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On properties of the space of quantum states and their application to the construction of entanglement monotones

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Abstract. We consider infinite-dimensional versions of the notions of the convex hull and convex roof of a function defined on the set of quantum states. We obtain sufficient conditions for the coincidence and continuity of restrictions of different convex hulls of a given lower semicontinuous function to the subset of states with bounded mean generalized energy (an affine lower semicontinuous non-negative function). These results are used to justify an infinite-dimensional generalization of the convex roof construction of entanglement monotones that is widely used in finite dimensions. We give several examples of entanglement monotones produced by the generalized convex roof construction. In particular, we consider an infinite-dimensional generalization of the notion of Entanglement of Formation and study its properties.

Keywords: convex hull and convex roof of a function, quantum state, entanglement monotone, entanglement of formation.

Introduction

The study of finite-dimensional quantum systems and channels makes wide-ranging use of such notions of convex analysis as the convex hull and convex closure (also called the convex envelope) of a function defined on the set of quantum states and the convex roof of a function defined on the set of pure quantum states. The last notion was introduced in [1] as a special convex extension of a function to the set of all quantum states. It plays a basic role in the construction of entanglement monotones, that is, those functions on the set of states of a composite quantum system that characterize the entanglement of these states [2], [3].

The main difficulty when using these functional constructions in the infinite-dimensional case arises from the necessity of applying them to functions with 'pathological' properties. For example, the von Neumann entropy (one of the main characteristics of quantum states) is a continuous bounded function in finite dimensions, but it takes the value $+\infty$ on a dense subset of the set of states of

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an infinite-dimensional quantum system. Other difficulties arise because the set of quantum states is non-compact and has no interior points (as a subset of the Banach space of trace class operators). All these features lead to very 'unnatural' behaviour of the above functional constructions: several 'elementary' results of convex analysis become false (for example, Jensen's well-known inequality may not hold for a measurable convex function). Thus a special analysis is required to overcome these difficulties. The main tools of this analysis are the following properties of the convex set $\mathfrak{S}(\mathcal{H})$ of quantum states in a separable Hilbert space $\mathcal{H}$:

1) the weak compactness of the set of measures whose barycentres form a compact set,
2) the openness of the barycentric map (in the weak topology).

These properties are established in [4] and [5] respectively and described in detail in § 1. They reflect the special relations between the topology and convex structure of the set of quantum states.

In § 2 we consider infinite-dimensional versions of the notions of the convex hull of a function defined on $\mathfrak{S}(\mathcal{H})$ and the convex roof of a function defined on the set $\operatorname{extr}\mathfrak{S}(\mathcal{H})$. We study their continuity properties and prove that the operation of convex closure is continuous on the class of lower semicontinuous lower-bounded functions on $\mathfrak{S}(\mathcal{H})$ with respect to monotone pointwise convergence.

In § 3 we obtain sufficient conditions for the continuity and coincidence of restrictions of different convex hulls of a given function to the set of states with bounded mean generalized energy (a non-negative lower semicontinuous affine function). This result yields several useful properties of the output Rényi entropy (in particular, of the output von Neumann entropy) of a quantum channel.

In § 4 we apply the results obtained to the theory of entanglement in composite quantum systems [6]. We consider two infinite-dimensional versions (discrete and continuous) of the convex roof construction of entanglement monotones, which is widely used in finite dimensions. It is shown that the discrete version may be 'false' in the sense that the resulting functions may not possess the main property of entanglement monotones (even if the generating function is bounded and lower semicontinuous) while the continuous version produces 'true' entanglement monotones under weak conditions on the generating functions. Therefore it is the last method that should be regarded as the generalized convex roof construction. It can be used to obtain an infinite-dimensional generalization of the Entanglement of Formation (EoF), which is one of the basic entanglement measures for finite-dimensional composite quantum systems [7]. In § 5 we compare this approach with another generalization of EoF that was proposed in [8].

§ 1. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, $\mathfrak{B}_h(\mathcal{H})$ the Banach space of bounded Hermitian operators on $\mathcal{H}$ containing the cone $\mathfrak{B}_+(\mathcal{H})$ of positive operators, and $\mathfrak{T}(\mathcal{H})$ (resp. $\mathfrak{T}_h(\mathcal{H})$) the separable Banach space of all trace-class operators on $\mathcal{H}$ (resp. all Hermitian trace-class operators) with the trace norm $\|\cdot\|_1 = \operatorname{Tr}|\cdot|$ (see [9]).

The closed convex subsets

$$\mathfrak{T}_1(\mathcal{H}) = \{A \in \mathfrak{T}(\mathcal{H}) \mid A \geqslant 0, \operatorname{Tr} A \leqslant 1\}, \qquad \mathfrak{S}(\mathcal{H}) = \{A \in \mathfrak{T}_1(\mathcal{H}) \mid \operatorname{Tr} A = 1\}$$

of $\mathfrak{T}(\mathcal{H})$ are complete separable metric spaces with metric defined by the trace norm. Every operator ρ in $\mathfrak{S}(\mathcal{H})$ determines a linear functional $A \mapsto \operatorname{Tr} A\rho$ on the algebra $\mathfrak{B}(\mathcal{H})$, and such functionals are called *states* in the theory of operator algebras. Thus, in what follows, we refer to operators in $\mathfrak{S}(\mathcal{H})$ as *states*. The *rank* of a positive operator (state) is the dimension of the orthogonal complement of its kernel.

We write $\operatorname{co}\mathcal{A}$ (resp. $\overline{\operatorname{co}}\,\mathcal{A}$) for the convex hull (resp. convex closure) of a set $\mathcal{A}$ [10]. The set of all extreme points of a convex set $\mathcal{A}$ is denoted by $\operatorname{extr}\mathcal{A}$.

We write $\mathcal{P}(\mathcal{A})$ for the set of all Borel probability measures on a complete separable metric space $\mathcal{A}$ and endow $\mathcal{P}(\mathcal{A})$ with the topology of weak convergence. This set may also be regarded as a complete separable metric space ([11], Ch. II, § 6). Let $\mathcal{P}^{\mathrm{f}}(\mathcal{A})$ be the subset of $\mathcal{P}(\mathcal{A})$ consisting of measures with finite support. We shall also use the abbreviations $\mathcal{P} = \mathcal{P}(\mathfrak{S}(\mathcal{H}))$ and $\widehat{\mathcal{P}} = \mathcal{P}(\operatorname{extr}\mathfrak{S}(\mathcal{H}))$.

The *barycentre* of a measure $\mu \in \mathcal{P}$ is the state defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \sigma\mu(d\sigma).$$

For an arbitrary subset $\mathcal{A} \subset \mathfrak{S}(\mathcal{H})$ let $\mathcal{P}_{\mathcal{A}}$ (resp. $\widehat{\mathcal{P}}_{\mathcal{A}}$) be the subset of $\mathcal{P}$ (resp. $\widehat{\mathcal{P}}$) consisting of all measures with barycentre in $\mathcal{A}$.

A finite or countable set $\{\rho_i\}$ of states with corresponding probability distribution $\{\pi_i\}$ is called an *ensemble* and denoted by $\{\pi_i, \rho_i\}$. In this paper we regard ensembles of states as particular cases of probability measures on the set of quantum states.

The *von Neumann entropy* of a state ρ and the *relative entropy* of states ρ and σ are defined by

$$H(\rho) = -\sum_i \langle i|\,\rho\log\rho\,|i\rangle, \qquad H(\rho\,\|\sigma) = \sum_i \langle i|\,(\rho\log\rho - \rho\log\sigma)\,|i\rangle$$

respectively, where $\{|i\rangle\}$ is a basis of eigenvectors of ρ and we put $H(\rho\,\|\sigma) = +\infty$ if the support of ρ (the orthogonal complement of its kernel) is not contained in the support of σ. The entropy and relative entropy are lower semicontinuous functions (of their arguments) with values in $[0, +\infty]$. The first is concave while the second is jointly convex [12].

An arbitrary unbounded positive operator H in $\mathcal{H}$ with discrete spectrum of finite multiplicity is called an $\mathfrak{H}$-*operator*.

The set of quantum states $\mathfrak{S}(\mathcal{H})$ has the following properties.

A) The set $\mathcal{P}_{\mathcal{A}}(\mathfrak{S}(\mathcal{H}))$ is compact for every compact subset $\mathcal{A} \subset \mathfrak{S}(\mathcal{H})$ (see [4]).

B) The barycentric map $\mathcal{P}(\mathfrak{S}(\mathcal{H})) \ni \mu \mapsto \bar{\rho}(\mu) \in \mathfrak{S}(\mathcal{H})$ is an open surjection (see [5], [13]).

Property A) enables one to prove that $\mathfrak{S}(\mathcal{H})$ possesses some well-known properties of compact convex sets (see [14], Lemma 1, or Propositions 1, 6 below). Hence it may be regarded as a kind of 'weak' compactness. In fact, this property is not purely topological: it reflects a special relation between the topology and convex structure of $\mathfrak{S}(\mathcal{H})$. Following [13], [15], we call it the *μ-compactness property*.

Note that the μ-compactness of the positive part of the unit ball is a specific feature of the Banach space of trace-class operators (the Shatten class of order $p=1$) within the family of Shatten classes of order $p\geqslant 1$. Moreover, it can be shown that $\mathfrak{T}_1(\mathcal{H})$ loses the μ-compactness property[1] when it is endowed with the $\|\cdot\|_p$-norm topology with $p>1$ and that the Shatten class of order $p=2$ (the Hilbert space of Hilbert–Schmidt operators) contains no non-compact μ-compact sets. These and other results about the μ-compactness property, as well as examples of μ-compact sets, are considered in [15].

Property B) reflects another relation between the topology and convex structure of $\mathfrak{S}(\mathcal{H})$. A characterization of this property for an arbitrary μ-compact convex set is obtained in [13], Theorem 1.[2] By this theorem, property B) is equivalent to the continuity of the convex hull of any continuous bounded function on $\mathfrak{S}(\mathcal{H})$ and the openness of the map

$$\mathfrak{S}(\mathcal{H})\times\mathfrak{S}(\mathcal{H})\times[0,1]\ni(\rho,\sigma,\lambda)\mapsto\lambda\rho+(1-\lambda)\sigma\in\mathfrak{S}(\mathcal{H}).$$

An analogue of the last property for any convex set seems to be the simplest for verification and (an equivalent but formally stronger form) is called the *stability* property (see [17], [18] and references therein).

§ 2. Convex hulls and convex roofs

In this section we consider some notions and constructions for functions defined on $\mathfrak{S}(\mathcal{H})$. Note that all the definitions are universal: they can be stated in terms of functions defined on any convex closed bounded subset $\mathcal{A}$ of a locally convex space (instead of $\mathfrak{S}(\mathcal{H})$). The main results in this section can also be proved in this extended context under certain conditions on $\mathcal{A}$ (which hold for $\mathfrak{S}(\mathcal{H})$). Possible generalizations of this kind are discussed in the Appendix.

2.1. Some notions of convexity of a function. In what follows we consider *semibounded* (lower- or upper-bounded) functions on $\mathfrak{S}(\mathcal{H})$ with values in $[-\infty,+\infty]$.

Besides the well-known notion of a convex function on $\mathfrak{S}(\mathcal{H})$, we shall use the following strengthened versions.

A semibounded function f on $\mathfrak{S}(\mathcal{H})$ is said to be *σ-convex* if

$$f\biggl(\sum_i\pi_i\rho_i\biggr)\leqslant\sum_i\pi_i f(\rho_i)$$

for any *countable* ensemble $\{\pi_i,\rho_i\}$ of states in $\mathfrak{S}(\mathcal{H})$.

[1] This shows that, unlike compactness, μ-compactness is not preserved under weakening of the topology.

[2] This theorem is a partial non-compact generalization of results in [16] about compact convex sets. A complete generalization of these results to the class of μ-compact convex sets is obtained in [15].

A semibounded universally measurable[3] function f on $\mathfrak{S}(\mathcal{H})$ is said to be *μ-convex* if

$$f\biggl(\int_{\mathfrak{S}(\mathcal{H})}\rho\mu(d\rho)\biggr)\leqslant\int_{\mathfrak{S}(\mathcal{H})}f(\rho)\mu(d\rho)$$

for any measure μ in $\mathcal{P}(\mathfrak{S}(\mathcal{H}))$.

The simplest example of a convex Borel function on $\mathfrak{S}(\mathcal{H})$ which is neither σ-convex nor μ-convex is the function equal to 0 on the convex set of finite-rank states and to $+\infty$ on the set of infinite-rank states. The difference between these convexity properties can also be illustrated by the functions in Examples 1, 2 below (the first is convex but not σ-convex while the second is σ-convex but not μ-convex).

Convexity implies σ-convexity for all upper-bounded functions on $\mathfrak{S}(\mathcal{H})$ (see Proposition A-1 in the Appendix).

By Jensen's integral inequality (Proposition A-2 in the Appendix), all these convexity properties are equivalent for lower semicontinuous functions and upper-bounded upper semicontinuous functions on $\mathfrak{S}(\mathcal{H})$.

2.2. Convex hulls and convex closure. The *convex hull* $\operatorname{co} f$ of a semibounded function f on $\mathfrak{S}(\mathcal{H})$ is defined as the greatest convex function majorized by f (see [20], § 2.8). Thus,

$$\operatorname{co} f(\rho)=\inf_{\{\pi_i,\rho_i\}\in\mathcal{P}^{\mathrm{f}}_{\{\rho\}}}\sum_i\pi_i f(\rho_i),\qquad\rho\in\mathfrak{S}(\mathcal{H})\tag{1}$$

(the infimum is taken over all finite ensembles $\{\pi_i,\rho_i\}$ of states with average state ρ).

The *σ-convex hull* σ-$\operatorname{co} f$ of a semibounded function f on $\mathfrak{S}(\mathcal{H})$ is defined by

$$\sigma\text{-}\operatorname{co} f(\rho)=\inf_{\{\pi_i,\rho_i\}\in\mathcal{P}_{\{\rho\}}}\sum_i\pi_i f(\rho_i),\qquad\rho\in\mathfrak{S}(\mathcal{H})\tag{2}$$

(the infimum is taken over all countable ensembles $\{\pi_i,\rho_i\}$ of states with average state ρ). The function σ-$\operatorname{co} f$ is σ-convex since for any countable ensemble $\{\lambda_i,\sigma_i\}$ with average state σ and any family $\{\{\pi_{ij},\rho_{ij}\}_j\}_i$ of countable ensembles such that $\sigma_i=\sum_j\pi_{ij}\rho_{ij}$ for all i, the countable ensemble $\{\lambda_i\pi_{ij},\rho_{ij}\}_{ij}$ has average state σ. Thus σ-$\operatorname{co} f$ is the greatest σ-convex function majorized by f.

