Entropy reduction of quantum measurements

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It is shown that the entropy reduction (the information gain in the initial terminology) of an efficient (ideal or pure) quantum measurement coincides with the generalized quantum mutual information of a quantum-classical channel mapping an *a priori* state to the corresponding *posteriori* probability distribution of the outcomes of the measurement. As a result the entropy reduction is defined for arbitrary *a priori* states (not only for states with finite von Neumann entropy) and its analytical properties is studied in detail by using general properties of the quantum mutual information. By using this approach it is shown that the entropy reduction of an efficient quantum measurement is a nonnegative lower semicontinuous concave function on the set of all *a priori* states having continuous restrictions to subsets on which the von Neumann entropy is continuous. Monotonicity and subadditivity of the entropy reduction are also easily proved by this method. A simple continuity condition for the entropy reduction and for the mean *posteriori* entropy considered as functions of a pair (*a priori* state, measurement) is obtained. A characterization of an irreducible measurement (in the Ozawa sense) which is not efficient is considered in the Appendix. © *2011 American Institute of Physics*. [doi:10.1063/1.3589831]

I. INTRODUCTION

The notion of a quantum measurement plays a key role in quantum theory. One of quantitative characteristics of a quantum measurement is the entropy reduction defined as a difference between the von Neumann entropy of an *a priori* (pre-measurement) state and the mean von Neumann entropy of the corresponding *posteriori* (post-measurement) states. This characteristic was originally called the information gain (cf., Refs. 6, 12, and 21) but then the terminology had been changed (some arguments explaining this change can be found in Ref. 4). Roughly speaking, the entropy reduction characterizes a degree of purifying ("gain in purity") of a state in a measurement process. More details about the information sense of this value can be found in Refs. 11, 15, and 21.

An interesting question concerns the sign of the entropy reduction. Groenewold has conjectured in Ref. 6 and Lindblad has proved in Ref. 12 that the entropy reduction is nonnegative for quantum measurements of the von Neumann-Luders type. The general case has been studied by Ozawa, who has proved in Ref. 21 that the entropy reduction is nonnegative if and only if the quantum measurement is *quasicomplete* (also called *irreducible* in Ref. 19) in the sense that for an arbitrary pure *a priori* state the corresponding family of *posteriori* states consists of pure states (for almost all outcomes). The class of quasicomplete (irreducible) quantum measurements contains the class of *efficient* or *pure* measurement (cf., Refs. 10, 11, and 15) described in Sec. III A. Quantum measurements belonging to the gap between these two classes are characterized in the Appendix as measurements with quite singular properties.

In this paper, we show that the entropy reduction of an efficient (pure) quantum measurement can be expressed via the (generalized) quantum mutual information of the quantum-classical channel mapping an *a priori* state to the corresponding *posteriori* probability distribution of the outcomes of

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the measurement. This makes it possible to define the entropy reduction for arbitrary *a priori* states (not only for states with finite von Neumann entropy), and to study its analytical properties by using results concerning the quantum mutual information of infinite-dimensional channels.[8]

The paper is organized as follows.

In Sec. II, we restrict attention to the case of quantum measurements with a discrete set of outcomes, which is more simple mathematically. In this case, the quantum-classical channel mapping an *a priori* state to the corresponding *posteriori* probability distribution of the outcomes can be considered as a quantum-quantum infinite-dimensional channel, and its quantum mutual information is a well studied value.[8] So, the obtained expression of the entropy reduction of an efficient quantum measurement with a discrete set of outcomes via this value directly shows that the entropy reduction can be uniquely extended to a nonnegative lower semicontinuous concave function on the set of all *a priori* states and that this function has several special properties such as continuity of restrictions to subsets on which the von Neumann entropy is continuous, etc., stated in Theorem 1.

In Sec. III, we consider the case of general quantum measurements described by completely positive instruments. In this case, we have to pay a special attention to the difference between the class of irreducible instruments (Definition 2) and its proper subclass consisting of efficient (or pure) instruments (Definition 3). As shown in Proposition 2, this difference becomes visible when we consider tensor product of an instrument with the "identity" instrument corresponding to the trivial measurement (similar to the difference between positive and completely positive maps). It is the class of efficient instruments to which the results of Sec. II can be extended.

The other problem in dealing with general quantum measurements consists in nonexistence of quantum-quantum modification for the quantum-classical channel mapping an *a priori* state to the corresponding *posteriori* probability distribution of the outcomes. This implies necessity to extend the notion of the quantum mutual information to channels taking values in the space of normal states of an arbitrary W^*-algebra (Definition 4). By using this extension the analog of the above representation of the entropy reduction via the quantum mutual information is proved (Corollary 1). This makes it possible to obtain generalizations of the assertions of Theorem 1 to the case of general quantum measurements, presented in Theorem 2.

