehat{A}_1 | x_j \rangle \langle y_i | A_2 | y_j \rangle = \frac{1}{4} \sum_{i,j=1}^4 t_{ij} a_{ij}^1 a_{k(i)k(j)}^2, \quad k(i) = 5 - i, \end{aligned} \tag{25}$$

where a_{ij}^n are elements of the matrix A_n , $n = 1, 2$. Since

$$M_1 \otimes M_2 | \psi \rangle = \frac{1}{2} \sum_{i=1}^4 s_i | A_1 x_i, C_1 x_i \rangle \otimes | \widehat{A}_2 y_i, C_2 y_i \rangle,$$

we have

$$\begin{aligned} \langle \psi | M_1 \otimes M_2 | \psi \rangle &= \frac{1}{4} \sum_{i,j=1}^4 s_i s_j \langle x_i, 0 | \otimes \langle y_i, 0 | \cdot | A_1 x_j, C_1 x_j \rangle \otimes | \widehat{A}_2 y_j, C_2 y_j \rangle \\ &= \frac{1}{4} \sum_{i,j=1}^4 s_i s_j \langle x_i | A_1 | x_j \rangle \langle y_i | \widehat{A}_2 | y_j \rangle \\ &= \frac{1}{4} \sum_{i,j=1}^4 s_i s_j t_{k(i)k(j)} a_{ij}^1 a_{k(i)k(j)}^2, \quad k(i) = 5 - i. \end{aligned}$$

The right hand side of this equality coincides with the right hand side of (25), since it is easy to verify that $t_{ij} = s_i s_j t_{k(i)k(j)}$. Hence (22) is valid. \square

Lemma 7. *There exists a transitive subspace \mathfrak{M} of \mathfrak{M}_4 with $\dim \mathfrak{M} = 7$ satisfying (12) such that the subspace $\mathfrak{M} \doteq \{ \widehat{A} \mid A \in \mathfrak{M} \}$, where $A \mapsto \widehat{A}$ is the above-defined isomorphism, is transitive (and satisfies (12)).*

Proof. The proof below is essentially based on the arguments from the proof of Theorem 1.2 in [6].

Consider the subspace $\mathfrak{N} \subset \mathfrak{M}_4$ consisting of the matrices

$$\begin{bmatrix} a + b + c & f + g & i & 0 \\ d + e & -a & 2f + g & i \\ h & 2d + e & -b & 3f + g \\ 0 & h & 3d + e & -c \end{bmatrix},$$

where $a, b, c, d, e, f, g, h, i$ are complex numbers.

This subspace does not contain 1-rank matrices. Indeed, a non-zero matrix N of \mathfrak{N} is non-zero on some diagonal. Consider the square submatrix containing the shortest non-zero diagonal of N as its main diagonal. This submatrix is triangular, and hence its rank is not less than the rank of its diagonal, which is at least 2. Hence $\text{rank } N \geq 2$.

Let $\mathfrak{M} = \mathfrak{N}^\perp \doteq \{ A \mid \text{Tr } AB = 0 \ \forall B \in \mathfrak{N} \}$. Since the subspace \mathfrak{N} is symmetric and consists of traceless matrices of rank $\neq 1$, \mathfrak{M} is a symmetric transitive subspace containing the unit matrix. Since $\dim \mathfrak{N} = 9$, $\dim \mathfrak{M} = 16 - 9 = 7$.

To complete the proof it suffices to show that the subspace $\widehat{\mathfrak{M}}$ is transitive. This can be done by checking that $\text{Tr} \widehat{A} \widehat{B} = \text{Tr} AB$ for any $A, B \in \mathfrak{M}_4$, which implies $\widehat{\mathfrak{M}} = [\widehat{\mathfrak{N}}]^\perp$, and by verifying that the subspace $\widehat{\mathfrak{N}}$ does not contain 1-rank matrices (in the same way as for \mathfrak{N}). \square

Theorem 2 and Proposition 2 immediately imply the following result.

Corollary 3. *There exists a pair of pseudo-diagonal channels $\Phi_i : \mathfrak{T}(\mathcal{H}_{A_i}) \rightarrow \mathfrak{T}(\mathcal{H}_{B_i})$ with $\dim \mathcal{H}_{A_i} = 8$, $\dim \mathcal{H}_{E_i} = 5$ and $\dim \mathcal{H}_{B_i} \leq 40$, $i = 1, 2$, for which extreme superactivation (18) holds.*

By using Corollary 1 one can obtain explicit expressions for channels Φ_1 and Φ_2 having the properties stated in Corollary 3.