The *μ-convex hull* μ-$\operatorname{co} f$ of a semibounded Borel function f on $\mathfrak{S}(\mathcal{H})$ is defined by

$$\mu\text{-}\operatorname{co} f(\rho)=\inf_{\mu\in\mathcal{P}_{\{\rho\}}}\int_{\mathfrak{S}(\mathcal{H})}f(\sigma)\mu(d\sigma),\qquad\rho\in\mathfrak{S}(\mathcal{H})\tag{3}$$

(the infimum is taken over all probability measures μ with barycentre ρ). If the function μ-$\operatorname{co} f$ is universally measurable[4] and μ-convex, then it is the greatest μ-convex function majorized by f. Propositions 1 and 2 below (along with the obvious convexity of the function μ-$\operatorname{co} f$ and Proposition A-2 in the Appendix) show that this holds if f is either lower-bounded and lower semicontinuous or upper-bounded and upper semicontinuous.

[3]This means that f is measurable with respect to any measure in $\mathcal{P}(\mathfrak{S}(\mathcal{H}))$ [19].

[4]One can deduce from results in [19] that μ-$\operatorname{co} f$ is universally measurable for any bounded Borel function f.

The *convex closure* $\overline{\mathrm{co}}f$ of a lower-bounded function f on $\mathfrak{S}(\mathcal{H})$ is defined as the greatest convex lower semicontinuous (closed) function majorized by f [20]. By Fenchel's theorem (see [10], [20], [21]), the function $\overline{\mathrm{co}}f$ coincides with the double Fenchel transformation of f, which means that[5]

$$\overline{\mathrm{co}}f(\rho) = f^{**}(\rho) = \sup_{A\in\mathfrak{B}_+(\mathcal{H})}[\mathrm{Tr}\,A\rho - f^*(A)], \qquad \rho\in\mathfrak{S}(\mathcal{H}), \tag{4}$$

where

$$f^*(A) = \sup_{\rho\in\mathfrak{S}(\mathcal{H})}[\mathrm{Tr}\,A\rho - f(\rho)], \qquad A\in\mathfrak{B}_+(\mathcal{H}).$$

It follows from the definitions and Proposition A-2 in the Appendix that

$$\overline{\mathrm{co}}f(\rho) \leqslant \mu\text{-}\mathrm{co}\,f(\rho) \leqslant \sigma\text{-}\mathrm{co}\,f(\rho) \leqslant \mathrm{co}\,f(\rho), \qquad \rho\in\mathfrak{S}(\mathcal{H}),$$

for any lower-bounded Borel function f on $\mathfrak{S}(\mathcal{H})$. One can prove (see Corollary 1 below) that these inequalities become equalities for all continuous bounded functions f on $\mathfrak{S}(\mathcal{H})$. The following examples show that this is not the case in general.

Example 1. Let H be the von Neumann entropy (see § 1) and ρ_0 be a state with $H(\rho_0) = +\infty$. Since the set of quantum states with finite entropy is convex, we have $\mathrm{co}\,H(\rho_0) = +\infty$, while the spectral theorem implies that $\sigma\text{-}\mathrm{co}\,H(\rho_0) = 0$.

Example 2. Let f be the indicator function of the complement of the closed set $\mathcal{A}_s$ of pure product states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})$, and let ω_0 be the separable state in $\overline{\mathrm{co}}\mathcal{A}_s$ constructed in [14] such that any measure in $\mathcal{P}_{\{\omega_0\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{H}))$ has no atoms in $\mathcal{A}_s$. It is easy to show that $\sigma\text{-}\mathrm{co}\,f(\omega_0) = 1$. By Lemma 1 in [14], there is a measure μ_0 in $\mathcal{P}_{\{\omega_0\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{H}))$ supported by $\mathcal{A}_s$. Hence $\mu\text{-}\mathrm{co}\,f(\omega_0) = 0$. Note that $\sigma\text{-}\mathrm{co}\,f$ is a μ_0-integrable σ-convex bounded function on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})$ for which Jensen's inequality does not hold:

$$1 = \sigma\text{-}\mathrm{co}\,f(\omega_0) > \int_{\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})}\sigma\text{-}\mathrm{co}\,f(\omega)\mu_0(d\omega) = 0$$

(since the functions $\sigma\text{-}\mathrm{co}\,f$ and f coincide on the support of the measure μ_0).

Example 3. Let f be the indicator function of a set consisting of one pure state. Then $\mu\text{-}\mathrm{co}\,f = f$ while $\overline{\mathrm{co}}f \equiv 0$.

Since $\mathfrak{S}(\mathcal{H})$ is μ-compact, Proposition 3 in [13] yields the following assertion.

Proposition 1. *Let f be a lower semicontinuous lower-bounded function on $\mathfrak{S}(\mathcal{H})$. Then the μ-convex hull of f is lower semicontinuous. Thus,*

$$\overline{\mathrm{co}}f(\rho) = \mu\text{-}\mathrm{co}\,f(\rho) = \inf_{\mu\in\mathcal{P}_{\{\rho\}}}\int_{\mathfrak{S}(\mathcal{H})}f(\sigma)\mu(d\sigma), \qquad \rho\in\mathfrak{S}(\mathcal{H}). \tag{5}$$

The infimum in (5) is achieved at some measure in $\mathcal{P}_{\{\rho\}}$.

[5] Since the space $\mathfrak{B}_h(\mathcal{H})$ is dual to $\mathfrak{T}_h(\mathcal{H})$, the deduction of this expression from Fenchel's theorem requires a consideration of the extension $\hat{f}$ of f to the real Banach space $\mathfrak{T}_h(\mathcal{H})$ by setting $\hat{f} = +\infty$ on $\mathfrak{T}_h(\mathcal{H})\setminus\mathfrak{S}(\mathcal{H})$.

The μ-compactness of the set $\mathfrak{S}(\mathcal{H})$ is an essential condition for the validity of the representation (5) for the convex closure ([15], Proposition 7). The representation (5) implies, in particular, that the convex closure of an arbitrary lower semicontinuous lower-bounded function on $\mathfrak{S}(\mathcal{H})$ coincides with that function on the set $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ of pure states. It is also essential in Proposition 1 that f is lower-bounded since Lemma 2 below shows that if a convex lower semicontinuous function is not lower-bounded on $\mathfrak{S}(\mathcal{H})$, then it is equal to $-\infty$ everywhere.

The stability of $\mathfrak{S}(\mathcal{H})$ enables us to prove the following result.

Proposition 2. *Let f be an upper semicontinuous function on $\mathfrak{S}(\mathcal{H})$. Then its convex hull* $\operatorname{co} f$ *is upper semicontinuous. If, in addition, f is upper-bounded, then the convex hull, the σ-convex hull and the μ-convex hull of f coincide:* $\operatorname{co} f = \sigma\text{-}\operatorname{co} f = \mu\text{-}\operatorname{co} f$.

Proof. The upper semicontinuity of $\operatorname{co} f$ can be deduced from the more general assertion in Lemma 4 below since, given any sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$ converging to a state ρ_0, we see from Lemma 3 in [4] that there is an $\mathfrak{H}$-operator H in $\mathcal{H}$ such that $\sup_{n\geqslant 0}\operatorname{Tr} H\rho_n < +\infty$.

When f is upper-bounded, the coincidence of $\operatorname{co} f$ and $\mu\text{-}\operatorname{co} f$ is easily proved using the upper semicontinuity of the functional $\mu \mapsto \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu(d\rho)$ on the set $\mathcal{P}(\mathfrak{S}(\mathcal{H}))$ and the density of measures with finite support in the set of all measures with a given barycentre ([4], Lemma 1).

Example 3 shows that the hypotheses of Proposition 2 do not imply the coincidence of the function $\overline{\operatorname{co}} f$ with the function $\mu\text{-}\operatorname{co} f = \sigma\text{-}\operatorname{co} f = \operatorname{co} f$.

Propositions 1 and 2 have the following obvious corollary.

Corollary 1. *Let f be a continuous lower-bounded function on $\mathfrak{S}(\mathcal{H})$. Then the convex hull* $\operatorname{co} f$ *is continuous on any subset of $\mathfrak{S}(\mathcal{H})$ where it coincides with the μ-convex hull* $\mu\text{-}\operatorname{co} f$.

If, in addition, f is bounded, then its convex hull, σ-convex hull, μ-convex hull, and convex closure coincide: $\operatorname{co} f = \sigma\text{-}\operatorname{co} f = \mu\text{-}\operatorname{co} f = \overline{\operatorname{co}} f$, *and this function is continuous.*

A necessary and sufficient condition for the coincidence of the functions $\operatorname{co} f$ and $\mu\text{-}\operatorname{co} f$ at a state $\rho_0 \in \mathfrak{S}(\mathcal{H})$ can easily be deduced from Proposition 1: the Jensen inequality $\operatorname{co} f(\rho_0) \leqslant \int \operatorname{co} f(\rho)\mu(d\rho)$ must hold for any measure μ in $\mathcal{P}_{\{\rho_0\}}$ (the convex function $\operatorname{co} f$ is Borel by Proposition 2). A sufficient condition for this coincidence is given in Corollary 6 below.

The second assertion of Corollary 1 shows that

$$\overline{\operatorname{co}} f(\rho) = \operatorname{co} f(\rho) = \inf_{\{\pi_i,\rho_i\}\in\mathcal{P}^{\mathrm{f}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \qquad \rho \in \mathfrak{S}(\mathcal{H}), \tag{6}$$

for any continuous bounded function f on $\mathfrak{S}(\mathcal{H})$. This representation for the convex closure is a non-compact generalization of Corollary I.3.6 in [22].

We shall use the following approximation result.

Lemma 1. *Let f be a lower-bounded Borel function on $\mathfrak{S}(\mathcal{H})$. For every state ρ_0 in $\mathfrak{S}(\mathcal{H})$ there is a sequence $\{\rho_n\}$ converging to ρ_0 such that*

$$\limsup_{n\to+\infty} \sigma\text{-co}\, f(\rho_n) \leqslant \limsup_{n\to+\infty} \operatorname{co} f(\rho_n) \leqslant \mu\text{-co}\, f(\rho_0).$$

If, in addition, f is lower semicontinuous, then

$$\lim_{n\to+\infty} \sigma\text{-co}\, f(\rho_n) = \lim_{n\to+\infty} \operatorname{co} f(\rho_n) = \mu\text{-co}\, f(\rho_0).$$

Proof. It suffices to consider the case when f is non-negative. For every positive integer n let μ_n be a measure in $\mathcal{P}_{\{\rho_0\}}$ such that

$$\mu\text{-co}\, f(\rho_0) \geqslant \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_n(d\rho) - \frac{1}{n}.$$

Since $\mathfrak{S}(\mathcal{H})$ is separable, there is a sequence $\{\mathcal{A}_i^n\}$ of Borel subsets of $\mathfrak{S}(\mathcal{H})$ with diameter $\leqslant 1/n$ such that $\mathfrak{S}(\mathcal{H}) = \bigcup_i \mathcal{A}_i^n$ and $\mathcal{A}_i^n \cap \mathcal{A}_j^n = \varnothing$ if $j \neq i$. Let $m = m(n)$ be such that $\sum_{i=m+1}^{+\infty} \mu_n(\mathcal{A}_i^n) < 1/n$. We may assume without loss of generality that $\mu_n(\mathcal{A}_i^n) > 0$ for $i = 1, \ldots, m$. For every i the set $\mathcal{A}_i^n$ contains a state ρ_i^n such that $f(\rho_i^n) \leqslant (\mu_n(\mathcal{A}_i^n))^{-1} \int_{\mathcal{A}_i^n} f(\rho)\mu_n(d\rho)$.

Let $\mathcal{B}_n = \bigcup_{i=1}^m \mathcal{A}_i^n$. Consider the state $\rho_n = (\mu_n(\mathcal{B}_n))^{-1} \sum_{i=1}^m \mu_n(\mathcal{A}_i^n)\rho_i^n$. We claim that

$$\lim_{n\to+\infty} \rho_n = \rho_0. \tag{7}$$

Indeed, for every i the state $\hat{\rho}_i^n = (\mu_n(\mathcal{A}_i^n))^{-1} \int_{\mathcal{A}_i^n} \rho\mu_n(d\rho)$ lies in the set $\overline{\text{co}}(\mathcal{A}_i^n)$ of diameter $\leqslant 1/n$. It follows that $\|\rho_i^n - \hat{\rho}_i^n\|_1 \leqslant 1/n$ for $i = 1, \ldots, m$. Since $\mu_n(\mathcal{B}_n) = \sum_{i=1}^m \mu_n(\mathcal{A}_i^n)$, we have

$$\begin{aligned}
\|\rho_n - \rho_0\|_1 &= \left\| (\mu_n(\mathcal{B}_n))^{-1} \sum_{i=1}^m \mu_n(\mathcal{A}_i^n)\rho_i^n - \sum_{i=1}^m \int_{\mathcal{A}_i^n} \rho\mu_n(d\rho) - \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{B}_n} \rho\mu_n(d\rho) \right\|_1 \\
&\leqslant \sum_{i=1}^m \mu_n(\mathcal{A}_i^n) \|(\mu_n(\mathcal{B}_n))^{-1}\rho_i^n - \hat{\rho}_i^n\|_1 + \left\| \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{B}_n} \rho\mu_n(d\rho) \right\|_1 \\
&\leqslant (1 - \mu_n(\mathcal{B}_n)) + \sum_{i=1}^m \mu_n(\mathcal{A}_i^n)\|\rho_i^n - \hat{\rho}_i^n\|_1 + \mu_n(\mathfrak{S}(\mathcal{H}) \setminus \mathcal{B}_n) < \frac{3}{n},
\end{aligned}$$

which proves (7).

By the choice of the states ρ_i^n we have

$$\begin{aligned}
\operatorname{co} f(\rho_n) &\leqslant (\mu_n(\mathcal{B}_n))^{-1} \sum_{i=1}^m \mu_n(\mathcal{A}_i^n) f(\rho_i^n) \leqslant (\mu_n(\mathcal{B}_n))^{-1} \sum_{i=1}^m \int_{\mathcal{A}_i^n} f(\rho)\mu_n(d\rho) \\
&\leqslant (\mu_n(\mathcal{B}_n))^{-1} \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_n(d\rho) \leqslant \left(1 - \frac{1}{n}\right)^{-1} \left(\mu\text{-co}\, f(\rho_0) + \frac{1}{n}\right).
\end{aligned}$$

This proves the first assertion of the lemma. The second follows from the first by Proposition 1 (since $\sigma\text{-co}\, f \geqslant \mu\text{-co}\, f = \overline{\text{co}} f$). The lemma is proved.

We also use the following corollary of the fact that $\mathfrak{S}(\mathcal{H})$ is a bounded subset of $\mathfrak{T}(\mathcal{H})$.

Lemma 2. *Let f be a concave upper semicontinuous function on a convex subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$. If f is finite at some point of $\mathcal{A}$, then it is upper-bounded on $\mathcal{A}$.*

Proof. Let ρ_0 be a state in $\mathcal{A}$ with $f(\rho_0) = c_0 \neq \pm\infty$. There is no loss of generality in assuming that $c_0 = 0$. If there is a sequence $\{\rho_n\} \subset \mathcal{A}$ with $\lim_{n\to+\infty} f(\rho_n) = +\infty$, then the sequence of states $\sigma_n = (1-\lambda_n)\rho_0 + \lambda_n\rho_n$ in $\mathcal{A}$, where $\lambda_n = (f(\rho_n))^{-1}$, converges to ρ_0 because $\mathcal{A}$ is bounded, and we have $f(\sigma_n) \geqslant \lambda_n f(\rho_n) = 1$ by the concavity of f. This contradicts the upper semicontinuity of f.