Section III is completed by considering a question of continuity of the entropy reduction with respect to simultaneous perturbations of a priory states and quantum instruments. Proposition 4 contains a sufficient continuity condition for the entropy reduction considered as a function of the pair (*a priory* state, quantum instrument), which is obtained by means of the above representation via the quantum mutual information. It implies a sufficient continuity condition for the mean *posteriori* entropy presented in Corollary 2.

In the Appendix, we consider a characterization of irreducible quantum instruments which are not efficient (Proposition 5). It shows quite singular properties of such instruments and provides a simple sufficient condition for efficiency of irreducible instruments (Corollary 3).

II. THE DISCRETE CASE

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ – the Banach space of all bounded operators in $\mathcal{H}$ with the operator norm $\| \cdot \|$, $\mathfrak{T}(\mathcal{H})$ – the Banach space of all trace-class operators in $\mathcal{H}$ with the trace norm $\| \cdot \|_1 = \mathrm{Tr}|\cdot|$, containing the cone $\mathfrak{T}_+(\mathcal{H})$ of all positive trace-class operators. The closed convex subset $\mathfrak{S}(\mathcal{H}) = \{A \in \mathfrak{T}_+(\mathcal{H}) \,|\, \mathrm{Tr}A = 1\}$ is a complete separable metric space with the metric defined by the trace norm. Operators in $\mathfrak{S}(\mathcal{H})$ are denoted $\rho, \sigma, \omega, \ldots$ and called density operators or states since each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$.[3]

The identity operator in a Hilbert space $\mathcal{H}$ and the identity transformation of the set $\mathfrak{T}(\mathcal{H})$ will be denoted $I_\mathcal{H}$ and $\mathrm{Id}_\mathcal{H}$ correspondingly.

For an arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ the partial states $\mathrm{Tr}_\mathcal{K}\omega$ and $\mathrm{Tr}_\mathcal{H}\omega$ will be denoted $\omega_\mathcal{H}$ and $\omega_\mathcal{K}$.

We will use the following natural extension of the von Neumann entropy $H(\rho) = -\mathrm{Tr}\rho \log \rho$ of a quantum state $\rho \in \mathfrak{S}(\mathcal{H})$ to the cone $\mathfrak{T}_+(\mathcal{H})$ of all positive trace-class operators

$$H(A) = \mathrm{Tr}AH\left(\frac{A}{\mathrm{Tr}A}\right) = \mathrm{Tr}\eta(A) - \eta(\mathrm{Tr}A), \quad \eta(x) = -x\log x,$$

where log denotes the natural logarithm.

The quantum relative entropy is defined for arbitrary operators A and B in $\mathfrak{T}_+(\mathcal{H})$ as follows:

$$H(A\,\|\,B) = \sum_i \langle i|\,(A\log A - A\log B + B - A)\,|i\rangle, \tag{1}$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of A and it is assumed that $H(A\,\|\,B) = +\infty$ if $\mathrm{supp}A$ is not contained in $\mathrm{supp}B$.[13]

A linear completely positive trace-preserving map $\Phi : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}')$ is called a *quantum-quantum channel*[9,17] (we use the term *quantum-quantum*, since we will use below the more general notion of a channel, in particular, the notion of a *quantum-classical* channel).

By the Stinespring dilation theorem, there exist a separable Hilbert space $\mathcal{H}''$ and an isometry $V : \mathcal{H} \to \mathcal{H}' \otimes \mathcal{H}''$ such that

$$\Phi(A) = \mathrm{Tr}_{\mathcal{H}''} VAV^*, \quad \forall A \in \mathfrak{T}(\mathcal{H}). \tag{2}$$

The quantum-quantum channel

$$\mathfrak{T}(\mathcal{H}) \ni A \mapsto \widetilde{\Phi}(A) = \mathrm{Tr}_{\mathcal{H}'} VAV^* \in \mathfrak{T}(\mathcal{H}'') \tag{3}$$

is called *complementary* to the channel Φ, it is uniquely defined up to unitary equivalence[7] (the channel $\widetilde{\Phi}$ is also called *conjugate* to the channel Φ[16]).