Since the subspaces \mathcal{L}_1 and \mathcal{L}_2 are not unitary equivalent, the above example of extreme superactivation is essentially nonsymmetric: $\Phi_1 \neq \Phi_2$. But they can be used to construct a symmetric example by applying the direct sum construction (see the proof of Theorem 1 in [8]).

Corollary 4. *There exists a quantum channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ with $\dim \mathcal{H}_A = 16$, $\dim \mathcal{H}_E = 10$ and $\dim \mathcal{H}_B \leq 40$, for which the following symmetric form of the extreme superactivation holds:*

$$\bar{C}_0(\Phi) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi \otimes \Phi) > 0.$$

This means that the channel Φ has vanishing one-shot classical zero-error capacity but positive two-shot quantum zero-error capacity.

4. On Channels Which Cannot be Superactivated

Park and Lee showed in [22] that superactivation of one-shot zero-error classical capacity (11) does not hold if either Φ_1 or Φ_2 is a qubit channel.⁷ Now we will show how to substantially extend this observation by using some results from [6] and [20], in particular, the following lemma (which is a reformulation of Corollary 6.13 in [6]).

Lemma 8. *Let \mathcal{L}_1 be a transitive subspace of $\mathfrak{B}(\mathcal{H}_1)$ which is contained in the weak-operator-topology closed linear span of its 1-rank elements. Then the spatial tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ is a transitive subspace of $\mathfrak{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ for any transitive subspace \mathcal{L}_2 of $\mathfrak{B}(\mathcal{H}_2)$.*

This observation is a strengthened infinite-dimensional version of the well known fact that the tensor product of any two unextendible product base is an unextendible product base [7].

Proposition 3. *Superactivation (11) of one-shot zero-error classical capacity does not hold for two channels $\Phi_i : \mathfrak{T}(\mathcal{H}_{A_i}) \rightarrow \mathfrak{T}(\mathcal{H}_{B_i})$, $i = 1, 2$ if the channel Φ_1 satisfies one of the following conditions (in which $\mathcal{G}(\Phi_1) \doteq \widehat{\Phi}_1^*(\mathfrak{B}(\mathcal{H}_{E_1}))$ is the non-commutative graph of Φ_1):*

(A) $\dim \mathcal{G}(\Phi_1) \geq [\dim \mathcal{H}_{A_1}]^2 - 1$ ($\dim \mathcal{H}_{A_1} < +\infty$);

⁷ In fact, one can prove that superactivation of one-shot zero-error classical capacity (11) does not hold if either Φ_1 or Φ_2 has input dimension ≤ 3 [9].

- (B) $\dim \mathcal{H}_{A_1} = 2$, in particular, Φ_1 is a qubit channel;
- (C) $\mathcal{G}(\Phi_1)$ is an algebra;
- (D) Φ_1 is a Bosonic Gaussian channel (described in Example 1);
- (E) Φ_1 is a finite-dimensional entanglement-breaking channel;
- (F) Φ_1 is an entanglement-breaking channel having Kraus representation (3) such that $\text{rank } V_k = 1$ for all k ,⁸

and the channel Φ_2 is arbitrary.

Proof. (A) If $\mathcal{G}(\Phi_1) = \mathfrak{B}(\mathcal{H}_{A_1})$ then this assertion follows from assertion C. If $\dim \mathcal{G}(\Phi_1) = [\dim \mathcal{H}_{A_1}]^2 - 1$ then $\dim \ker \widehat{\Phi}_1 = 1$. If the one-shot zero-error classical capacity of the channel Φ_1 is zero then, by Lemma 1, the minimal rank of all nonzero operators in $\ker \widehat{\Phi}_1$ is not less than 2. By [20, Theorem 1.1] this implies that the subspace $\ker \widehat{\Phi}_1$ is reflexive, which means that $\mathcal{G}(\Phi_1) = [\ker \widehat{\Phi}_1]^\perp$ is spanned by its one rank elements [20, Claim 3.1].

If Φ_2 is an arbitrary channel with zero one-shot zero-error classical capacity then $\mathcal{G}(\Phi_2)$ is a transitive subspace (by Lemma 2). Lemma 8 shows that $\mathcal{G}(\Phi_1 \otimes \Phi_2) = \mathcal{G}(\Phi_1) \otimes \mathcal{G}(\Phi_2)$ is a transitive subspace and hence the one-shot zero-error classical capacity of the channel $\Phi_1 \otimes \Phi_2$ is zero (by Lemma 2).