2.3. Convex roofs. If $\dim\mathcal{H} < +\infty$, then any state in $\mathfrak{S}(\mathcal{H})$ can be represented as the average state of some finite ensemble of pure states. Therefore any function f defined on the set $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ of pure states has the following well-defined convex extension to $\mathfrak{S}(\mathcal{H})$:

$$f_*(\rho) = \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}^{\mathrm{f}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \qquad \rho \in \mathfrak{S}(\mathcal{H}) \tag{8}$$

(the infimum is taken over all finite ensembles $\{\pi_i, \rho_i\}$ of *pure* states with average state ρ). Following [1], we call this extension the *convex roof* of f. The notion of a convex roof plays an essential role in quantum information theory, where it is used, in particular, to construct entanglement monotones (see § 4).

In the case $\dim\mathcal{H} = +\infty$ one can consider the following generalizations of this construction.

Let f be a semibounded function on the set $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ of pure states. The *σ-convex roof* f_*^σ of f is defined as

$$f_*^\sigma(\rho) = \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \qquad \rho \in \mathfrak{S}(\mathcal{H}) \tag{9}$$

(the infimum is taken over all countable ensembles $\{\pi_i, \rho_i\}$ of *pure* states with average state ρ). As in the case of σ-co f, it is easy to show that f_*^σ is σ-convex. Thus f_*^σ is the greatest σ-convex extension of f to $\mathfrak{S}(\mathcal{H})$.

Let f be a semibounded Borel function on the set $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ of pure states. The *μ-convex roof* f_*^μ of f is defined as

$$f_*^\mu(\rho) = \inf_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}} \int_{\operatorname{extr}\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma), \qquad \rho \in \mathfrak{S}(\mathcal{H}) \tag{10}$$

(the infimum is taken over all probability measures μ supported by *pure* states with barycentre ρ). If f_*^μ is universally measurable[6] and μ-convex, then it is the greatest μ-convex extension of f to $\mathfrak{S}(\mathcal{H})$. Propositions 3 and 4 below (along with the obvious convexity of the function f_*^μ and Proposition A-2 in the Appendix) show that this holds if f is either lower-bounded and lower semicontinuous or upper-bounded and upper semicontinuous.

[6] One can deduce from results in [19] that f_*^μ is universally measurable for any bounded Borel function f.

Note that the notions of the σ-convex roof and μ-convex roof can be reduced to the notions of the σ-convex hull and μ-convex hull (respectively) introduced in § 2.2. Indeed, it is easy to see that $f_*^\sigma = \sigma\text{-co}\,\hat{f}$ and $f_*^\mu = \mu\text{-co}\,\hat{f}$ for any function f on $\operatorname{extr}\mathfrak{S}(\mathcal{H})$, where

$$\hat{f}(\rho) = \begin{cases} f(\rho), & \rho \in \operatorname{extr}\mathfrak{S}(\mathcal{H}), \\ +\infty, & \rho \in \mathfrak{S}(\mathcal{H}) \setminus \operatorname{extr}\mathfrak{S}(\mathcal{H}). \end{cases}$$

Since the lower semicontinuity of a function f on $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ implies that $\hat{f}$ is lower semicontinuous on $\mathfrak{S}(\mathcal{H})$, Proposition 1 yields the following result (which can also be derived from part A of Theorem 2 in [13] since the set $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ is μ-compact).

Proposition 3. *Let f be a lower semicontinuous lower-bounded function on the set* $\operatorname{extr}\mathfrak{S}(\mathcal{H})$. *Then the function f_*^μ is the greatest lower semicontinuous convex extension of f to $\mathfrak{S}(\mathcal{H})$ and, for every state ρ in $\mathfrak{S}(\mathcal{H})$, the infimum in the definition* (10) *of $f_*^\mu(\rho)$ is achieved at some measure in $\widehat{\mathcal{P}}_{\{\rho\}}$.*

The importance of the μ-compactness property of $\mathfrak{S}(\mathcal{H})$ in the proof of this proposition is illustrated by the examples in [15].

By Theorem 1 in [13], the stability of $\mathfrak{S}(\mathcal{H})$ implies that the map $\mathcal{P}(\operatorname{extr}\mathfrak{S}(\mathcal{H})) \ni \mu \mapsto \bar{\rho}(\mu) \in \mathfrak{S}(\mathcal{H})$ is open.[7] Hence part B of Theorem 2 in [13] yields the following result.

Proposition 4. *Let f be an upper semicontinuous upper-bounded function on* $\operatorname{extr}\mathfrak{S}(\mathcal{H})$. *Then the σ-convex roof and μ-convex roof of f coincide: $f_*^\sigma = f_*^\mu$, and the function $f_*^\sigma = f_*^\mu$ is upper semicontinuous on $\mathfrak{S}(\mathcal{H})$ and coincides with the greatest upper-bounded convex extension of f to $\mathfrak{S}(\mathcal{H})$.*

Propositions 3 and 4 have the following obvious corollary.

Corollary 2. *Let f be a continuous bounded function on* $\operatorname{extr}\mathfrak{S}(\mathcal{H})$. *Then its σ-convex roof and its μ-convex roof coincide and the function $f_*^\sigma = f_*^\mu$ is continuous on the set $\mathfrak{S}(\mathcal{H})$.*

By this corollary, an arbitrary continuous bounded function on the set of pure states has at least one continuous bounded convex extension to the set of all states.

2.4. The convex hulls of concave functions. In the case when $\dim\mathcal{H} < +\infty$, it is easy to show that the convex hull of any concave function f defined on $\mathfrak{S}(\mathcal{H})$ coincides with the convex roof of the restriction $f|_{\operatorname{extr}\mathfrak{S}(\mathcal{H})}$ of that function to the set $\operatorname{extr}\mathfrak{S}(\mathcal{H})$. Since $\mathfrak{S}(\mathcal{H})$ is stable, the continuity of f implies that the function $\operatorname{co} f = (f|_{\operatorname{extr}\mathfrak{S}(\mathcal{H})})_*$ is continuous.

The following proposition establishes an analogue of this observation in the case when $\dim\mathcal{H} = +\infty$.

Proposition 5. *If f is a concave lower-bounded function on $\mathfrak{S}(\mathcal{H})$, then $\sigma\text{-co}\, f = \big(f|_{\operatorname{extr}\mathfrak{S}(\mathcal{H})}\big)_*^\sigma$. If, in addition, f is lower semicontinuous, then $\mu\text{-co}\, f = \big(f|_{\operatorname{extr}\mathfrak{S}(\mathcal{H})}\big)_*^\mu$*

[7] By the generalized Vesterstrom–O'Brien theorem (proved in [15]), the openness of this map is equivalent to the stability of $\mathfrak{S}(\mathcal{H})$.

and this function is lower semicontinuous. If f is a concave upper semicontinuous (*resp. concave, continuous and lower-bounded*) *function on* $\mathfrak{S}(\mathcal{H})$, *then*

$$\mathrm{co}\, f = \sigma\text{-}\mathrm{co}\, f = \mu\text{-}\mathrm{co}\, f = (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\sigma}_{*} = (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\mu}_{*}$$

and this function is upper semicontinuous (*resp. continuous*).

Proof. To show the coincidence of σ-co f and $(f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\sigma}_{*}$ (resp. μ-co f and $(f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\mu}_{*}$), it suffices to prove that $\sigma\text{-co}\, f \geqslant (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\sigma}_{*}$ (resp. $\mu\text{-co}\, f \geqslant (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\mu}_{*}$).

The first inequality for concave lower-bounded functions f follows directly from Jensen's discrete inequality (Proposition A-1 in the Appendix).

Let f be a lower-bounded lower semicontinuous concave function and ρ_0 an arbitrary state. By Lemma 1 there is a sequence $\{\rho_n\}$ converging to ρ_0 such that $\lim_{n\to+\infty} \sigma\text{-co}\, f(\rho_n) = \mu\text{-co}\, f(\rho_0)$. By the first part of the proposition we have

$$\sigma\text{-co}\, f(\rho_n) = (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\sigma}_{*}(\rho_n) \geqslant (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\mu}_{*}(\rho_n) \qquad \forall\, n.$$

Passing to the limit as $n \to +\infty$ in this inequality and using Proposition 3, we get the inequality $\mu\text{-co}\, f(\rho_0) \geqslant (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\mu}_{*}(\rho_0)$.

Let f be an upper semicontinuous concave function taking a finite value on at least one state. By Lemma 2 this function is upper-bounded. Propositions 2 and 4 respectively imply that $\mathrm{co}\, f = \sigma\text{-co}\, f = \mu\text{-co}\, f$ and $(f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\sigma}_{*} = (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\mu}_{*}$ and these functions are upper semicontinuous. Since $\mathrm{co}\, f \geqslant (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\sigma}_{*}$ by Proposition A-2 in the Appendix and $\mu\text{-co}\, f \leqslant (f|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H})})^{\sigma}_{*}$ in view of the definitions, we obtain the main part of the second assertion of the proposition.

The assertion concerning concave continuous lower-bounded functions f follows from the previous ones.

2.5. A result concerning the convex closure. It is well known[8] that, for an arbitrary increasing sequence $\{f_n\}$ of continuous functions on a convex compact set $\mathcal{A}$ converging pointwise to a continuous function f_0, the corresponding sequence $\{\overline{\mathrm{co}}f_n\}$ converges to the function $\overline{\mathrm{co}}f_0$. The μ-compactness of the non-compact set $\mathfrak{S}(\mathcal{H})$ enables us to prove an analogous assertion for $\mathfrak{S}(\mathcal{H})$.

Proposition 6. *The following inequality holds for any increasing sequence $\{f_n\}$ of lower semicontinuous lower-bounded functions on $\mathfrak{S}(\mathcal{H})$ and any convergent sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$:*

$$\liminf_{n\to+\infty} \overline{\mathrm{co}}f_n(\rho_n) \geqslant \overline{\mathrm{co}}f_0(\rho_0),$$

where $f_0 = \lim_{n\to+\infty} f_n$ and $\rho_0 = \lim_{n\to+\infty} \rho_n$. In particular,

$$\lim_{n\to+\infty} \overline{\mathrm{co}}f_n(\rho) = \overline{\mathrm{co}}f_0(\rho) \qquad \forall\, \rho \in \mathfrak{S}(\mathcal{H}).$$

[8]This follows from Dini's lemma. The importance of the compactness condition is illustrated by the sequence $f_n(x) = \exp(-x^2/n)$ on $\mathbb{R}$, which converges to $f_0(x) \equiv 1$ but has $\overline{\mathrm{co}}f_n(x) \equiv 0$ for all n.

In fact, the μ-compactness of $\mathfrak{S}(\mathcal{H})$ is *equivalent* to the validity of the last assertion of Proposition 6 (see [23]).

Proof. For an arbitrary Borel function g on the set $\mathfrak{S}(\mathcal{H})$ and any measure $\mu \in \mathcal{P}$ we introduce the notation

$$\mu(g) = \int_{\mathfrak{S}(\mathcal{H})} g(\sigma)\mu(d\sigma).$$

We may assume without loss of generality that the sequence $\{f_n\}$ consists of non-negative functions. Suppose that there is a sequence $\{\rho_n\}$ converging to ρ_0 such that

$$\overline{\mathrm{co}} f_n(\rho_n) + \Delta \leqslant \overline{\mathrm{co}} f_0(\rho_0), \qquad \Delta > 0, \qquad \forall\, n.$$

We assume that $\overline{\mathrm{co}} f_0(\rho_0) < +\infty$. The case $\overline{\mathrm{co}} f_0(\rho_0) = +\infty$ is treated similarly.

By the representation (4) there is a continuous affine function α on the set $\mathfrak{S}(\mathcal{H})$ such that

$$\alpha(\rho) \leqslant f_0(\rho) \quad \forall\, \rho \in \mathfrak{S}(\mathcal{H}), \qquad \overline{\mathrm{co}} f_0(\rho_0) \leqslant \alpha(\rho_0) + \frac{1}{4}\Delta. \tag{11}$$

Let N be such that $|\alpha(\rho_n) - \alpha(\rho_0)| < \frac{1}{4}\Delta$ for all $n \geqslant N$.

By Proposition 1, for every n there is a measure $\mu_n \in \mathcal{P}_{\{\rho_n\}}$ such that $\overline{\mathrm{co}} f_n(\rho_n) = \mu_n(f_n)$. Since the function α is affine, we have

$$\begin{aligned}\mu_n(\alpha) - \mu_n(f_n) &= \alpha(\rho_n) - \overline{\mathrm{co}} f_n(\rho_n) \\ &= [\alpha(\rho_n) - \alpha(\rho_0)] + [\alpha(\rho_0) - \overline{\mathrm{co}} f_0(\rho_0)] + [\overline{\mathrm{co}} f_0(\rho_0) - \overline{\mathrm{co}} f_n(\rho_n)] \\ &\geqslant -\frac{1}{4}\Delta - \frac{1}{4}\Delta + \Delta = \frac{1}{2}\Delta \qquad \forall\, n \geqslant N.\end{aligned} \tag{12}$$

The μ-compactness of $\mathfrak{S}(\mathcal{H})$ implies that the sequence $\{\mu_n\}$ is relatively compact. By Prokhorov's theorem (see [24], § 6) this sequence is *tight*: for every $\varepsilon > 0$ there is a compact subset $\mathcal{K}_\varepsilon \subset \mathfrak{S}(\mathcal{H})$ such that $\mu_n(\mathfrak{S}(\mathcal{H}) \setminus \mathcal{K}_\varepsilon) < \varepsilon$ for all n.

Put $M = \sup_{\rho \in \mathfrak{S}(\mathcal{H})} |\alpha(\rho)|$ and $\varepsilon_0 = \frac{\Delta}{4M}$. By (12), for all $n \geqslant N$ we have

$$\int_{\mathcal{K}_{\varepsilon_0}} (\alpha(\rho) - f_n(\rho))\mu_n(d\rho) \geqslant \frac{1}{2}\Delta - \int_{\mathfrak{S}(\mathcal{H}) \setminus \mathcal{K}_{\varepsilon_0}} (\alpha(\rho) - f_n(\rho))\mu_n(d\rho) \geqslant \frac{1}{4}\Delta.$$

Hence the set $\mathcal{C}_n = \{\rho \in \mathcal{K}_{\varepsilon_0} \mid \alpha(\rho) \geqslant f_n(\rho) + \frac{1}{4}\Delta\}$ is non-empty for all $n \geqslant N$.

Since the sequence $\{f_n\}$ is increasing, the sequence $\{\mathcal{C}_n\}$ of *closed* subsets of the *compact* set $\mathcal{K}_{\varepsilon_0}$ is monotone: $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n \ \forall\, n$. Hence there exists $\rho_* \in \bigcap_n \mathcal{C}_n$. This means that $\alpha(\rho_*) \geqslant f_n(\rho_*) + \frac{1}{4}\Delta$ for all n, and hence $\alpha(\rho_*) > f_0(\rho_*)$, contrary to (11).