The *quantum mutual information* is an important entropic characteristic of a channel $\Phi : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}')$ related to the entanglement-assisted classical capacity of this channel.[9,17] In finite dimensions, it is defined at arbitrary state $\rho \in \mathfrak{S}(\mathcal{H})$ by the expression (cf., Ref. 1)

$$I(\rho, \Phi) = H(\rho) + H(\Phi(\rho)) - H(\widetilde{\Phi}(\rho)). \tag{4}$$

In infinite dimensions, this expression may contain uncertainty "$\infty - \infty$," but it can be modified to avoid this problem as follows:

$$I(\rho, \Phi) = H\left(\Phi \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}) \| \Phi \otimes \mathrm{Id}_{\mathcal{K}}(\rho \otimes \varrho)\right), \tag{5}$$

where $\mathcal{K}$ is a Hilbert space isomorphic to $\mathcal{H}$, $\hat{\rho}$ is a purification of the state ρ in the space $\mathcal{H} \otimes \mathcal{K}$ (this means that $\mathrm{Tr}_{\mathcal{K}}\hat{\rho} = \rho$), and $\varrho = \mathrm{Tr}_{\mathcal{H}}\hat{\rho}$ is a state in $\mathfrak{S}(\mathcal{K})$ isomorphic to ρ. Analytical properties of the function $(\rho, \Phi) \mapsto I(\rho, \Phi)$ defined by (5) in the infinite dimensional case are studied in Ref. 8.

A general quantum measurement $\mathfrak{M}$ with a countable outcome set $X = \{x_i\}_{i \in I}$ is described by a set $\{V_{ij}\}_{i \in I, j \in J_i}$ of operators in $\mathfrak{B}(\mathcal{H})$ such that $\sum_{i \in I} \sum_{j \in J_i} V_{ij}^* V_{ij} = I_{\mathcal{H}}$. A measurement $\mathfrak{M}$ is called *efficient* if for each i the index set J_i degenerates to a single point, so it is described by a set $\{V_i\}_{i \in I}$ of operators in $\mathfrak{B}(\mathcal{H})$ such that $\sum_{i \in I} V_i^* V_i = I_{\mathcal{H}}$.[15,17] Applying this measurement to an arbitrary *a priori* state $\rho \in \mathfrak{S}(\mathcal{H})$ results in the *posteriori* probability distribution $\{\pi_i(\rho)\}_{i \in I}$, where $\pi_i(\rho) = \mathrm{Tr}V_i\rho V_i^*$ is the probability of the outcome x_i, and the corresponding family of *posteriori* states $\{\rho_i\}_{i \in I}$, where $\rho_i = (\pi_i(\rho))^{-1} V_i\rho V_i^*$. Thus, $\sum_{i \in I} \pi_i(\rho)H(\rho_i) = \sum_{i \in I} H(V_i\rho V_i^*)$ is the mean entropy of *posteriori* states. The *entropy reduction* of the quantum measurement $\mathfrak{M}$ at an *a priori* state ρ with finite entropy is the following value:

$$ER(\rho, \mathfrak{M}) \doteq H(\rho) - \sum_{i \in I} \pi_i(\rho)H(\rho_i) = H(\rho) - \sum_{i \in I} H(V_i\rho V_i^*).$$

Let $\mathcal{H}_X$ be a Hilbert space having dimension coinciding with the cardinality of the outcome set X. Consider the quantum-quantum channel

$$\mathfrak{T}(\mathcal{H}) \ni A \mapsto \Pi_{\mathfrak{M}}(A) = \sum_{i \in I} \mathrm{Tr}\left[V_i A V_i^*\right] |\varphi_i\rangle\langle\varphi_i| \in \mathfrak{T}(\mathcal{H}_X), \tag{6}$$

where $\{|\varphi_i\rangle\}_{i \in I}$ is a particular orthonormal basis in $\mathcal{H}_X$. The channel $\Pi_{\mathfrak{M}}$ is a quantum-quantum modification of the quantum-classical channel mapping a state ρ to the probability distribution $\{\pi_i(\rho)\}_{i \in I}$. It is essential that

$$ER(\rho, \mathfrak{M}) = I(\rho, \Pi_{\mathfrak{M}}) \tag{7}$$

for any state ρ in $\mathfrak{S}(\mathcal{H})$ with finite entropy. If $H(\{\pi_i(\rho)\}_{i \in I}) < +\infty$ this equality directly follows from (4), since $\widetilde{\Pi}_{\mathfrak{M}}(\cdot) = \sum_{i \in I} U_i V_i(\cdot) V_i^* U_i^*$, where $\{U_i\}_{i \in I}$ is a family of isometrical embedding of $\mathcal{H}$ into $\bigoplus_{i \in I} \mathcal{H}_i$, $\mathcal{H}_i \cong \mathcal{H}$, such that $U_i \mathcal{H} = \mathcal{H}_i$. In general case, it can be easily deduced from Ref. [8, (Proposition 3 and Theorem 1)] or can be proved directly by the obvious modification of the proof of Proposition 3 in Sec. III B.