- (B) If $\dim \mathcal{H}_{A_1} = 2$ and $\bar{C}_0(\Phi_1) = 0$ then, by Lemma 1, all nonzero operators in $\ker \widehat{\Phi}_1$ have rank = 2, i.e they are invertible. This implies that $\dim \ker \widehat{\Phi}_1 \leq 1$. Indeed, if T, S are invertible operators in $\ker \widehat{\Phi}_1$ and λ is an eigenvalue of the operator TS^{-1} then

$$T - \lambda S = (TS^{-1} - \lambda)S$$

is a non-invertible operator in $\ker \widehat{\Phi}_1$ and hence $T = \lambda S$. So, this assertion follows from the previous one.

- (C) If $\mathcal{G}(\Phi_1)$ is an algebra and $\bar{C}_0(\Phi_1) = 0$ then Proposition 1 and basic results of the von Neumann algebras theory (cf.[18]) imply that $\mathcal{G}(\Phi_1)$ is dense in $\mathfrak{B}(\mathcal{H}_{A_1})$ in the weak-operator topology. Hence to prove that $\bar{C}_0(\Phi_1 \otimes \Phi_2) = 0$ for any channel Φ_2 with $\bar{C}_0(\Phi_2) = 0$ it suffices, by Lemma 2, to show transitivity of the subspace $\mathfrak{B}(\mathcal{H}_{A_1}) \otimes \mathcal{L}$ for any transitive subspace \mathcal{L} of $\mathfrak{B}(\mathcal{H}_{A_2})$.

This assertion is obvious if $n = \dim \mathcal{H}_{A_1} < +\infty$, since in this case the subspace $\mathfrak{B}(\mathcal{H}_{A_1}) \otimes \mathcal{L}$ can be identified with the subspace of all $n \times n$ matrices with entries in \mathcal{L} (considered as operators in $\bigoplus_{k=1}^n \mathcal{H}_k$, where \mathcal{H}_k is a copy of \mathcal{H}_{A_2} for all k). Assume that $\dim \mathcal{H}_{A_1} = +\infty$ and there is a vector $|\varphi\rangle = \sum_{i=1}^{+\infty} c_i |e_i\rangle \otimes |f_i\rangle$ in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ (where $c_1 \neq 0$, $\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ are orthonormal base in \mathcal{H}_{A_1} and in \mathcal{H}_{A_2}) such that all the vectors $C|\varphi\rangle$, $C \in \mathfrak{B}(\mathcal{H}_{A_1}) \otimes \mathcal{L}$, belong to a proper subspace \mathcal{K} of $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$. Let \mathcal{H}_n be the subspace of \mathcal{H}_{A_1} spanned by the vectors $|e_1\rangle, \dots, |e_n\rangle$ and $|\varphi_n\rangle = \sum_{i=1}^n c_i |e_i\rangle \otimes |f_i\rangle$. By the above observation the set $\{C|\varphi_n\rangle \mid C \in \mathfrak{B}(\mathcal{H}_n) \otimes \mathcal{L}\}$ is dense in $\mathcal{H}_n \otimes \mathcal{H}_{A_2}$. But it is easy to see that

$$C|\varphi_n\rangle = C|\varphi\rangle$$

for any $C \in \mathfrak{B}(\mathcal{H}_n) \otimes \mathcal{L}$. Since $\mathfrak{B}(\mathcal{H}_n) \otimes \mathcal{L} \subseteq \mathfrak{B}(\mathcal{H}_{A_1}) \otimes \mathcal{L}$ this implies $\mathcal{H}_n \otimes \mathcal{H}_{A_2} \subseteq \mathcal{K}$ for any n , that is a contradiction.

- (D) This assertion follows from the previous one, since the noncommutative graph of a Bosonic Gaussian channel is an algebra (see Example 1).

⁸ This means that $\Phi \otimes \text{Id}_{\mathcal{K}}(\omega)$ is a countably-decomposable separable state in $\mathfrak{S}(\mathcal{H}_B \otimes \mathcal{K})$ for any state $\omega \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{K})$, see Remark 2 below.

- (E) If Φ_1 is a finite-dimensional entanglement-breaking channel then it has Kraus representation (3) such that $\text{rank } V_k = 1$ for all k [15]. So, this assertion follows from assertion F.
- (F) In this case the noncommutative graph $\mathcal{G}(\Phi_1) \doteq \widehat{\Phi}_1^*(\mathfrak{B}(\mathcal{H}_{E_1}))$ is spanned by the 1-rank operators $V_k^* V_l$ (this follows from expression (4)). So, this assertion follows from Lemmas 2 and 8. \square

Proposition 3 directly implies the following two observations.