Corollary 3. *The following inequality holds for any increasing sequence $\{f_n\}$ of lower semicontinuous lower-bounded functions on* $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ *and any convergent sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$:*

$$\liminf_{n \to +\infty} (f_n)_*^\mu(\rho_n) \geqslant (f_0)_*^\mu(\rho_0),$$

where $f_0 = \lim_{n\to+\infty} f_n$ *and* $\rho_0 = \lim_{n\to+\infty} \rho_n$. *In particular,*

$$\lim_{n\to+\infty} (f_n)_*^\mu(\rho) = (f_0)_*^\mu(\rho) \qquad \forall\, \rho \in \mathfrak{S}(\mathcal{H}).$$

Proof. By Theorems 1 and 2 in [13], if f is any lower semicontinuous lower-bounded function on $\operatorname{extr}\mathfrak{S}(\mathcal{H})$, then the function

$$f^*(\rho) \doteq \sup_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}} \int_{\operatorname{extr}\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma) = \sup_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \qquad \rho \in \mathfrak{S}(\mathcal{H}),$$

is a lower semicontinuous lower-bounded concave extension of f to $\mathfrak{S}(\mathcal{H})$. It is clear that, given any increasing sequence $\{f_n\}$ of lower semicontinuous lower-bounded functions on $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ converging pointwise to f_0, the corresponding increasing sequence $\{f_n^*\}$ converges pointwise to f_0^* on $\mathfrak{S}(\mathcal{H})$. Thus the corollary can be derived from Proposition 6 using Propositions 1 and 5.

Remark 1. The μ-convex roof cannot be replaced by the σ-convex roof in Corollary 3. Indeed, let f be the indicator function of the set $\operatorname{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})\setminus\mathcal{A}_s$ and let ω_0 be the separable state considered in Example 2. This function f can be represented as the limit of an increasing sequence $\{f_n\}$ of continuous bounded functions on $\operatorname{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})$. Since we have $(f_n)_*^\sigma = (f_n)_*^\mu$ for all n by Corollary 2, it follows from Corollary 3 and the properties of ω_0 that $\lim_{n\to+\infty}(f_n)_*^\sigma(\omega_0) = (f_0)_*^\mu(\omega_0) = 0$ and $(f_0)_*^\sigma(\omega_0) = 1$.

Remark 2. The monotone convergence theorem yields the following results dual to the second assertions of Proposition 6 and Corollary 3.

1) *For any decreasing sequence* $\{f_n\}$ *of upper-bounded Borel functions on* $\mathfrak{S}(\mathcal{H})$ *we have*

$$\lim_{n\to+\infty} \mu\text{-co}\, f_n(\rho) = \mu\text{-co}\, f_0(\rho), \quad \forall\, \rho \in \mathfrak{S}(\mathcal{H}), \qquad \textit{where} \quad f_0 = \lim_{n\to+\infty} f_n.$$

2) *For any decreasing sequence* $\{f_n\}$ *of upper-bounded Borel functions on* $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ *we have*

$$\lim_{n\to+\infty} (f_n)_*^\mu(\rho) = (f_0)_*^\mu(\rho), \qquad \forall\, \rho \in \mathfrak{S}(\mathcal{H}), \qquad \textit{where} \quad f_0 = \lim_{n\to+\infty} f_n.$$

The following result is easily proved using Corollary 1, Proposition 6, the first assertion of Remark 2 and Dini's lemma.

Corollary 4. *Let* $\{f_t\}_{t\in\mathrm{T}\subseteq\mathbb{R}}$ *be a family of continuous bounded functions on* $\mathfrak{S}(\mathcal{H})$ *such that*

1) $f_{t_1}(\rho) \leqslant f_{t_2}(\rho)$ *for all* $\rho \in \mathfrak{S}(\mathcal{H})$ *and all* $t_1, t_2 \in \mathrm{T}$ *with* $t_1 < t_2$,

2) *the function* $\mathrm{T} \ni t \mapsto f_t(\rho)$ *is continuous for all* $\rho \in \mathfrak{S}(\mathcal{H})$.

Then the function $\mathfrak{S}(\mathcal{H}) \times \mathrm{T} \ni (\rho, t) \mapsto \operatorname{co} f_t(\rho)$ *is continuous.*

One can prove an analogous result for the μ-convex roof of a family of continuous bounded functions on $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ using Corollary 2, Corollary 3, the second assertion of Remark 2 and Dini's lemma.

§ 3. The main theorem

Let α be a lower semicontinuous affine function on $\mathfrak{S}(\mathcal{H})$ with values in $[0,+\infty]$. Consider the family of closed subsets

$$\mathcal{A}_c = \{\rho \in \mathfrak{S}(\mathcal{H}) \mid \alpha(\rho) \leqslant c\}, \qquad c \in \mathbb{R}_+, \tag{13}$$

of the set $\mathfrak{S}(\mathcal{H})$. The following theorem describes the properties of the restrictions of the convex hulls of a given function to the subsets in this family.

Theorem 1. *Let f be a lower-bounded Borel function on $\mathfrak{S}(\mathcal{H})$ and α an affine function as above. If f has an upper semicontinuous bounded restriction to $\mathcal{A}_c$ for every $c > 0$ and*

$$\limsup_{c\to+\infty} c^{-1} \sup_{\rho\in\mathcal{A}_c} f(\rho) < +\infty, \tag{14}$$

then

$$\mathrm{co}\, f(\rho) = \sigma\text{-}\mathrm{co}\, f(\rho) = \mu\text{-}\mathrm{co}\, f(\rho)$$

for all $\rho \in \bigcup_{c>0} \mathcal{A}_c$, and the common restriction of these functions to $\mathcal{A}_c$ is upper semicontinuous for every $c > 0$.

If, in addition, f is lower semicontinuous on $\mathfrak{S}(\mathcal{H})$, then

$$\mathrm{co}\, f(\rho) = \sigma\text{-}\mathrm{co}\, f(\rho) = \mu\text{-}\mathrm{co}\, f(\rho) = \overline{\mathrm{co}} f(\rho)$$

for all $\rho \in \bigcup_{c>0} \mathcal{A}_c$ and the common restriction of these functions to the set $\mathcal{A}_c$ is continuous for every $c > 0$.

Proof. We can assume without loss of generality that f is a non-negative function.

Let ρ_0 be a state such that $\alpha(\rho_0) = c_0 < +\infty$. By hypothesis, $\mu\text{-}\mathrm{co}\, f(\rho_0) \leqslant f(\rho_0) < +\infty$. For arbitrary $\varepsilon > 0$ let μ_0 be a measure in $\mathcal{P}_{\{\rho_0\}}$ such that

$$\int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_0(d\rho) < \mu\text{-}\mathrm{co}\, f(\rho_0) + \varepsilon.$$

The condition (14) implies that there are positive numbers c_* and M such that $f(\rho) \leqslant M\alpha(\rho)$ for all $\rho \in \mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_{c_*}$.

Note that $\lim_{c\to+\infty} \mu_0(\mathcal{A}_c) = 1$. Indeed, the inequality

$$c\mu_0(\mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_c) \leqslant \int_{\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho) + \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho) = \alpha(\rho_0) = c_0,$$

obtained using Corollary A-1 in the Appendix, implies that

$$\mu_0(\mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_c) \leqslant \frac{c_0}{c}.$$

Thus the monotone convergence theorem implies that

$$\lim_{c\to+\infty} \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho) = \lim_{c\to+\infty} \biggl(\alpha(\rho_0) - \int_{\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho)\biggr) = 0.$$

Let $c^* > c_*$ be such that $\int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_{c^*}} \alpha(\rho)\mu_0(d\rho) < \varepsilon$. By Lemma 3 below there is a sequence $\{\mu_n\}$ of measures in $\mathcal{P}^{\mathrm{f}}_{\{\rho_0\}}$ converging weakly to μ_0 such that $\mu_n(\mathcal{A}_{c^*}) = \mu_0(\mathcal{A}_{c^*})$ and $\int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_{c^*}} \alpha(\rho)\mu_n(d\rho) < \varepsilon$ for all n. Since f is upper semicontinuous and bounded on $\mathcal{A}_{c^*}$, we have (see [24], § 2)

$$\limsup_{n\to+\infty} \int_{\mathcal{A}_{c^*}} f(\rho)\mu_n(d\rho) \leqslant \int_{\mathcal{A}_{c^*}} f(\rho)\mu_0(d\rho).$$

Hence, by noting that

$$\int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_{c^*}} f(\rho)\mu_n(d\rho) \leqslant M \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_{c^*}} \alpha(\rho)\mu_n(d\rho) < M\varepsilon, \qquad n = 0, 1, 2, \dots,$$

we obtain

$$\begin{aligned} \operatorname{co} f(\rho_0) &\leqslant \liminf_{n\to+\infty} \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_n(d\rho) \leqslant \limsup_{n\to+\infty} \int_{\mathcal{A}_{c^*}} f(\rho)\mu_n(d\rho) + M\varepsilon \\ &\leqslant \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_0(d\rho) + M\varepsilon \leqslant \mu\text{-}\operatorname{co} f(\rho_0) + \varepsilon(M+1). \end{aligned}$$

Since ε is arbitrary, it follows that $\operatorname{co} f(\rho_0) = \mu\text{-}\operatorname{co} f(\rho_0)$.

The proof of the first assertion of the theorem is completed by applying Lemma 4 below. The second follows from the first by Proposition 1.

Lemma 3. *Let α be a lower semicontinuous affine function on $\mathfrak{S}(\mathcal{H})$ with values in $[0, +\infty]$ and let μ_0 be a measure in $\mathcal{P}$ such that $\alpha(\bar{\rho}(\mu_0)) < +\infty$. For every $c > 0$ there is a sequence $\{\mu_n\}$ of measures in $\mathcal{P}^{\mathrm{f}}_{\{\bar{\rho}(\mu_0)\}}$ converging to μ_0 such that*

$$\mu_n(\mathcal{A}_c) = \mu_0(\mathcal{A}_c), \qquad \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_c} \alpha(\rho)\mu_n(d\rho) = \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho)$$

for all n, where $\mathcal{A}_c$ is the subset of $\mathfrak{S}(\mathcal{H})$ defined by (13).

Proof. This lemma can be proved using a simple modification of the proof of Lemma 1 in [4]. Namely, for every n we find a decomposition $\{\mathcal{A}_i^n\}_{i=1}^{m+2}$ of the set $\mathfrak{S}(\mathcal{H})$ into $m+2$ $(m = m(n))$ disjoint Borel subsets such that

1) the set $\mathcal{A}_i^n$ has diameter $< 1/n$ for $i = 1, \dots, m$;

2) $\mu_0(\mathcal{A}_{m+1}^n) < 1/n$ and $\mu_0(\mathcal{A}_{m+2}^n) < 1/n$;

3) the set $\mathcal{A}_i^n$ is contained in either $\mathcal{A}_c$ or $\mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_c$ for $i = 1, \dots, m+2$.

An important role in this construction is played by the implication

$$\mathcal{A} \subseteq \mathcal{B} \;\Rightarrow\; (\mu_0(\mathcal{A}))^{-1} \int_{\mathcal{A}} \rho\mu_0(d\rho) \in \mathcal{B}, \qquad \mathcal{B} = \mathcal{A}_c, \quad \mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_c$$

and the equality

$$\int_{\mathcal{A}} \alpha(\rho)\mu_0(d\rho) = \mu_0(\mathcal{A})\, \alpha\biggl(\frac{1}{\mu_0(\mathcal{A})} \int_{\mathcal{A}} \rho\mu_0(d\rho)\biggr), \qquad \mathcal{A} \subseteq \mathfrak{S}(\mathcal{H}), \quad \mu_0(\mathcal{A}) \neq 0,$$

which are easily obtained using Corollary A-1 in the Appendix. The lemma is proved.

The following lemma enables us to use the stability of $\mathfrak{S}(\mathcal{H})$ in the proof of Theorem 1.

Lemma 4. *Let α be a lower semicontinuous affine function on $\mathfrak{S}(\mathcal{H})$ with values in $[0,+\infty]$ and let f be a function on $\mathfrak{S}(\mathcal{H})$ having an upper semicontinuous restriction to the set $\mathcal{A}_c$ defined by (13) for each $c>0$. Then the function $\operatorname{co} f$ has an upper semicontinuous restriction to the set $\mathcal{A}_c$ for each $c>0$.*

Proof. Take $\rho_0 \in \mathcal{A}_{c_0}$ and let $\{\rho_n\} \subset \mathcal{A}_{c_0}$ be an arbitrary sequence converging to ρ_0. Suppose that we have

$$\lim_{n\to+\infty} \operatorname{co} f(\rho_n) > \operatorname{co} f(\rho_0). \tag{15}$$

Given any $\varepsilon>0$, let $\{\pi_i^0,\rho_i^0\}_{i=1}^m$ be an ensemble in $\mathcal{P}^{\mathrm{f}}_{\{\rho_0\}}$ such that $\sum_{i=1}^m \pi_i^0 f(\rho_i^0) < \operatorname{co} f(\rho_0)+\varepsilon$. Since $\mathfrak{S}(\mathcal{H})$ is stable (see [5]), there is a sequence $\{\{\pi_i^n,\rho_i^n\}_{i=1}^m\}_n$ of ensembles such that $\sum_{i=1}^m \pi_i^n\rho_i^n = \rho_n$ for each n, $\lim_{n\to+\infty}\pi_i^n = \pi_i^0$ and $\lim_{n\to+\infty}\rho_i^n = \rho_i^0$ for all $i=1,\ldots,m$. Put $\pi_* = \min_{1\leqslant i\leqslant m}\pi_i^0$. Then there is N such that $\pi_i^n \geqslant \pi_*/2$ for all $n\geqslant N$ and $i=1,\ldots,m$. It follows from the inequality $\sum_{i=1}^m \pi_i^n\alpha(\rho_i^n) = \alpha(\rho_n)\leqslant c_0$ that $\rho_i^n\in\mathcal{A}_{2c_0/\pi_*}$ for all $n\geqslant N$ and $i=1,\ldots,m$. Since f is upper semicontinuous on $\mathcal{A}_{2c_0/\pi_*}$, we have

$$\limsup_{n\to+\infty}\operatorname{co} f(\rho_n)\leqslant \limsup_{n\to+\infty}\sum_{i=1}^m \pi_i^n f(\rho_i^n)\leqslant \sum_{i=1}^m \pi_i^0 f(\rho_i^0) < \operatorname{co} f(\rho_0)+\varepsilon.$$

This contradicts (15) since ε is arbitrary.

Remark 3. If f is a concave function, then the condition (14) follows from the boundedness of the restriction of f to the set $\mathcal{A}_c$ for each c. Indeed, for any nonnegative lower semicontinuous affine function α, the concavity of f on $\mathfrak{S}(\mathcal{H})$ implies that the function $c\mapsto \sup_{\rho\in\mathcal{A}_c} f(\rho)$ is concave on $\mathbb{R}_+$. Since this function is finite, we see that (14) holds.

By Remark 3, we obtain the following result from Theorem 1, Lemma 2 and Proposition 5.