Equality (7) obtained under the condition $H(\rho) < +\infty$ makes it possible to consider the entropy reduction of an efficient quantum measurement with a countable outcome set as a function on the whole space of *a priori* states.

Definition 1: The entropy reduction of an efficient quantum measurement $\mathfrak{M} = \{V_i\}_{i \in I}$ *at an arbitrary* a priori *state* ρ *is defined as follows:*

$$ER(\rho, \mathfrak{M}) \doteq I(\rho, \Pi_{\mathfrak{M}}),$$

where $\Pi_{\mathfrak{M}}$ *is the quantum-quantum channel defined by (6).*

By equality (7), this definition is consistent with the conventional one. Its main advantage consists in possibility to study the function $\rho \mapsto ER(\rho, \mathfrak{M})$ on the whole space of a priori states by using properties of the quantum mutual information (many of them follow from the corresponding properties of the quantum relative entropy).

Theorem 1: *Let* $\mathfrak{M}$ *be an efficient measurement in a Hilbert space* $\mathcal{H}$ *with a countable outcome set. The function* $\rho \mapsto ER(\rho, \mathfrak{M})$ *is nonnegative concave and lower semicontinuous on the set* $\mathfrak{S}(\mathcal{H})$. *It has the following properties:*

(1) $\{ER(\rho, \mathfrak{M}) = 0\} \Leftrightarrow \{\rho_i \cong \rho \;\; \forall i \;\; s.t. \;\; \pi_i(\rho) \neq 0\}$, *where* $\{\pi_i(\rho)\}$ *and* $\{\rho_i\}$ *are respectively the posteriori probability distribution and the family of posteriori states corresponding to an a priori state* ρ;

(2) *continuity on any subset of* $\mathfrak{S}(\mathcal{H})$ *on which the von Neumann entropy is continuous:*

$$\lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty \;\; \Rightarrow \;\; \lim_{n \to +\infty} ER(\rho_n, \mathfrak{M}) = ER(\rho_0, \mathfrak{M}) < +\infty$$

for any sequence $\{\rho_n\}$ *of states converging to a state* ρ_0;

(3) *monotonicity: for arbitrary efficient measurements* $\mathfrak{M} = \{V_i\}_{i \in I}$ *and* $\mathfrak{N} = \{U_j\}_{j \in J}$ *in a Hilbert space* $\mathcal{H}$ *with the outcome sets* X *and* Y *the inequality*

$$ER(\rho, \mathfrak{N} \circ \mathfrak{M}) \geq ER(\rho, \mathfrak{M}) \tag{8}$$

holds for any $\rho \in \mathfrak{S}(\mathcal{H})$, *where* $\mathfrak{N} \circ \mathfrak{M}$ *is the measurement in the space* $\mathcal{H}$ *with the outcome set* $X \times Y$ *determined by the family* $\{U_j V_i\}_{i \in I, j \in J}$;

(4) *subadditivity: for arbitrary efficient measurements* $\mathfrak{M} = \{V_i\}_{i \in I}$ *and* $\mathfrak{N} = \{U_j\}_{j \in J}$ *in Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$ *with the outcome sets* X *and* Y *the inequality*

$$ER(\omega, \mathfrak{M} \otimes \mathfrak{N}) \leq ER(\omega_{\mathcal{H}}, \mathfrak{M}) + ER(\omega_{\mathcal{K}}, \mathfrak{N}) \tag{9}$$

holds for any $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, *where* $\mathfrak{M} \otimes \mathfrak{N}$ *is the measurement in the space* $\mathcal{H} \otimes \mathcal{K}$ *with the outcome set* $X \times Y$ *determined by the family* $\{V_i \otimes U_j\}_{i \in I, j \in J}$.

Proof: (1) Note that the equality

$$ER(\rho, \mathfrak{M}) \doteq I(\rho, \Pi_{\mathfrak{M}}) \doteq H(\Pi_{\mathfrak{M}} \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}) \| \Pi_{\mathfrak{M}}(\rho) \otimes \varrho) = 0,$$

where $\hat{\rho}$ is a purification of ρ and $\varrho = \mathrm{Tr}_{\mathcal{H}} \hat{\rho}$, means

$$\Pi_{\mathfrak{M}} \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}) = \Pi_{\mathfrak{M}}(\rho) \otimes \varrho, \tag{10}$$

by the well known property of the relative entropy.