Corollary 5. *If a quantum channel Φ satisfies one of conditions A–F from Proposition 3 then $C_0(\Phi) = 0$ if and only if $\bar{C}_0(\Phi) = 0$.*

Corollary 6. *Superactivation of asymptotic classical zero-error capacity (property (11) with \bar{C}_0 replaced by C_0) does not hold for channels Φ_1 and Φ_2 , if Φ_1 satisfies one of conditions A–F from Proposition 3 and Φ_2 is arbitrary.*

Remark 2. The question about validity of the assertions of Proposition 3 and Corollaries 5–6 for arbitrary infinite-dimensional entanglement-breaking channel Φ_1 remains open, since the existence of countably nondecomposable separable states in an infinite-dimensional bipartite quantum system implies the existence of entanglement-breaking channels which don't have Kraus representation (3) with 1-rank operators V_k [14].

Proposition 4. *Superactivation (19) of one-shot zero-error quantum capacity does not hold for two channels $\Phi_i : \mathfrak{T}(\mathcal{H}_{A_i}) \rightarrow \mathfrak{T}(\mathcal{H}_{B_i})$, $i = 1, 2$ if one of the following conditions holds (in which $\mathcal{G}(\Phi_i) \doteq \widehat{\Phi}_i^*(\mathfrak{B}(\mathcal{H}_{E_i}))$ is the noncommutative graph of Φ_i):*

- (A) $\mathcal{G}(\Phi_1)$ contains a maximal commutative $*$ -subalgebra of \mathfrak{M}_{n_1} , where $n_1 = \dim \mathcal{H}_{A_1} < +\infty$, and Φ_2 is an arbitrary channel;
- (B) $\dim \mathcal{H}_{A_1} = 2$ (in particular, when Φ_1 is a qubit channel) and Φ_2 is an arbitrary channel;
- (C) $\mathcal{G}(\Phi_1)$ and $\mathcal{G}(\Phi_2)$ are algebras;
- (D) Φ_1 and Φ_2 are Bosonic Gaussian channels (described in Example 1).

Proof. (A) Since a maximal commutative $*$ -subalgebra of \mathfrak{M}_{n_1} consists of all matrices which are diagonal with respect to some orthonormal basis, the noncommutative graph $\mathcal{G}(\Phi_1 \otimes \Phi_2)$ contains the subspace of all block-diagonal matrices of the form $\text{diag}(a_1 A, \dots, a_{n_1} A)$, where $a_1, \dots, a_{n_1} \in \mathbb{C}$ and $A \in \mathcal{G}(\Phi_2)$. So, the assumption $\bar{Q}_0(\Phi_1 \otimes \Phi_2) > 0$ implies, by Lemma 3, the existence of unit vectors $|\varphi\rangle = (x_1, \dots, x_{n_1})$ and $|\psi\rangle = (y_1, \dots, y_{n_1})$, where $x_k, y_k \in \mathcal{H}_{A_2}$, such that

$$\sum_{k=1}^{n_1} a_k \langle y_k | A | x_k \rangle = 0 \quad \text{and} \quad \sum_{k=1}^{n_1} a_k \langle x_k | A | x_k \rangle = \sum_{k=1}^{n_1} a_k \langle y_k | A | y_k \rangle$$

for all $a_1, \dots, a_{n_1} \in \mathbb{C}$ and all $A \in \mathcal{G}(\Phi_2)$. It follows that

$$\langle y_k | A | x_k \rangle = 0 \quad \text{and} \quad \langle x_k | A | x_k \rangle = \langle y_k | A | y_k \rangle \tag{26}$$

for all k and all $A \in \mathcal{G}(\Phi_2)$. Since $\mathcal{G}(\Phi_2)$ contains the identity operator, (26) shows that $\|x_k\| = \|y_k\|$ for all k and hence there exists k_0 such that $\|x_{k_0}\| = \|y_{k_0}\| \neq 0$. Thus, (26) with $k = k_0$ implies, by Lemma 3, that $\bar{Q}_0(\Phi_2) > 0$.