Corollary 5. *Let f be a concave lower semicontinuous lower-bounded function on $\mathfrak{S}(\mathcal{H})$ and let α be a lower semicontinuous affine function on $\mathfrak{S}(\mathcal{H})$ with values in $[0,+\infty]$. If f has a continuous restriction to the set $\mathcal{A}_c$ defined by (13) for each $c>0$, then*

$$\operatorname{co} f(\rho) = \sigma\text{-}\operatorname{co} f(\rho) = \mu\text{-}\operatorname{co} f(\rho) = \overline{\operatorname{co}} f(\rho) = (f|_{\operatorname{extr}\mathfrak{S}(\mathcal{H})})^{\sigma}_*(\rho) = (f|_{\operatorname{extr}\mathfrak{S}(\mathcal{H})})^{\mu}_*(\rho)$$

for all $\rho\in\bigcup_{c>0}\mathcal{A}_c$ and the common restriction of these functions to $\mathcal{A}_c$ is continuous for each $c>0$.

Theorem 1 yields the following sufficient conditions for the coincidence and continuity of convex hulls.

Corollary 6. *Let f be a lower-bounded Borel function on $\mathfrak{S}(\mathcal{H})$ and ρ_0 an arbitrary state in $\mathfrak{S}(\mathcal{H})$. Suppose that there is an affine lower semicontinuous function α on $\mathfrak{S}(\mathcal{H})$ with values in $[0,+\infty]$ such that $\alpha(\rho_0)<+\infty$, f has an upper* *semicontinuous bounded restriction to the set* $\mathcal{A}_c$ *defined by* (13) *for each* $c > 0$, *and condition* (14) *holds. Then*

$$\operatorname{co} f(\rho_0) = \sigma\text{-}\operatorname{co} f(\rho_0) = \mu\text{-}\operatorname{co} f(\rho_0).$$

Corollary 7. *Let* f *be a lower semicontinuous lower-bounded function on* $\mathfrak{S}(\mathcal{H})$ *and* $\{\rho_n\}$ *an arbitrary sequence of states in* $\mathfrak{S}(\mathcal{H})$ *converging to a state* ρ_0. *Suppose that there is an affine lower semicontinuous function* α *on the set* $\mathfrak{S}(\mathcal{H})$ *with values in* $[0, +\infty]$ *such that* $\sup_n \alpha(\rho_n) < +\infty$, f *has a continuous bounded restriction to the set* $\mathcal{A}_c$ *defined by* (13) *for each* $c > 0$, *and condition* (14) *holds. Then*

$$\operatorname{co} f(\rho_n) = \sigma\text{-}\operatorname{co} f(\rho_n) = \mu\text{-}\operatorname{co} f(\rho_n) = \overline{\operatorname{co}} f(\rho_n), \qquad n = 0, 1, 2, \dots, \tag{16}$$

$$\lim_{n \to +\infty} \operatorname{co} f(\rho_n) = \operatorname{co} f(\rho_0). \tag{17}$$

Remark 4. If f is a concave function, then condition (14) in Corollaries 6 and 7 can be omitted by Remark 3.

Example 4. An important role in the study of the informational properties of a quantum channel is played by the output Rényi entropy, in particular, the output von Neumann entropy, and their convex closures [25].

Let $\Phi\colon \mathfrak{T}(\mathcal{H}) \mapsto \mathfrak{T}(\mathcal{H}')$ be a quantum channel (a linear completely positive trace-preserving map; see [9], § 3.1) and let $\mathfrak{S}(\mathcal{H}) \ni \rho \mapsto (R_p \circ \Phi)(\rho) = \frac{\log \operatorname{Tr} \Phi(\rho)^p}{1-p}$ be the output Rényi entropy of this channel of order $p \in (0, +\infty]$ (the case $p = 1$ corresponds to the output von Neumann entropy $-\operatorname{Tr} \Phi(\rho) \log \Phi(\rho)$ and the case $p = +\infty$ to the function $-\log \lambda_{\max}(\Phi(\rho))$, where $\lambda_{\max}(\Phi(\rho))$ is the maximal eigenvalue of the state $\Phi(\rho)$). For $p \in (0, 1]$ the function $R_p \circ \Phi$ is lower semicontinuous and concave and takes values in $[0, +\infty]$, while for $p \in (1, +\infty]$ it is continuous and finite but not concave. The output von Neumann entropy $H \circ \Phi = R_1 \circ \Phi$ is the supremum (pointwise limit as $p \to 1 + 0$) of the monotone family $\{R_p \circ \Phi\}_{p>1}$ of continuous functions. By Proposition 6, the convex closure $\overline{\operatorname{co}}(H \circ \Phi)$ of the output von Neumann entropy coincides with the supremum (pointwise limit as $p \to 1 + 0$) of the monotone family of functions $\{\overline{\operatorname{co}}(R_p \circ \Phi)\}_{p>1}$.

Corollary 6 enables us to show that

$$\begin{gathered} \operatorname{co}(R_p \circ \Phi)(\rho_0) = \sigma\text{-}\operatorname{co}(R_p \circ \Phi)(\rho_0) = \mu\text{-}\operatorname{co}(R_p \circ \Phi)(\rho_0) = \overline{\operatorname{co}}(R_p \circ \Phi)(\rho_0) \\ \forall\, p \in [1, +\infty] \end{gathered} \tag{18}$$

for any state ρ_0 with $(H \circ \Phi)(\rho_0) < +\infty$. Indeed, the condition $H(\Phi(\rho_0)) < +\infty$ implies that there is an $\mathfrak{H}$-operator H' in $\mathcal{H}'$ such that

$$\mathrm{g}(H') = \inf\{\lambda > 0 \mid \operatorname{Tr} \exp(-\lambda H') < +\infty\} < +\infty$$

and $\operatorname{Tr} H' \Phi(\rho_0) < +\infty$. By Proposition 1 in [26], the hypotheses of Corollary 6 hold for the function $f(\rho) = (R_p \circ \Phi)(\rho) \leqslant (H \circ \Phi)(\rho)$ with $p \in [1, +\infty]$ provided that $\alpha(\rho) = \operatorname{Tr} H' \Phi(\rho)$. Note that if $(H \circ \Phi)(\rho_0) = +\infty$, then (18) need not hold (see [25], Proposition 7).

By Corollary 1, it follows from the coincidence (proved above) of the convex hulls and the continuity of the Rényi entropy for $p>1$ that the function $\operatorname{co}(R_p\circ\Phi)$ is continuous on the convex subset $\{\rho\in\mathfrak{S}(\mathcal{H})\mid (H\circ\Phi)(\rho)<+\infty\}$ for $p>1$.

If the output von Neumann entropy $H\circ\Phi$ is continuous on some set $\mathcal{A}\subseteq\mathfrak{S}(\mathcal{H})$, then Theorem 1 in [25] yields that its convex closure $\overline{\operatorname{co}}(H\circ\Phi)$ is also continuous and coincides with the convex hull $\operatorname{co}(H\circ\Phi)$ on that set. If $\mathcal{A}$ is compact, then the continuity (proved above) of the function $\operatorname{co}(R_p\circ\Phi)$ and Dini's lemma imply that the continuous functions $\operatorname{co}(R_p\circ\Phi)|_{\mathcal{A}}=\overline{\operatorname{co}}(R_p\circ\Phi)|_{\mathcal{A}}$ converge uniformly to the continuous function $\operatorname{co}(H\circ\Phi)|_{\mathcal{A}}=\overline{\operatorname{co}}(H\circ\Phi)|_{\mathcal{A}}$ as $p\to1+0$. This shows, in particular, that the Holevo capacity[9] of the $\mathcal{A}$-constrained channel Φ (see [4]) satisfies

$$\overline{C}(\Phi,\mathcal{A})=\lim_{p\to1+0}\sup_{\rho\in\mathcal{A}}\bigl((R_p\circ\Phi)(\rho)-\operatorname{co}(R_p\circ\Phi)(\rho)\bigr).$$

This formula can be used to approximate the Holevo capacity (since the Rényi entropy for $p>1$ is often more 'computable' than the von Neumann entropy) and analyze the continuity of the Holevo capacity as a function of the channel (since the Rényi entropy is continuous for $p>1$).

§ 4. Entanglement monotones

4.1. Basic properties. Entanglement is an essential feature of quantum systems. It may be regarded as a special quantum correlation having no classical analogue. It is this property that provides a base for the construction of various quantum algorithms and cryptographic protocols (see [6], Ch. 3). One of the basic problems in the theory of entanglement is to find appropriate quantitative characteristics of the entanglement of a state in composite systems and study their properties (see [3], [27] and the references therein). Entanglement monotones form an important class of such characteristics [2]. In this section we consider an infinite-dimensional generalization of the 'convex roof construction' of entanglement monotones and investigate its properties. This generalization is based on results appearing in the previous sections.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. A state $\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ is said to be *separable* or *non-entangled* if it belongs to the convex closure of the set of all pure product states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. Otherwise it is said to be *entangled*.

A key role in entanglement theory is played by the notion of a LOCC-*operation* in a composite quantum system. This is a composite of Local Operations on each of the subsystems and Classical Communications between these subsystems [3], [27]. The action of a *selective* LOCC-operation on any state of the composite system results in an *ensemble*, that is, a set of states of this system along with the corresponding probability distribution (which is generally a probability measure on that set of states). A typical example of a selective LOCC-operation is a quantum measurement on one of the subsystems, which 'transforms' an arbitrary *a priori* state to the set of *a posteriori* states corresponding to the outcomes of the measurement along with the probability distribution of these outcomes ([9], Ch. 2). Averaging the output ensemble of a selective LOCC-operation gives the corresponding *non-selective*

[9] This is closely related to the classical capacity of a quantum channel (see [6], Ch. 5).

LOCC-operation. Thus the action of a non-selective LOCC-operation on any state of a composite system results in a particular *state* of that system. In the example above, this averaging corresponds to a quantum measurement in which the result of the measurement is ignored (but the measured state may be changed).

An *entanglement monotone* is any non-negative function E on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ having the following two properties (see [2], [3]).

EM-1) $\{E(\omega)=0\} \Leftrightarrow \{$*the state* ω *is separable*$\}$.

EM-2a) E *is monotone under non-selective* LOCC-*operations.* This means that

$$E(\omega) \geqslant E\left(\sum_i \pi_i\omega_i\right) \tag{19}$$

for any state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ and any LOCC-operation mapping ω to the finite or countable ensemble $\{\pi_i,\omega_i\}$.

This requirement is often strengthened as follows.

EM-2b) E *is monotone under selective* LOCC-*operations.* This means that

$$E(\omega) \geqslant \sum_i \pi_i E(\omega_i) \tag{20}$$

for any state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ and any LOCC-operation mapping ω to the finite or countable ensemble $\{\pi_i,\omega_i\}$.

In infinite dimensions the last requirement is naturally generalized as follows.

EM-2c) E *is monotone under generalized selective* LOCC-*operations.* This means that for every state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ and any local instrument[10] $\mathfrak{M}$ with set $\mathcal{X}$ of outcomes, the function $x \mapsto E(\sigma(x|\,\omega))$ is μ_ω-measurable on $\mathcal{X}$ and

$$E(\omega) \geqslant \int_{\mathcal{X}} E(\sigma(x|\,\omega))\mu_\omega(dx), \tag{21}$$

where $\mu_\omega(\cdot) = \operatorname{Tr}\mathfrak{M}(\cdot)[\omega]$ and $\{\sigma(x|\,\omega)\}_{x\in\mathcal{X}}$ are respectively the probability measure on $\mathcal{X}$ describing the results of the measurement and the family of *a posteriori* states corresponding to the *a priori* state ω; see [9], [28].

Remark 5. By definition, the function $x \mapsto \sigma(x|\,\omega)$ is μ_ω-measurable with respect to the minimal σ-algebra on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ for which all the linear functionals $\omega \mapsto \operatorname{Tr} A\omega$, $A \in \mathfrak{B}(\mathcal{H}\otimes\mathcal{K})$, are measurable. By Corollary 1 in [29], this σ-algebra coincides with the Borel σ-algebra on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. Thus the function $x \mapsto \sigma(x|\,\omega)$ is μ_ω-measurable with respect to the Borel σ-algebra on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ and, therefore, the function $x \mapsto E(\sigma(x|\,\omega))$ is μ_ω-measurable for any Borel function $\omega \mapsto E(\omega)$.

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[10] An *instrument* in the set $\mathfrak{S}(\mathcal{H})$ of states with a measurable space $\mathcal{X}$ of outcomes is a set function $\mathfrak{M}$ defined on the σ-algebra $\mathfrak{B}(\mathcal{X})$ and satisfying the following conditions (see [9], Ch. 4): $\mathfrak{M}(B)$ is a linear completely positive trace-non-increasing transformation of the space $\mathfrak{T}(\mathcal{H})$ for any $B \in \mathfrak{B}(\mathcal{X})$; $\mathfrak{M}(\mathcal{X})$ is a trace-preserving transformation; if $\{B_j\} \subset \mathfrak{B}(\mathcal{X})$ is a finite or countable disjoint decomposition of $B \in \mathfrak{B}(\mathcal{X})$, then $\mathfrak{M}(B)[T] = \sum_j \mathfrak{M}(B_j)[T]$, $T \in \mathfrak{T}(\mathcal{H})$, where the series converges in the norm of $\mathfrak{T}(\mathcal{H})$.

According to [3], an entanglement monotone E is called an *entanglement measure* if $E(\omega)=H(\mathrm{Tr}_{\mathcal{K}}\,\omega)$ for any pure state ω in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, where H is the von Neumann entropy.

The following requirement is sometimes included in the definition of an entanglement monotone (see [27]).

EM-3a) *E is convex on the set* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. This means that

$$E\biggl(\sum_i \pi_i\omega_i\biggr)\leqslant\sum_i \pi_i E(\omega_i)$$

for any finite ensemble $\{\pi_i,\omega_i\}$ of states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.

This requirement comes from the observation that one cannot increase the entanglement by taking convex mixtures (which describe the classical noise in the preparation of a quantum state).

The following stronger forms of the convexity requirement are motivated by the need to consider countable and continuous ensembles of states when dealing with infinite-dimensional quantum systems (see [4]).

EM-3b) *E is σ-convex on* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. This means that

$$E\biggl(\sum_i \pi_i\omega_i\biggr)\leqslant\sum_i \pi_i E(\omega_i)$$

for any countable ensemble $\{\pi_i,\omega_i\}$ of states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.

If this requirement holds, then EM-2b $\Rightarrow$ EM-2a.

EM-3c) *E is μ-convex on* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. This means that E is universally measurable and

$$E\biggl(\int_{\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})}\omega\mu(d\omega)\biggr)\leqslant\int_{\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})}E(\omega)\mu(d\omega)$$

for any Borel probability measure μ on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. One may regard μ as a generalized (continuous) ensemble of states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.

It is shown in § 2 that these convexity properties are generally inequivalent.

EM-4) *E is subadditive.* This means that

$$E(\omega_1\otimes\omega_2)\leqslant E(\omega_1)+E(\omega_2) \tag{22}$$

for any states $\omega_1\in\mathfrak{S}(\mathcal{H}_1\otimes\mathcal{K}_1)$ and $\omega_2\in\mathfrak{S}(\mathcal{H}_2\otimes\mathcal{K}_2)$.