Let $\rho = \sum_k \lambda_k |k\rangle\langle k|$ and $\hat{\rho} = \sum_{j,k} \sqrt{\lambda_j}\sqrt{\lambda_k}|j\rangle\langle k| \otimes |j\rangle\langle k|$. It is easy to see that (10) is equivalent to

$$\langle k|V_i^* V_i|j\rangle = \delta_{jk}\mathrm{Tr}V_i^* V_i\rho = \delta_{jk}\pi_i(\rho) \quad \text{for all} \ \ i, j, k. \tag{11}$$

"$\Leftarrow$" Note that this implication is obvious only if the state ρ has finite entropy. Let $\varrho_i = (\pi_i(\rho))^{-1}\mathrm{Tr}_{\mathcal{H}}V_i \otimes I_{\mathcal{K}} \cdot \hat{\rho} \cdot V_i^* \otimes I_{\mathcal{K}}$. Then $\varrho_i \cong \rho_i$ and hence $\varrho_i \cong \varrho$, since $\varrho \cong \rho$. By noting that $\varrho = \sum_{i \in I} \pi_i(\rho)\varrho_i$ we conclude from Lemma 6 in the Appendix that $\varrho_i = \varrho$ for all i. Since $\varrho_i = (\pi_i(\rho))^{-1}\sum_{j,k}\sqrt{\lambda_j}\sqrt{\lambda_k}[\mathrm{Tr}V_i|j\rangle\langle k|V_i^*]|j\rangle\langle k|$ and $\varrho = \sum_k \lambda_k |k\rangle\langle k|$, we obtain (11).

"$\Rightarrow$" Relations (11) mean that $PV_i^* V_i P = \pi_i(\rho)P$ for each i, where $P = \sum_k |k\rangle\langle k|$ is the projector on the support of the state ρ. Thus, $(\pi_i(\rho))^{-1/2}V_i P$ is a partial isometry and hence $\rho_i = (\pi_i(\rho))^{-1}V_i P\rho PV_i^* \cong \rho$ for each i such that $\pi_i(\rho) \neq 0$.

(2) This directly follows from Proposition 4 in Ref. 8.

(3) This follows from the 1-st chain rule for the quantum mutual information (property 3 in Proposition 1 in Ref. 8). Indeed,

$$\Pi_{\mathfrak{N}\circ\mathfrak{M}}(A) = \sum_{i \in I, j \in J} \mathrm{Tr}\left[V_i^* U_j^* U_j V_i A\right]|i\rangle\langle i| \otimes |j\rangle\langle j|, \quad A \in \mathfrak{T}(\mathcal{H}),$$

where $\{|i\rangle\}_{i \in I}$ and $\{|j\rangle\}_{j \in J}$ are particular orthonormal bases in the spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$, and hence $\Pi_{\mathfrak{M}}(A) = \mathrm{Tr}_{\mathcal{H}_Y}\Pi_{\mathfrak{N}\circ\mathfrak{M}}(A)$.

(4) This follows from subadditivity of the quantum mutual information (property 5 in Proposition 1 in Ref. 8), since $\Pi_{\mathfrak{M}\otimes\mathfrak{N}} = \Pi_{\mathfrak{M}} \otimes \Pi_{\mathfrak{N}}$. □

Consider a question of continuity of the entropy reduction with respect to "perturbation" of quantum measurements.

Let $\mathbb{M}(\mathcal{H})$ be the set of all efficient quantum measurements in the Hilbert space $\mathcal{H}$ with finite or countable set of outcomes identified with the set of all sequences $\{V_i\}_{i=1}^{+\infty}$ of operators in $\mathfrak{B}(\mathcal{H})$ such that $\sum_{i=1}^{+\infty} V_i^* V_i = I_{\mathcal{H}}$ endowed with the topology of coordinate-wise strong operator convergence. Proposition 1 in Ref. 8 and Corollary 2 in Ref. 8 imply the following assertion.

Proposition 1: The function

$$(\rho, \mathfrak{M}) \mapsto ER(\rho, \mathfrak{M}) \tag{12}$$

is lower semicontinuous on the set $\mathfrak{S}(\mathcal{H}) \times \mathbb{M}(\mathcal{H})$. Let $\mathcal{A}$ be an arbitrary subset of $\mathfrak{S}(\mathcal{H})$ on which the von Neumann entropy is continuous. Then function (12) is continuous on the set $\mathcal{A} \times \mathbb{M}(\mathcal{H})$.

By Proposition 1 and Proposition 6.6 in Ref. 18 function (12) is continuous on the set $\mathcal{K}_{H,h} \times \mathbb{M}(\mathcal{H})$, where $\mathcal{K}_{H,h}$ is the set of states with the mean energy $\mathrm{Tr}H\rho$ not exceeding $h > 0$ provided the Hamiltonian H of the quantum system satisfies the condition $\mathrm{Tr}e^{-\lambda H} < +\infty$ for all $\lambda > 0$ (which holds, for example, for the Hamiltonian of the system of quantum oscillators).