(B) follows from assertion A, since the noncommutative graph $\mathcal{G}(\Phi_1)$ of any non-reversible channel Φ_1 with $\dim \mathcal{H}_{A_1} = 2$ contains a maximal commutative $*$ -subalgebra

of \mathfrak{M}_2 . Indeed, since $\mathcal{G}(\Phi_1)$ contains an operator $T \neq \lambda I_2$, it contains a self-adjoint operator $T' \neq \lambda I_2$ which is diagonal in a particular basis. The operators T' and I_2 generate a maximal commutative $*$ -subalgebra of \mathfrak{M}_2 contained in $\mathcal{G}(\Phi_1)$.

(C) follows from Proposition 1, since

$$[\mathcal{G}(\Phi_1 \otimes \Phi_2)]' = [\mathcal{G}(\Phi_1)]' \bar{\otimes} [\mathcal{G}(\Phi_2)]',$$

where $\bar{\otimes}$ denotes a tensor product of von Neumann algebras [18, Chap. 10].

(D) This assertion follows from the previous one, since the noncommutative graph of a Bosonic Gaussian channel is an algebra (see Example 1). \square

Proposition 4 and its proof imply the following two observations.

Corollary 7. *If a quantum channel Φ satisfies one of conditions A–D from Proposition 4 then $Q_0(\Phi) = 0$ if and only if $\bar{Q}_0(\Phi) = 0$.*

Corollary 8. *Superactivation of asymptotic quantum zero-error capacity (property (19) with \bar{Q}_0 replaced by Q_0) does not hold for channels Φ_1 and Φ_2 satisfying one of conditions A–D from Proposition 4.*

5. Relations to Reversibility Properties of a Channel

5.1. Reversibility of a single channel and one-shot zero-error capacities. Reversibility (sufficiency) of a quantum channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ with respect to a family \mathfrak{S} of states in $\mathfrak{S}(\mathcal{H}_A)$ means the existence of a quantum channel $\Psi : \mathfrak{T}(\mathcal{H}_B) \rightarrow \mathfrak{T}(\mathcal{H}_A)$ such that $\Psi(\Phi(\rho)) = \rho$ for all $\rho \in \mathfrak{S}$ [16, 17].

The notion of reversibility of a channel naturally arises in analysis of different general questions of quantum information theory, in particular, of conditions for preserving entropic characteristics of quantum states under the action of a channel. In particular, it follows from Petz’s theorem that the Holevo quantity⁹ of an ensemble $\{\pi_i, \rho_i\}$ of quantum states is preserved under the action of a quantum channel Φ , i.e.

$$\chi(\{\pi_i, \Phi(\rho_i)\}) = \chi(\{\pi_i, \rho_i\}),$$

if and only if the channel Φ is reversible with respect to the family $\{\rho_i\}$ [16].

A general criterion for reversibility of a quantum channel (in the von Neumann algebras theory settings) is obtained in [16]. Several conditions for reversibility expressed in terms of a complementary channel are derived from this criterion in [23], where a complete characterization of reversibility with respect to families of pure states is given. The case of families of pure states is of special interest in quantum information theory, since many capacity-like characteristics of a quantum channel can be determined as extremal values of functionals depending on ensembles of pure states [12, 21].

To describe reversibility properties of a channel Φ the *reversibility index*

$$\text{ri}(\Phi) = [\text{ri}_1(\Phi), \text{ri}_2(\Phi)]$$

is introduced in [23], in which the components $\text{ri}_1(\Phi)$ and $\text{ri}_2(\Phi)$ take the values 0, 1, 2. The first component $\text{ri}_1(\Phi)$ characterizes reversibility of the channel Φ with respect to (w.r.t.) complete¹⁰ families \mathfrak{S} of pure states as follows

⁹ The Holevo quantity provides an upper bound for accessible classical information which can be obtained by applying a quantum measurement [12, 21].

¹⁰ A family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ of pure states in $\mathfrak{S}(\mathcal{H})$ is called complete if the linear hull of the family $\{|\varphi_\lambda\rangle\}_{\lambda \in \Lambda}$ is dense in \mathcal{H} .

- $ri_1(\Phi) = 0$ if Φ is not reversible w.r.t. any complete family \mathfrak{S} ;
- $ri_1(\Phi) = 1$ if Φ is reversible w.r.t. a complete orthogonal family \mathfrak{S} but it is not reversible w.r.t. any complete nonorthogonal family \mathfrak{S} ;
- $ri_1(\Phi) = 2$ if Φ is reversible w.r.t. a complete nonorthogonal family \mathfrak{S} .

The second component $ri_2(\Phi)$ characterizes reversibility of the channel Φ with respect to noncomplete families of pure states and is defined similarly to $ri_1(\Phi)$ with the term “complete” replaced by “noncomplete”.