This property guarantees the existence of a regularization

$$E^*(\omega)=\lim_{n\to+\infty}\frac{E(\omega^{\otimes n})}{n},\qquad \omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}).$$

In the finite-dimensional case it is natural to require that any entanglement monotone E be continuous on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. In infinite dimensions this requirement is very restrictive. Moreover, the discontinuity of the von Neumann entropy implies that any entanglement measure on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ is discontinuous in this case. Nevertheless, some weaker continuity requirements may be considered.

EM-5a) *E is lower semicontinuous on* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. This means that

$$\liminf_{n\to+\infty} E(\omega_n) \geqslant E(\omega_0)$$

for any sequence $\{\omega_n\}$ of states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ converging to a state ω_0 or, equivalently, that the set of states defined by the inequality $E(\omega)\leqslant c$ is closed for any $c>0$. This requirement is motivated by the natural physical observation that one cannot increase an entanglement by an approximation procedure. It is essential that the lower semicontinuity of E guarantees that this function is Borel and that the requirements EM-3a – EM-3c are equivalent for this function (by Proposition A-2 in the Appendix).

From the physical point of view it is natural to require that entanglement monotones must be continuous on the set of states produced in a physical experiment. This leads to the following requirement.

EM-5b) *E is continuous on subsets of* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ *with bounded mean energy*. Let $H_{\mathcal{H}}$ and $H_{\mathcal{K}}$ be the Hamiltonians of the quantum systems associated with the spaces $\mathcal{H}$ and $\mathcal{K}$ respectively ([9], § 1.2). Then the Hamiltonian of the composite system has the form $H_{\mathcal{H}}\otimes I_{\mathcal{K}}+I_{\mathcal{H}}\otimes H_{\mathcal{K}}$ and hence the set of states of the composite system whose mean energy does not exceed h is defined by the inequality

$$\mathrm{Tr}(H_{\mathcal{H}}\otimes I_{\mathcal{K}}+I_{\mathcal{H}}\otimes H_{\mathcal{K}})\omega\leqslant h.$$

Requirement EM-5b means that the restrictions of E to these subsets of $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ are continuous for all $h>0$.

The strongest continuity requirement is as follows.

EM-5c) *E is continuous on the set* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.

Despite the infinite-dimensionality, there is a non-trivial class of entanglement monotones for which this requirement holds (see Example 5 in the next subsection).

4.2. Generalized convex roof constructions. A general method of producing entanglement monotones in the finite-dimensional case is the 'convex roof construction' ([3], [27], [30]). This construction starts with a given concave continuous non-negative function f on $\mathfrak{S}(\mathcal{H})$ satisfying

$$f^{-1}(0)=\mathrm{extr}\,\mathfrak{S}(\mathcal{H}),\qquad f(\rho)=f(U\rho U^*) \tag{23}$$

for any state ρ in $\mathfrak{S}(\mathcal{H})$ and any unitary operator U in $\mathcal{H}$. The corresponding entanglement monotone E^f is defined as the convex roof $(f\circ\Theta|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})})_*$ of the restriction of the function $f\circ\Theta$ to the set $\mathrm{extr}\,\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, where $\Theta\colon\omega\mapsto\mathrm{Tr}_{\mathcal{K}}\,\omega$ is the partial trace. Taking the von Neumann entropy for f, we obtain one of the most important entanglement measures: the Entanglement of Formation E_F (see [7]).

In the infinite-dimensional case there are two possible generalizations of this construction: the σ-convex roof $(f\circ\Theta|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})})_*^{\sigma}$ and the μ-convex roof $(f\circ\Theta|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})})_*^{\mu}$ of the function $f\circ\Theta|_{\mathrm{extr}\,\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})}$. To simplify the notation, we omit the symbol of restriction and denote these functions by $(f\circ\Theta)_*^{\sigma}$ and $(f\circ\Theta)_*^{\mu}$ respectively.

The results of the previous sections enable us to prove the following assertions concerning the main properties of the generalized convex roof constructions.

Theorem 2. *Let f be a non-negative concave function on $\mathfrak{S}(\mathcal{H})$ satisfying the condition* (23).

A-1) *If f is upper semicontinuous, then*

$$(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}=\mu\text{-co}(f\circ\Theta)=\sigma\text{-co}(f\circ\Theta)=\text{co}(f\circ\Theta),$$

and the function $(f\circ\Theta)^{\mu}_{*}=(f\circ\Theta)^{\sigma}_{*}$ is upper semicontinuous and satisfies requirements EM-1, EM-2c *and* EM-3c.

A-2) *If f is lower semicontinuous, then the function $(f\circ\Theta)^{\sigma}_{*}$ satisfies requirements*[11] EM-2b *and* EM-3b *while the function $(f\circ\Theta)^{\mu}_{*}$ coincides with $\overline{\text{co}}(f\circ\Theta)$ and satisfies requirements* EM-1, EM-2c, EM-3c *and* EM-5a.

B) *If f is subadditive,*[12] *then the functions $(f\circ\Theta)^{\sigma}_{*}$ and $(f\circ\Theta)^{\mu}_{*}$ satisfy requirement* EM-4.

C) *Let $H_{\mathcal{H}}$ be a positive operator in the space $\mathcal{H}$. If f is lower semicontinuous and has a finite continuous restriction to the subset $\mathcal{K}_{H_{\mathcal{H}},h}=\{\rho\in\mathfrak{S}(\mathcal{H})\mid \operatorname{Tr}H_{\mathcal{H}}\rho\leqslant h\}$ for every $h>0$, then*

$$(f\circ\Theta)^{\mu}_{*}(\omega)=(f\circ\Theta)^{\sigma}_{*}(\omega)=\overline{\text{co}}(f\circ\Theta)(\omega)=\text{co}(f\circ\Theta)(\omega)\qquad\forall\,\omega\in\bigcup_{h>0}\mathcal{K}_{H_{\mathcal{H}}\otimes I_{\mathcal{K}},h},$$

where $\mathcal{K}_{H_{\mathcal{H}}\otimes I_{\mathcal{K}},h}=\{\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})\mid \operatorname{Tr}(H_{\mathcal{H}}\otimes I_{\mathcal{K}})\omega\leqslant h\}$, and the common restriction of these functions to the set $\mathcal{K}_{H_{\mathcal{H}}\otimes I_{\mathcal{K}},h}$ is continuous for every $h>0$. In particular, if $H_{\mathcal{H}}$ is the Hamiltonian of the quantum system associated with $\mathcal{H}$, then the functions $(f\circ\Theta)^{\mu}_{*}$ and $(f\circ\Theta)^{\sigma}_{*}$ satisfy requirement EM-5b.

D) *If f is continuous on $\mathfrak{S}(\mathcal{H})$, then*

$$(f\circ\Theta)^{\mu}_{*}=(f\circ\Theta)^{\sigma}_{*}=\overline{\text{co}}(f\circ\Theta)=\mu\text{-co}(f\circ\Theta)=\sigma\text{-co}(f\circ\Theta)=\text{co}(f\circ\Theta)$$

and the function $(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}$ satisfies requirement EM-5c.

Proof. A) By Lemma 2, the upper semicontinuity and concavity of f guarantees its boundedness while Proposition 5 implies that

$$(f\circ\Theta)^{\mu}_{*}=(f\circ\Theta)^{\sigma}_{*}=\mu\text{-co}(f\circ\Theta)=\sigma\text{-co}(f\circ\Theta)=\text{co}(f\circ\Theta)$$

and this function is upper semicontinuous. Proposition A-2 in the Appendix shows that requirement EM-3c holds for the function $(f\circ\Theta)^{\mu}_{*}=(f\circ\Theta)^{\sigma}_{*}$.

By Proposition 3, the lower semicontinuity of f implies that $(f\circ\Theta)^{\mu}_{*}$ is lower semicontinuous (so that requirement EM-5a holds). Hence Proposition A-2 in the Appendix shows that requirement EM-3c holds for the function $(f\circ\Theta)^{\mu}_{*}$.

Requirement EM-3b holds for the function $(f\circ\Theta)^{\sigma}_{*}$ by its definition.

Arguing as in the proof of the LOCC-monotonicity of the convex roof of $f\circ\Theta$ in the finite-dimensional case (see [3], [7]) and using Jensen's discrete inequality

[11] The example in Remark 6 below shows that $(f\circ\Theta)^{\sigma}_{*}$ may not satisfy requirements EM-1, EM-3c or EM-5a even for bounded lower semicontinuous functions f.

[12] This means that $f(\rho_1\otimes\rho_2)\leqslant f(\rho_1)+f(\rho_2)$ for any states $\rho_1\in\mathfrak{S}(\mathcal{H}_1)$ and $\rho_2\in\mathfrak{S}(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces (we make implicit use of the fact that all such spaces are isomorphic).

(Proposition A-1 in the Appendix), we see that requirement EM-2b holds for the function $(f \circ \Theta)^{\sigma}_{*}$.

Consider requirement EM-2c. Let $\mathfrak{M}$ be an arbitrary instrument acting in the subsystem associated with $\mathcal{K}$. If f is lower (resp. upper) semicontinuous, then the function $(f \circ \Theta)^{\mu}_{*}$ is lower (resp. upper) semicontinuous and hence it is Borel. By Remark 5, this guarantees the μ_{ω}-measurability of the function $x \mapsto E(\sigma(x|\,\omega))$ for any state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$.

Let ω be a pure state. By the local nature of the instrument $\mathfrak{M}$ we have

$$\Theta(\omega) = \int_{\mathcal{X}} \Theta(\sigma(x|\,\omega))\mu_{\omega}(dx).$$

Since f is non-negative, concave and lower or upper semicontinuous, Proposition A-2 in the Appendix implies that

$$f \circ \Theta(\omega) \geqslant \int_{\mathcal{X}} f \circ \Theta(\sigma(x|\,\omega))\mu_{\omega}(dx) \geqslant \int_{\mathcal{X}} (f \circ \Theta)^{\mu}_{*}(\sigma(x|\,\omega))\mu_{\omega}(dx),$$

where the last inequality follows from Proposition 5.

Let ω be a mixed state. We first prove that

$$(f \circ \Theta)^{\sigma}_{*}(\omega) \geqslant \int_{\mathcal{X}} (f \circ \Theta)^{\mu}_{*}(\sigma(x|\,\omega))\mu_{\omega}(dx). \tag{24}$$

For any given $\varepsilon > 0$, let $\{\pi_i, \omega_i\}$ be an ensemble in $\widehat{\mathcal{P}}_{\{\omega\}}(\mathfrak{S}(\mathcal{H} \otimes \mathcal{K}))$ such that

$$(f \circ \Theta)^{\sigma}_{*}(\omega) > \sum_i \pi_i f \circ \Theta(\omega_i) - \varepsilon.$$

By the above observation concerning a pure state ω, we have

$$(f \circ \Theta)^{\sigma}_{*}(\omega) > \sum_i \pi_i \int_{\mathcal{X}} (f \circ \Theta)^{\mu}_{*}(\sigma(x|\,\omega_i))\mu_{\omega_i}(dx) - \varepsilon. \tag{25}$$

By the Radon–Nicodým theorem, the decomposition

$$\mu_{\omega}(\cdot) = \mathrm{Tr}\ \mathfrak{M}(\cdot)[\omega] = \sum_i \pi_i \,\mathrm{Tr}\ \mathfrak{M}(\cdot)[\omega_i] = \sum_i \pi_i \mu_{\omega_i}(\cdot)$$

yields the existence of a family $\{p_i\}$ of μ_{ω}-measurable functions on $\mathcal{X}$ such that

$$\pi_i \mu_{\omega_i}(\mathcal{X}_0) = \int_{\mathcal{X}_0} p_i(x)\mu_{\omega}(dx)$$

for any μ_{ω}–measurable subset $\mathcal{X}_0 \subseteq \mathcal{X}$ and $\sum_i p_i(x) = 1$ for μ_{ω}-almost all x in $\mathcal{X}$. Since

$$\int_{\mathcal{X}_0} \sigma(x|\,\omega)\mu_{\omega}(dx) = \sum_i \pi_i \int_{\mathcal{X}_0} \sigma(x|\,\omega_i)\mu_{\omega_i}(dx) = \sum_i \int_{\mathcal{X}_0} \sigma(x|\,\omega_i)p_i(x)\mu_{\omega}(dx)$$

for any μ_ω-measurable subset $\mathcal{X}_0 \subseteq \mathcal{X}$, we have

$$\sum_i p_i(x)\sigma(x|\,\omega_i) = \sigma(x|\,\omega)$$

for μ_ω-almost all x in $\mathcal{X}$.

Note that the function $(f\circ\Theta)^\mu_*$ is σ-convex in both cases. Indeed, when f is upper semicontinuous, this follows from its coincidence with $(f\circ\Theta)^\sigma_*$. But when f is lower semicontinuous, the convex function $(f\circ\Theta)^\mu_*$ is lower semicontinuous and hence μ-convex (by Proposition A-2 in the Appendix).

Using (25) and the σ-convexity of $(f\circ\Theta)^\mu_*$, we obtain

$$\begin{aligned}(f\circ\Theta)^\sigma_*(\omega) &> \int_{\mathcal{X}}\sum_i p_i(x)(f\circ\Theta)^\mu_*(\sigma(x|\,\omega_i))\mu_\omega(dx)-\varepsilon\\ &\geqslant \int_{\mathcal{X}}(f\circ\Theta)^\mu_*(\sigma(x|\,\omega))\mu_\omega(dx)-\varepsilon,\end{aligned}$$

which proves (24) since ε is arbitrary.

If f is upper semicontinuous, then $(f\circ\Theta)^\sigma_* = (f\circ\Theta)^\mu_*$ and (24) is equivalent to (21) for the function $E=(f\circ\Theta)^\sigma_*=(f\circ\Theta)^\mu_*$.

If f is lower semicontinuous and $\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ is an arbitrary state, then Lemma 1 and Proposition 5 yield a sequence $\{\omega_n\}\subset\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ converging to ω such that

$$\lim_{n\to+\infty}(f\circ\Theta)^\sigma_*(\omega_n) = (f\circ\Theta)^\mu_*(\omega).$$

Inequality (21) for the function $E=(f\circ\Theta)^\mu_*$ can be proved by applying inequality (24) for each ω_n and passing to the limit as $n\to+\infty$ if we use Lemma A-1 in the Appendix and the lower semicontinuity of $(f\circ\Theta)^\mu_*$.

Consider requirement EM-1. Note that a state ω is separable if and only if there is a measure μ in $\widehat{\mathcal{P}}_{\{\omega\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$ supported by pure product states [14].

Let f be a lower semicontinuous function. By Proposition 3, for an arbitrary state ω in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ there is a measure μ_ω in $\widehat{\mathcal{P}}_{\{\omega\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$ such that $(f\circ\Theta)^\mu_*(\omega) = \int f\circ\Theta(\sigma)\mu_\omega(d\sigma)$. Hence requirement EM-1 holds for the function $(f\circ\Theta)^\mu_*$ by the above characterization of the set of separable states.

Let f be an upper semicontinuous function. Then the function $(f\circ\Theta)^\sigma_*=(f\circ\Theta)^\mu_*$ vanishes on the set of separable states by the above characterization of this set.

Suppose that this function vanishes at some entangled state ω_0. Then there is a local operation Λ such that the state $\Lambda(\omega_0)$ is entangled and has reduced states of finite rank. By the LOCC-monotonicity (proved above) of the function $(f\circ\Theta)^\sigma_*=(f\circ\Theta)^\mu_*$, this function vanishes at the entangled state $\Lambda(\omega_0)$.