III. THE GENERAL CASE

A. On properties of efficient (pure) instruments

A general quantum measurement in a Hilbert space $\mathcal{H}$ with the measurable outcome set $\{X, \mathcal{F}\}$ is described by a special mathematical object called *instrument*, which was introduced by Davis and Lewis.[5] An instrument $\mathfrak{M}$ (in a space of states) is a σ-additive measure on $\{X, \mathcal{F}\}$ taking values within the set of quantum operations – completely positive trace-non-increasing linear transformations of $\mathfrak{T}(\mathcal{H})$ such that $\mathfrak{M}(X)$ is a channel (see the detailed definition in Refs. 2, 9, and 14).

Let ρ be an arbitrary *a priori* state in $\mathfrak{S}(\mathcal{H})$. Then the outcome of the measurement $\mathfrak{M}$ is contained in a set $F \in \mathcal{F}$ with probability $\mathrm{Tr}\mathfrak{M}(F)[\rho]$. If this probability is nonzero then $(\mathrm{Tr}\mathfrak{M}(F)[\rho])^{-1}\mathfrak{M}(F)[\rho]$ is the corresponding *posteriori* state of the system. Thus, $F \mapsto \mu_\rho(F) \doteq \mathrm{Tr}\mathfrak{M}(F)[\rho]$ is the *posteriori* probability measure on the outcome set $\{X, \mathcal{F}\}$ corresponding to the *a priori* state ρ.

Ozawa proved in Ref. 20 existence of a family $\{\rho_x\}_{x \in X}$ of *posteriori* states defined for μ_ρ-almost all x, such that the function $x \mapsto \mathrm{Tr} A \rho_x$ is $\mathcal{F}$-measurable for any $A \in \mathfrak{B}(\mathcal{H})$ and

$$\int_F \rho_x \mu_\rho(dx) = \mathfrak{M}(F)[\rho] \quad \forall F \in \mathcal{F}. \quad \text{(Bochner integral)}$$

By using the family $\{\rho_x\}_{x \in X}$, one can consider the mean entropy of *posteriori* state $\int_X H(\rho_x)\mu_\rho(dx)$, and assuming that $H(\rho) < +\infty$ one can define the entropy reduction as follows:

$$ER(\rho, \mathfrak{M}) = H(\rho) - \int_X H(\rho_x)\mu_\rho(dx).$$

This is a natural generalization of the entropy reduction considered in Sec. II for the class of measurements with a countable set of outcomes.

Ozawa proved in Ref. 21 that $ER(\rho, \mathfrak{M})$ is nonnegative if and only if the instrument $\mathfrak{M}$ is irreducible in the sense of the following definition.[24]

Definition 2: An instrument $\mathfrak{M}$ is called irreducible if for an arbitrary pure a priori *state ρ the posteriori states ρ_x are pure for μ_ρ-almost all x.*

An arbitrary instrument $\mathfrak{M}$ in a Hilbert space $\mathcal{H}$ can be represented as follows:

$$\mathfrak{M}^*(F)[A] = V^* \cdot A \otimes P(F) \cdot V, \quad A \in \mathfrak{B}(\mathcal{H}), \tag{13}$$

where $\mathfrak{M}^*(F)$ is a dual map to the map $\mathfrak{M}(F)$, V is an isometry from $\mathcal{H}$ into $\mathcal{H} \otimes \mathcal{H}_0$ and $P(F)$ is a spectral measure in $\mathcal{H}_0$[19] (see also Ref. 10).

The following notion introduced in Ref. 10 is a natural generalization of the notion of an efficient measurement with a countable outcome set.

Definition 3: An instrument $\mathfrak{M}$ is called efficient or pure if it has representation (13) with the spectral measure $P(F)$ of multiplicity one.

Below, we will use the term *efficient* to be consistent with the accepted terminology.

In Ref. 10, it is shown that an efficient instrument is irreducible. The converse assertion is not true (see Example 1 in the Appendix). A characterization of a quantum instrument, which is irreducible but not efficient, and a simple sufficient condition providing equivalence of efficiency and irreducibility are presented respectively in Proposition 5 and in Corollary 3 in the Appendix. This characterization shows that a quantum instrument, which is irreducible but not efficient, has quite singular properties.