So that $ri(\Phi) = 01$ means that the channel Φ is not reversible with respect to any family of pure states which is either complete or nonorthogonal, but it is reversible with respect to some noncomplete orthogonal family.

A channel Φ with given $ri(\Phi)$ can be characterized by properties of the set $\ker \widehat{\Phi}$ [23, Corollary 2]. This characterization and Lemmas 1,3 shows that

$$ri_2(\Phi) = 0 \Leftrightarrow \bar{C}_0(\Phi) = 0, \quad ri_2(\Phi) = 2 \Leftrightarrow \bar{Q}_0(\Phi) > 0,$$

while $ri_2(\Phi) = 1$ means that $\bar{C}_0(\Phi) > 0$ but $\bar{Q}_0(\Phi) = 0$.

5.2. *On reversibility of a tensor product channel.* Let $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ and $\Psi : \mathfrak{T}(\mathcal{H}_C) \rightarrow \mathfrak{T}(\mathcal{H}_D)$ be arbitrary quantum channels. It is easy to see that reversibility of the channels Φ and Ψ with respect to particular families \mathfrak{S}_Φ and \mathfrak{S}_Ψ imply reversibility of the channel $\Phi \otimes \Psi$ with respect to the family $\mathfrak{S}_\Phi \otimes \mathfrak{S}_\Psi = \{\rho \otimes \sigma \mid \rho \in \mathfrak{S}_\Phi, \sigma \in \mathfrak{S}_\Psi\}$. It follows that

$$ri_1(\Phi \otimes \Psi) \geq \min\{ri_1(\Phi), ri_1(\Psi)\} \tag{27}$$

and

$$ri_2(\Phi \otimes \Psi) \geq \max\{ri_2(\Phi), ri_2(\Psi)\}. \tag{28}$$

An interesting question concerns the possibility of a strict inequality in (27) and in (28). This question is nontrivial, since the channel $\Phi \otimes \Psi$ may be reversible with respect to families consisting of *entangled* pure states in $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_C)$ (and the corresponding reversing channel may not be of the tensor product form).

As to inequality (27) this question has a simple solution.

Proposition 5. *An equality holds in (27) for any channels Φ and Ψ .*

Proof. This follows from Corollary 2 in [23], since it is easy to show that $\widehat{\Phi} \otimes \widehat{\Psi}$ is a discrete c-q channel if and *only if* $\widehat{\Phi}$ and $\widehat{\Psi}$ are discrete c-q channels.¹¹ \square

By the remark at the end of Sect. 5.1 the validity of a strict inequality in (28) means a particular form of superactivation of one-shot zero-error capacities. For example, the superactivation of one-shot zero-error classical capacity is equivalent to the existence of two channels Φ_1 and Φ_2 such that

$$ri_2(\Phi_1) = ri_1(\Phi_2) = 0, \quad \text{but} \quad ri_2(\Phi_1 \otimes \Phi_2) = 1,$$

¹¹ A channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called discrete classical-quantum (discrete c-q) if it has the representation $\Phi(\rho) = \sum_{i=1}^{\dim \mathcal{H}_A} |i\rangle\langle i| \sigma_i$, where $\{|i\rangle\}$ is an orthonormal basis in \mathcal{H}_A and $\{\sigma_i\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$ [12].

while the extreme form of superactivation means the existence of two channels Φ_1 and Φ_2 such that

$$\text{ri}_2(\Phi_1) = \text{ri}_2(\Phi_2) = 0, \quad \text{but} \quad \text{ri}_2(\Phi_1 \otimes \Phi_2) = 2.$$

These effects can be also called *superactivation of reversibility* of a channel.

So, we see that reversibility of a channel with respect to *noncomplete* families of pure states can be superactivated by tensor products in contrast to reversibility with respect to *complete* families of pure states (this follows from Proposition 5).

Proposition 3 shows that

$$\text{ri}_2(\Phi_1) = \text{ri}_2(\Phi_2) = 0 \quad \Rightarrow \quad \text{ri}_2(\Phi_1 \otimes \Phi_2) = 0$$

for any channel Φ_1 satisfying one of the conditions of this proposition and arbitrary channel Φ_2 .

Proposition 4 shows that

$$\max\{\text{ri}_2(\Phi_1), \text{ri}_2(\Phi_2)\} < 2 \quad \Rightarrow \quad \text{ri}_2(\Phi_1 \otimes \Phi_2) < 2$$

for any channels Φ_1 and Φ_2 satisfying one of the conditions of this proposition.

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