Let $\mathcal{H}_0$ be the finite-dimensional support of the state $\mathrm{Tr}_{\mathcal{K}}\Lambda(\omega_0)$. Then the upper semicontinuous concave function f satisfying condition (23) has a continuous restriction to $\mathfrak{S}(\mathcal{H}_0)$. Indeed, the continuity of this restriction at any pure state in $\mathfrak{S}(\mathcal{H}_0)$ follows from the upper semicontinuity of the non-negative function f and condition (23), while the continuity at any mixed state in $\mathfrak{S}(\mathcal{H}_0)$ can easily be derived from the well-known fact that any concave bounded function is continuous at every interior point of a convex subset of a Banach space ([21], Proposition 3.2.3). Since

$$(f\circ\Theta|_{\mathfrak{S}(\mathcal{H}_0\otimes\mathcal{K})})^{\mu}_{*}=(f\circ\Theta)^{\mu}_{*}|_{\mathfrak{S}(\mathcal{H}_0\otimes\mathcal{K})},$$

we can apply the previous observation concerning lower semicontinuous functions f to show that the equality $(f\circ\Theta)^{\mu}_{*}(\Lambda(\omega_0))=0$ implies the separability of the state $\Lambda(\omega_0)$, contrary to assumption.

B) If f is subadditive, then so is $f\circ\Theta$. Take arbitrary measures $\mu_i\in\widehat{\mathcal{P}}_{\{\omega_i\}}(\mathfrak{S}(\mathcal{L}_i))$, where $\mathcal{L}_i=\mathcal{H}_i\otimes\mathcal{K}_i$, $i=1,2$. The set of product states in $\operatorname{extr}\mathfrak{S}(\mathcal{L}_1\otimes\mathcal{L}_2)$ may be regarded as the Cartesian product of the sets $\operatorname{extr}\mathfrak{S}(\mathcal{L}_1)$ and $\operatorname{extr}\mathfrak{S}(\mathcal{L}_2)$. Hence one can define the Cartesian product $\mu_1\otimes\mu_2$ of the measures μ_1 and μ_2 on this set and regard it as a measure in $\widehat{\mathcal{P}}_{\{\omega_1\otimes\omega_2\}}(\mathfrak{S}(\mathcal{L}_1\otimes\mathcal{L}_2))$ supported by the set of product states. Using this construction, one can easily prove that the function $(f\circ\Theta)^{\mu}_{*}$ is subadditive. The same argument with atomic measures μ_1 and μ_2 yields[13] that the function $(f\circ\Theta)^{\sigma}_{*}$ is subadditive.

C) If f is lower semicontinuous and satisfies the additional hypotheses in part C of the theorem, then $f\circ\Theta$ satisfies the hypotheses of Corollary 5 with an affine function $\alpha(\omega)=\operatorname{Tr}(H_{\mathcal{H}}\otimes I_{\mathcal{K}})\omega$.

D) Assertion D follows from Proposition 5.

Remark 6. The function $(f\circ\Theta)^{\sigma}_{*}$ may not satisfy the basic requirement EM-1 even for bounded lower semicontinuous f (see part A-2 of Theorem 2). Indeed, let f be the indicator function of the set of all mixed states in $\mathfrak{S}(\mathcal{H})$ and let ω_0 be a separable state such that any measure in $\widehat{\mathcal{P}}_{\{\omega_0\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$ has no atoms in the set of separable states [14]. Then it is easy to see that $(f\circ\Theta)^{\sigma}_{*}(\omega_0)=1$ (while $(f\circ\Theta)^{\mu}_{*}(\omega_0)=0$).

The function $(f\circ\Theta)^{\sigma}_{*}$ in Remark 6 does not satisfy requirements EM-3c, EM-5a. This is a general feature of all σ-convex roofs which do not coincide with the corresponding μ-convex roofs.

Remark 6 and Theorem 2 show that the function $(f\circ\Theta)^{\sigma}_{*}$ either coincides with the function $(f\circ\Theta)^{\mu}_{*}$ (if f is upper semicontinuous) or may not satisfy the basic requirement EM-1 of entanglement monotones (if f is lower semicontinuous). Thus the μ-convex roof construction seems to be the *preferable* candidate for the role of an infinite-dimensional generalization of the convex roof construction of entanglement monotones. Thus we will use the notation

$$E^{f}=(f\circ\Theta)^{\mu}_{*}$$

for any function f satisfying the hypotheses of Theorem 2.

Example 5. Generalizing an observation in [30] to the infinite-dimensional case, we consider the family of functions

$$f_{\alpha}(\rho)=2(1-\operatorname{Tr}\rho^{\alpha}),\qquad \alpha>1,$$

on the set $\mathfrak{S}(\mathcal{H})$ with $\dim\mathcal{H}=+\infty$. The functions of this family are non-negative, concave, continuous and satisfy conditions (23). By Theorem 2, $E^{f_{\alpha}}$ is an entanglement monotone satisfying requirements EM-1, EM-2c, EM-3c and EM-5c. In the

[13] In this case the measure $\mu_1\otimes\mu_2$ corresponds to the tensor product of the countable ensembles of pure states corresponding to μ_1 and μ_2.

case $\alpha = 2$, the entanglement monotone E^{f_2} may be regarded as the infinite-dimensional generalization of the I-tangle [31]. The function $(\omega, \alpha) \mapsto E^{f_\alpha}(\omega)$ is continuous on $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \times [1, +\infty)$ by Corollary 4. The least upper bound of the monotone family $\{E^{f_\alpha}\}_{\alpha>1}$ of continuous entanglement monotones coincides with the indicator function of the set of entangled states by Corollary 3.

Example 6. Let $R_p(\rho) = \frac{\log \operatorname{Tr} \rho^p}{1-p}$ be the Rényi entropy of the state $\rho \in \mathfrak{S}(\mathcal{H})$ of order $p \in [0, 1]$ (the case $p = 0$ corresponds to the function $\log \operatorname{rank}(\rho)$ and the case $p = 1$ to the von Neumann entropy). Then R_p is a concave lower semicontinuous subadditive function on $\mathfrak{S}(\mathcal{H})$ with range $[0, +\infty]$ satisfying condition (23). By Theorem 2, the function E^{R_p} is an entanglement monotone satisfying requirements EM-1, EM-2c, EM-3c, EM-4 and EM-5a. In the case $p = 0$ the entanglement monotone E^{R_0} is an infinite-dimensional generalization of the Schmidt measure [27]. In the case $p = 1$, the entanglement monotone $E^{R_1} = E^H$ is an entanglement measure which may be regarded as an infinite-dimensional generalization of the Entanglement of Formation [7] (see the next section). If $\mathrm{g}(H_{\mathcal{H}}) = \inf\{\lambda > 0 \mid \operatorname{Tr} \exp(-\lambda H_{\mathcal{H}}) < +\infty\} = 0$, then Theorem 2, C) implies that the entanglement measure $E^{R_1} = E^H$ satisfies requirement EM-5b since the von Neumann entropy $H = R_1$ is continuous on the set $\mathcal{K}_{H_{\mathcal{H}},h}$ (see [12] or [26], Proposition 1).

4.3. Approximation of entanglement monotones. The entanglement monotones produced by the μ-convex roof construction are generally unbounded and discontinuous (being only lower or upper semicontinuous), and this may lead to analytical difficulties in dealing with these functions. One can overcome some of these difficulties using the following approximation result.

Proposition 7. *Let f be a concave non-negative lower semicontinuous (resp. upper semicontinuous) function on $\mathfrak{S}(\mathcal{H})$ satisfying condition (23) and representable as the pointwise limit of some increasing (resp. decreasing) sequence $\{f_n\}$ of concave continuous non-negative functions on $\mathfrak{S}(\mathcal{H})$ satisfying condition (23). Then the entanglement monotone E^f is the pointwise limit of the increasing (resp. decreasing) sequence $\{E^{f_n}\}_n$ of continuous entanglement monotones.*

If, in addition, f satisfies condition C in Theorem 2, then the sequence $\{E^{f_n}\}$ converges to the entanglement monotone E^f uniformly on compact subsets of $\mathcal{K}_{H_{\mathcal{H}} \otimes I_{\mathcal{K}},h}$ for every $h > 0$.

Proof. The first assertion follows from Theorem 2, Corollary 3 and Remark 2. The second assertion follows from the first and Dini's lemma.

§ 5. Entanglement of Formation

5.1. The two definitions. The Entanglement of Formation (EoF) of a state ω of a finite-dimensional composite system is defined in [7] as the minimal possible average entanglement over all pure-state *discrete finite* decompositions of ω (the entanglement of a pure state is defined as the von Neumann entropy of its reduced state). In our notation this means that

$$E_F = (H \circ \Theta)_* = \overline{\mathrm{co}}(H \circ \Theta) = \mathrm{co}(H \circ \Theta).$$

A generalization of this notion is considered in [8], where the Entanglement of Formation of a state ω of an infinite-dimensional composite system is defined as the minimal possible average entanglement over all pure-state *discrete countable* decompositions of ω, which means that $E_F^d = (H \circ \Theta)_*^{\sigma}$.

The generalized convex roof construction in § 4.2 with the von Neumann entropy H taken for f leads to the definition of EoF proposed in [25]: $E_F^c = E^H = (H \circ \Theta)_*^{\mu} = \overline{\mathrm{co}}(H \circ \Theta)$. Here the Entanglement of Formation of a state ω of an infinite-dimensional composite system is defined as the minimal possible average entanglement over all pure-state *continuous* decompositions of ω.

An interesting open question is the relation between E_F^d and E_F^c. It follows from the definitions that

$$E_F^d(\omega) \geqslant E_F^c(\omega) \qquad \forall\, \omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}).$$

It is shown in [25] that

$$E_F^d(\omega) = E_F^c(\omega) \tag{26}$$

for any state ω satisfying either $H(\mathrm{Tr}_{\mathcal{H}}\, \omega) < +\infty$ or $H(\mathrm{Tr}_{\mathcal{K}}\, \omega) < +\infty$. Equality (26) obviously holds for all pure states and all non-entangled states, but its validity for an arbitrary state ω has not been proved (as far as I know). The example in Remark 6 shows that this question cannot be solved using only the analytical properties of the von Neumann entropy: concavity and lower semicontinuity. Note that the question of the coincidence of the functions E_F^d and E_F^c is equivalent to that of the lower semicontinuity of E_F^d since E_F^c is the greatest lower semicontinuous convex function that coincides with the von Neumann entropy on the set of pure states.

Although the definition of E_F^d seems more reasonable from the physical point of view (since it involves optimization over ensembles of quantum states rather than measures), the assumption of the existence of a state ω_0 such that $E_F^d(\omega_0) \neq E_F^c(\omega_0)$ leads to the following 'non-physical' property of the function E_F^d. For every positive integer n consider the local measurement $\{M_k^n\}_{k\in\mathbb{N}}$, where

$$M_1 = \left(\sum_{i=1}^{n} |i\rangle\langle i|\right) \otimes I_{\mathcal{K}}, \qquad M_k = |n+k-1\rangle\langle n+k-1| \otimes I_{\mathcal{K}}, \quad k > 1.$$

It is clear that the sequence $\{\Phi_n\}_n$ of non-selective local operations $\Phi_n = \{M_k^n\}_{k\in\mathbb{N}}$ tends to the identity transformation (in the strong operator topology). Since the functions E_F^d and E_F^c satisfy requirements EM-2b and EM-3b, we have

$$E_F^d(\omega_0) \geqslant \sum_{k=1}^{+\infty} \pi_k^n E_F^d(\omega_k^n) \geqslant E_F^d\left(\sum_{k=1}^{+\infty} \pi_k^n \omega_k^n\right) = E_F^d(\Phi_n(\omega_0)),$$

$$E_F^c(\omega_0) \geqslant \sum_{k=1}^{+\infty} \pi_k^n E_F^c(\omega_k^n)$$

for all n, where $\pi_k^n = \mathrm{Tr}\, M_k^n \omega_0 M_k^n$ is the probability of the kth outcome and $\omega_k^n = (\pi_k^n)^{-1} M_k^n \omega_0 M_k^n$ is the *a posteriori* state corresponding to this outcome ([9], Ch. 4).

Since the state $\mathrm{Tr}_{\mathcal{K}}\,\omega_k^n$ has finite rank for all n and k, the result in [25] mentioned above implies that $E_F^d(\omega_k^n) = E_F^c(\omega_k^n)$. Thus the two last inequalities show that

$$E_F^d(\Phi_n(\omega_0)) = E_F^d\left(\sum_{k=1}^{+\infty} \pi_k^n \omega_k^n\right) \leqslant E_F^c(\omega_0)$$

for all n, and hence

$$\limsup_{n\to+\infty} E_F^d(\Phi_n(\omega_0)) \leqslant E_F^d(\omega_0) - \Delta, \qquad \Delta = E_F^d(\omega_0) - E_F^c(\omega_0) > 0,$$

despite the fact that the sequence $\{\Phi_n\}_n$ of non-selective local operations tends to the identity transformation. In contrast to this, the lower semicontinuity and LOCC-monotonicity of the function E_F^c imply that

$$\lim_{n\to+\infty} E_F^c(\Phi_n(\omega_0)) = E_F^c(\omega_0)$$

for any state ω_0 and any sequence $\{\Phi_n\}_n$ of non-selective LOCC-operations tending to the identity transformation.

Another advantage of the function E_F^c is its generalized LOCC-monotonicity (the validity of requirement EM-2c), which follows from Theorem 2. On the other hand, the assumption that $E_F^d \neq E_F^c$ means that the function E_F^d is not lower semicontinuous, which is a real obstacle to proving the analogous property for this function.

5.2. The approximation of EoF. For every integer $n > 1$ we define a function H_n on $\mathfrak{S}(\mathcal{H})$ by

$$H_n(\rho) = \sup \sum_i \pi_i H(\rho_i),$$

where the supremum is taken over all countable ensembles $\{\pi_i, \rho_i\}$ of states of rank at most n with $\sum_i \pi_i \rho_i = \rho$. It is easy to see that the function H_n is concave, satisfies condition (23), has range $[0, \log n]$ and coincides with the von Neumann entropy on the subset of $\mathfrak{S}(\mathcal{H})$ consisting of states of rank at most n. Using a strengthened version of the stability property of $\mathfrak{S}(\mathcal{H})$, it was shown in [32] that the function H_n is continuous on $\mathfrak{S}(\mathcal{H})$ and that the increasing sequence $\{H_n\}$ converges pointwise to the von Neumann entropy on $\mathfrak{S}(\mathcal{H})$.