It is the class of efficient instruments to which the results of Sec. II can be extended. We will use the following characterization of such instruments, where $\mathfrak{M} \otimes \mathfrak{I}_{\mathcal{K}}$ denotes the instrument $\mathfrak{M}(\cdot) \otimes \mathrm{Id}_{\mathcal{K}}$ in the Hilbert space $\mathcal{H} \otimes \mathcal{K}$ with the outcome set of the instrument $\mathfrak{M}$.

Proposition 2: Let $\mathfrak{M}$ be an instrument in a Hilbert space $\mathcal{H}$ with the outcome set $\{X, \mathcal{F}\}$. The following statements are equivalent:

(i)   *the instrument $\mathfrak{M}$ is efficient;*
(ii)   *the instrument $\mathfrak{M} \otimes \mathfrak{I}_{\mathcal{K}}$ is irreducible for 2-D Hilbert space $\mathcal{K}$;*
(iii) *the instrument $\mathfrak{M} \otimes \mathfrak{I}_{\mathcal{K}}$ is irreducible for a separable Hilbert space $\mathcal{K}$;*
(iv)   *there exist a positive σ-finite measure μ on $\{X, \mathcal{F}\}$, a dense domain $\mathcal{D} \subset \mathcal{H}$ and a function $x \mapsto V(x)$ defined for μ-almost all x, such that $V(x)$ is a linear operator from $\mathcal{D}$ to $\mathcal{H}$, satisfying*

$$\langle \varphi | \mathfrak{M}^*(F)[A]\varphi \rangle = \int_F \langle V(x)\varphi | A V(x)\varphi \rangle \mu(dx),$$

   for any $\varphi \in \mathcal{D}$, $F \in \mathcal{F}$, $A \in \mathfrak{B}(\mathcal{H})$.
(v)   *statement* (iv) *holds with $\mathcal{D} = \mathrm{lin}\{|\varphi_i\rangle\}$, where $\{|\varphi_i\rangle\}$ is a given arbitrary orthonormal basis in $\mathcal{H}$.*

Proof: (i) $\Rightarrow$ (iii). It is easy to see that the instruments $\mathfrak{M}$ and $\mathfrak{M} \otimes \mathfrak{J}_{\mathcal{K}}$ have representation (13) with the same spectral measure. Hence, efficiency of $\mathfrak{M}$ is equivalent to efficiency of $\mathfrak{M} \otimes \mathfrak{J}_{\mathcal{K}}$. As mentioned after the proof of Theorem 1 in Ref. 10 any efficient instrument is irreducible.

(iii) $\Rightarrow$ (ii) is obvious, since the instrument in (ii) can be considered as a restriction of the instrument in (iii).

(ii) $\Rightarrow$ (v). By Theorem 1 in Ref. 10 and its proof, there exist a positive σ-finite measure μ on $\{X, \mathcal{F}\}$ and a countable family $\{x \mapsto V_k(x)\}_k$ of functions defined for μ-almost all x, such that $V_k(x)$ is a linear operator from $\mathcal{D} = \lin(\{|\varphi_i\rangle\})$ to $\mathcal{H}$, satisfying

$$\langle \varphi| \mathfrak{M}^*(F)[A]\varphi\rangle = \int_F \sum_k \langle V_k(x)\varphi|A V_k(x)\varphi\rangle \mu(dx),$$

for any $\varphi \in \mathcal{D}$, $F \in \mathcal{F}$, $A \in \mathfrak{B}(\mathcal{H})$. Consider the family $\{\widehat{V}_k(x) = V_k(x) \otimes \mathrm{Id}_{\mathcal{K}}\}$ of linear operators from $\widehat{\mathcal{D}} = \lin(\{\varphi_i \otimes \phi_j\}_{ij})$ to $\mathcal{H} \otimes \mathcal{K}$, where $\{|\phi_j\rangle\}$ is an orthonormal basis of the space $\mathcal{K}$. By using the polarization identity it is easy to show that

$$\langle \hat{\varphi}| \mathfrak{M}^*(F) \otimes \mathrm{Id}_{\mathcal{K}}[C] \hat{\varphi}\rangle = \int_F \sum_k \langle \widehat{V}_k(x)\hat{\varphi}|C\widehat{V}_k(x)\hat{\varphi}\rangle \mu(dx),$$

for any $\hat{\varphi} \in \widehat{\mathcal{D}}$, $F \in \mathcal{F}$, $C \in \mathfrak{B}(\mathcal{H} \otimes \mathcal{K})$. Hence, for the instrument $\mathfrak{M} \otimes \mathfrak{J}_{\mathcal{K}}$ and a pure *a priori* state $\hat{\rho} = |\hat{\varphi}\rangle\langle\hat{\varphi}|$ (where $\hat{\varphi} \in \widehat{\mathcal{D}}$) we have