By Theorem 2, the function $E_F^n = (H_n \circ \Theta)_*^\mu$ is an entanglement monotone satisfying requirements EM-1, EM-2c, EM-3c, EM-4 and EM-5c. It is easy to see that E_F^n has range $[0, \log n]$ and coincides with E_F^c on the set

$$\{\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K}) \mid \min\{\mathrm{rank}\,\mathrm{Tr}_{\mathcal{K}}\,\omega, \mathrm{rank}\,\mathrm{Tr}_{\mathcal{H}}\,\omega\} \leqslant n\}.$$

By Proposition 7, the increasing sequence $\{E_F^n\}$ converges pointwise to E_F^c on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, and this convergence is uniform on each compact set of continuity of the function E_F^c, in particular, on compact subsets of the set $\mathcal{K}_{H_{\mathcal{H}}\otimes I_{\mathcal{K}},h}$ for all $h > 0$, where $H_{\mathcal{H}}$ is an $\mathfrak{H}$-operator in the space $\mathcal{H}$ such that $\mathrm{Tr}\, e^{-\lambda H_{\mathcal{H}}} < +\infty$ for any $\lambda > 0$. Conditions for the continuity of the function E_F^c are considered in the next subsection.

5.3. Continuity conditions for EoF. Theorem 1 in [25] yields the following continuity condition for the function E_F^c, which can also be stated as a continuity condition for E_F^d since it implies the coincidence of these functions.

Proposition 8. *The function E_F^c has a continuous restriction to a set $\mathcal{A} \subset \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ if either the function $\omega \mapsto H(\operatorname{Tr}_{\mathcal{H}} \omega)$ or the function $\omega \mapsto H(\operatorname{Tr}_{\mathcal{K}} \omega)$ has a continuous restriction to $\mathcal{A}$.*

This condition enables us to prove the result mentioned in Example 6 (the validity of requirement EM-5b) as well as the following observation.

Corollary 8. *Let ρ be a state in $\mathfrak{S}(\mathcal{H})$. Then the function E_F^c has a continuous restriction to the set $\{\omega \mid \operatorname{Tr}_{\mathcal{K}} \omega = \rho\}$ if and only if $H(\rho) < +\infty$.*

Proof. It suffices to note that when $H(\rho) = +\infty$, there is a pure state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that $\operatorname{Tr}_{\mathcal{K}} \omega = \rho$.

By Corollary 8, the function $t \mapsto E_F^c(\Psi_t(\omega))$ is continuous for any continuous family $\{\Psi_t\}_t$ of local operations on the quantum system associated with the space $\mathcal{K}$ and any state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ with $\operatorname{Tr}_{\mathcal{K}} \omega < +\infty$.

For an arbitrary state σ let $\mathrm{d}(\sigma) = \inf\{\lambda \in \mathbb{R} \mid \operatorname{Tr} \sigma^{\lambda} < +\infty\}$ be the characteristic of the spectrum of this state. Clearly, $\mathrm{d}(\sigma) \in [0, 1]$. Proposition 8, Proposition 2 in [26] and the monotonicity of the relative entropy yield the following condition for the continuity of E_F^c with respect to the convergence defined by the relative entropy (which is stronger than the convergence defined by the trace norm).

Corollary 9. *Let ω_0 be a state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that either $\mathrm{d}(\operatorname{Tr}_{\mathcal{H}} \omega) < 1$ or $\mathrm{d}(\operatorname{Tr}_{\mathcal{K}} \omega) < 1$. If $\{\omega_n\}$ is a sequence such that $\lim_{n\to+\infty} H(\omega_n \| \omega_0) = 0$, then $\lim_{n\to+\infty} E_F^c(\omega_n) = E_F^c(\omega_0)$.*

§ 6. Possible generalizations

The definitions of σ-convexity and μ-convexity generalize naturally to functions defined on an arbitrary convex closed subset of a locally convex space if any probability measure on this set has a well-defined barycentre. The definitions of σ-convex and μ-convex roofs also admit such generalization, but it is necessary to impose certain conditions for these constructions to be well defined.

There is a class of convex subsets of locally convex spaces, including all metrizable compact sets and several non-compact sets (in particular, the set $\mathfrak{S}(\mathcal{H})$ of quantum states), to which the main results obtained in §§ 2, 3 can be extended. This class of subsets (which are referred to as μ-compact in [13]) is studied in detail in [15], where several well-known results about convex compact sets (in particular, Choquet's theorem on barycentric decomposition and the Versterstrem–O'Brien theorem [16]) are extended to μ-compact sets. The last theorem states that the stability of a convex μ-compact set (which means the openness of the convex mixture map) is equivalent to several other properties, including the openness of the barycentric map and of the restriction of this map to the set of measures supported by extreme points.

Using the results in [13], [15], it is easy to show that all the assertions in §§ 2, 3 hold if we replace $\mathfrak{S}(\mathcal{H})$ by any convex stable μ-compact set $\mathcal{A}$ with

$\mathcal{A} = \sigma\text{-}\operatorname{co}(\operatorname{extr}\mathcal{A})$. The stability of $\mathcal{A}$ is used only in the proofs of Propositions 2, 4, Corollaries 1, 2, 4, the second part of Proposition 5, and Theorem 1 and its corollaries, while in the proofs of Proposition 3 and Corollary 3 it can be replaced by the weaker requirement that the set $\operatorname{extr}\mathcal{A}$ be closed, which is necessary for the definition of the μ-convex roof. The condition $\mathcal{A} = \sigma\text{-}\operatorname{co}(\operatorname{extr}\mathcal{A})$ is necessary for the definition of the σ-convex roof and is used in the proofs of all the assertions related to this construction.

Appendix

A1. Jensen's inequality for functions on Banach spaces. Here we give sufficient conditions for Jensen's inequality to hold (in its discrete and integral forms) for convex functions on Banach spaces with values in $[-\infty, +\infty]$. A simple example showing the importance of these conditions in the propositions below is the affine Borel function on the simplex of all probability distributions with a countable number of outcomes taking the value 0 on finite-rank distributions and the value $+\infty$ on infinite-rank distributions. Other examples are considered in § 2.

The following assertion is easily proved using Jensen's inequality for finite convex combinations and a simple approximation argument.

Proposition A-1 (Jensen's discrete inequality). *Let f be a convex upper-bounded function on a closed convex bounded subset $\mathcal{A}$ of a Banach space. Then the following inequality holds for every countable set $\{x_i\} \subset \mathcal{A}$ with corresponding probability distribution $\{\pi_i\}$:*

$$f\Bigl(\sum_{i=1}^{+\infty}\pi_i x_i\Bigr) \leqslant \sum_{i=1}^{+\infty}\pi_i f(x_i).$$

Proposition A-2 (Jensen's integral inequality). *Let f be a convex function on a closed bounded convex subset $\mathcal{A}$ of a separable Banach space. Suppose that f is either lower semicontinuous or upper-bounded and upper semicontinuous. Then the following inequality holds for any Borel probability measure μ on $\mathcal{A}$:*

$$f\Bigl(\int_{\mathcal{A}} x\mu(dx)\Bigr) \leqslant \int_{\mathcal{A}} f(x)\mu(dx). \tag{27}$$

(If $\mathcal{A}$ is a subset of $\mathbb{R}^n$, then inequality (27) holds for any Borel function f with values in $[-\infty, +\infty]$ and any Borel measure μ [33].)

Proof. Let μ_0 be an arbitrary probability measure on $\mathcal{A}$.

Let f be an upper-bounded upper semicontinuous function. Then the functional $\mu \mapsto \int_{\mathcal{A}} f(x)\mu(dx)$ is upper semicontinuous on the set $\mathcal{P}(\mathcal{A})$ of all Borel probability measures on $\mathcal{A}$ endowed with the weak convergence topology ([24], § 2). Let $\{\mu_n\}$ be a sequence of measures with finite support having the same barycentre as μ_0 and converging weakly to μ_0. Since f is convex, inequality (27) holds with $\mu = \mu_n$ for every n. Since the functional $\mu \mapsto \int_{\mathcal{A}} f(x)\mu(dx)$ is upper semicontinuous, we can pass to the limit as $n \to +\infty$ in this inequality and get inequality (27) with $\mu = \mu_0$.

Let f be a lower semicontinuous function. Arguing as in the proof of Lemma 2, one can show that f either is lower-bounded or takes only infinite values. It suffices to consider the first case. Suppose that $\int_{\mathcal{A}} f(x)\mu(dx) < +\infty$. Using the construction in the proof of Lemma 1, we obtain a sequence $\{\mu_n\}$ of measures on $\mathcal{A}$ with finite support such that

$$\limsup_{n\to+\infty} \int_{\mathcal{A}} f(x)\mu_n(dx) \leqslant \int_{\mathcal{A}} f(x)\mu_0(dx), \qquad \lim_{n\to+\infty} \int_{\mathcal{A}} x\mu_n(dx) = \int_{\mathcal{A}} x\mu_0(dx).$$

Since f is convex, inequality (27) holds with $\mu = \mu_n$ for each n. Since f is lower semicontinuous, we can pass to the limit as $n \to +\infty$ and get inequality (27) with $\mu = \mu_0$.

Corollary A-1. *Let f be an affine lower semicontinuous function on a closed bounded convex subset $\mathcal{A}$ of a separable Banach space. Then the following equality holds for any Borel probability measure μ on $\mathcal{A}$:*

$$f\left(\int_{\mathcal{A}} x\mu(dx)\right) = \int_{\mathcal{A}} f(x)\mu(dx). \tag{28}$$

A2. A property of a posteriori states. Let $\mathfrak{M}$ be an arbitrary instrument on $\mathfrak{S}(\mathcal{H})$ with set $\mathcal{X}$ of outcomes ([9], Ch. 4). For any state $\rho \in \mathfrak{S}(\mathcal{H})$ let $\mu_\rho(\cdot) = \operatorname{Tr}\mathfrak{M}(\cdot)[\rho]$ be the *a posteriori* measure on $\mathcal{X}$ and $\{\sigma(x|\rho)\}_{x\in\mathcal{X}}$ the family of *a posteriori* states corresponding to the *a priori* state ρ [9], [28].

Lemma A-1. *The following inequality holds for any convex lower semicontinuous function f on $\mathfrak{S}(\mathcal{H})$ and any sequence $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H})$ converging to ρ_0:*

$$\liminf_{n\to+\infty} \int_{\mathcal{X}} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) \geqslant \int_{\mathcal{X}} f(\sigma(x|\rho_0))\mu_{\rho_0}(dx).$$

Proof. It is sufficient to show that the assumption

$$\lim_{n\to+\infty} \int_{\mathcal{X}} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) \leqslant \int_{\mathcal{X}} f(\sigma(x|\rho_0))\mu_{\rho_0}(dx) - \Delta, \qquad \Delta > 0, \tag{29}$$

leads to a contradiction.

Let $\nu_0 = \mu_{\rho_0} \circ \sigma^{-1}(\,\cdot\,|\rho_0)$ be the image of the measure μ_{ρ_0} under the map $x \mapsto \sigma(x|\rho_0)$. Clearly, $\nu_0 \in \mathcal{P}$ (see Remark 5) and

$$\int_{\mathcal{X}} f(\sigma(x|\rho_0))\mu_{\rho_0}(dx) = \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\nu_0(d\rho).$$

Since $\mathfrak{S}(\mathcal{H})$ is separable, for given m one can find a family $\{\mathcal{B}_i^m\}_i$ of Borel subsets of $\mathfrak{S}(\mathcal{H})$ such that $\nu_0(\mathcal{B}_i^m) > 0$ for all i and the sequence of measures

$$\nu_m = \left\{\nu_0(\mathcal{B}_i^m), \frac{1}{\nu_0(\mathcal{B}_i^m)} \int_{\mathcal{B}_i^m} \rho\nu_0(d\rho)\right\}_i$$

converges weakly to ν_0 (see the proof of Lemma 1 in [4]). The lower semicontinuity of the functional $\mu \mapsto \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu(d\rho)$ implies the existence of m_0 such that

$$\begin{aligned}\sum_i \nu_0(\mathcal{B}_i^{m_0}) f\left(\frac{1}{\nu_0(\mathcal{B}_i^{m_0})} \int_{\mathcal{B}_i^{m_0}} \rho\nu_0(d\rho)\right) \\ = \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\nu_{m_0}(d\rho) \geqslant \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\nu_0(d\rho) - \frac{1}{3}\Delta.\end{aligned} \tag{30}$$

Using the finite family $\{\mathcal{X}_i\}$ of μ_{ρ_0}-measurable subsets $\mathcal{X}_i = \sigma^{-1}(\mathcal{B}_i^{m_0}|\rho_0)$ of $\mathcal{X}$, we can construct a family $\{\mathcal{X}'_i\}$ consisting of the same number of Borel subsets of $\mathcal{X}$ and such that $\mu_{\rho_0}((\mathcal{X}'_i \setminus \mathcal{X}_i) \cup (\mathcal{X}_i \setminus \mathcal{X}'_i)) = 0$ and $\bigcup_i \mathcal{X}'_i = \mathcal{X}$. For each i the state

$$\sigma_0^i = \frac{1}{\nu_0(\mathcal{B}_i^{m_0})} \int_{\mathcal{B}_i^{m_0}} \rho\nu_0(d\rho) = \frac{1}{\mu_{\rho_0}(\mathcal{X}'_i)} \int_{\mathcal{X}'_i} \sigma(x|\rho_0)\mu_{\rho_0}(dx) = \frac{\mathfrak{M}(\mathcal{X}'_i)[\rho_0]}{\operatorname{Tr} \mathfrak{M}(\mathcal{X}'_i)[\rho_0]}$$

is the *a posteriori* state corresponding to the set $\mathcal{X}'_i$ of outcomes and the *a priori* state ρ_0.

For every i let $\sigma_n^i = \frac{\mathfrak{M}(\mathcal{X}'_i)[\rho_n]}{\operatorname{Tr} \mathfrak{M}(\mathcal{X}'_i)[\rho_n]}$ be the *a posteriori* state corresponding to the set $\mathcal{X}'_i$ of outcomes and the *a priori* state ρ_n.[14] Since f is lower semicontinuous and $\lim_{n\to+\infty} \mathfrak{M}(\mathcal{X}'_i)[\rho_n] = \mathfrak{M}(\mathcal{X}'_i)[\rho_0]$, we have

$$\sum_i \mu_{\rho_n}(\mathcal{X}'_i) f(\sigma_n^i) \geqslant \sum_i \mu_{\rho_0}(\mathcal{X}'_i) f(\sigma_0^i) - \frac{1}{3}\Delta \tag{31}$$

for all sufficiently large n.

By Jensen's inequality (Proposition A-2) it follows from the convexity and lower semicontinuity of f that

$$\mu_{\rho_n}(\mathcal{X}'_i) f(\sigma_n^i) \leqslant \int_{\mathcal{X}'_i} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) \qquad \forall\, i, n. \tag{32}$$

Using (30)–(32), we obtain

$$\int_{\mathcal{X}} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) = \sum_i \int_{\mathcal{X}'_i} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) \geqslant \sum_i \mu_{\rho_n}(\mathcal{X}'_i) f(\sigma_n^i)$$
$$\geqslant \sum_i \mu_{\rho_0}(\mathcal{X}'_i) f(\sigma_0^i) - \frac{1}{3}\Delta \geqslant \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\nu_0(d\rho) - \frac{2}{3}\Delta$$

for all sufficiently large n, which contradicts (29).

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[14] Since $\operatorname{Tr} \mathfrak{M}(\mathcal{X}'_i)[\rho_0] = \mu_{\rho_0}(\mathcal{X}'_i) > 0$, the state σ_n^i is well defined for all sufficiently large n.

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