$$\mu_{\hat{\rho}}(dx) = \sum_k \|\widehat{V}_k(x)\hat{\varphi}\|^2 \mu(dx), \tag{14}$$

while the *posteriori* state corresponding to the outcome $x \in X \setminus X^s_{\hat{\varphi}}$ is

$$\hat{\rho}_x = \frac{\sum_k |\widehat{V}_k(x)\hat{\varphi}\rangle\langle\widehat{V}_k(x)\hat{\varphi}|}{\sum_k \|\widehat{V}_k(x)\hat{\varphi}\|^2}, \tag{15}$$

where $X^s_{\hat{\varphi}} = \{x \in X \mid \sum_k \|\widehat{V}_k(x)\hat{\varphi}\|^2 = 0\}$ is a set such that $\mu_{\hat{\rho}}(X^s_{\hat{\varphi}}) = 0$.[10]

By noting that the instrument $\mathfrak{M} \otimes \mathfrak{J}_{\mathcal{K}}$ is irreducible and by using (14), one can show existence of a set $X_{\hat{\varphi}} \in \mathcal{F}$ such that $\mu(X \setminus X_{\hat{\varphi}}) = 0$ and the above state $\hat{\rho}_x$ is pure for any $x \in X_{\hat{\varphi}} \setminus X^s_{\hat{\varphi}}$.

Let $\widehat{\mathcal{D}}_0$ be a countable subset of $\widehat{\mathcal{D}}$ consisting of finite linear combinations of the vectors of the family $\{\varphi_i \otimes \phi_j\}_{ij}$ with rational coefficients. Let $X_0 = \bigcap_{\hat{\varphi} \in \widehat{\mathcal{D}}_0} X_{\hat{\varphi}} \in \mathcal{F}$. Then $\mu(X \setminus X_0) = 0$ and the state $\hat{\rho}_x$ defined by (15) is pure for all $\hat{\varphi} \in \widehat{\mathcal{D}}_0$ and all $x \in X_0 \setminus X^s_{\hat{\varphi}}$. Hence, the family $\{|\widehat{V}_k(x)\hat{\varphi}\rangle\}_k$ consists of collinear vectors for all $\hat{\varphi} \in \widehat{\mathcal{D}}_0$ and all $x \in X_0$. Since the rank of the operator $\widehat{V}_k(x)$ is either 0 or > 1, Lemma 5 in the Appendix shows that $\widehat{V}_k(x) = \lambda_k(x)\widehat{V}_1(x)$, and hence $V_k(x) = \lambda_k(x)V_1(x)$, where $\lambda_k(x) \in \mathbb{C}$, for all k and all $x \in X_0$.

Consider the linear operator $V(x) = \sqrt{\sum_k |\lambda_k(x)|^2}\, V_1(x)$ defined on the set $\mathcal{D}$ for all $x \in X_0$. It is easy to see that

$$\langle \varphi|\mathfrak{M}^*(F)[A]\varphi\rangle = \int_F \langle V(x)\varphi|A V(x)\varphi\rangle \mu(dx),$$

for any $\varphi \in \mathcal{D}$, $F \in \mathcal{F}$, $A \in \mathfrak{B}(\mathcal{H})$.

(v) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). By the condition, the linear operator

$$\mathcal{D} \ni \varphi \mapsto V(x)\varphi \in L_2(X, \mathcal{F}, \mu, \mathcal{H}) \cong \mathcal{H} \otimes L_2(X, \mathcal{F}, \mu)$$

is isometrical, and hence it can be extended to the isometry V from $\mathcal{H}$ into $\mathcal{H} \otimes L_2(X, \mathcal{F}, \mu)$. A direct verification shows that

$$\langle \varphi|\mathfrak{M}^*(F)[A]\varphi\rangle = \langle V\varphi|(A \otimes P(F))V\varphi\rangle = \langle \varphi|V^*(A \otimes P(F))V\varphi\rangle,$$

for any vector φ in $\mathcal{H}$, where $P(\cdot)$ is the spectral measure defined as follows $(P(F)f)(x) = \chi_F(x)f(x)$ for any $f \in L_2(X, \mathcal{F}, \mu)$, where $\chi_F(\cdot)$ is the indicator function of the set $F \in \mathcal{F}$. Thus, the instrument $\mathfrak{M}$ is efficient. $\qquad\square$

We will also use the following simple observations.

Lemma 1: (1) *For arbitrary efficient instruments $\mathfrak{M}$ and $\mathfrak{N}$ in a separable Hilbert space $\mathcal{H}$ with the outcome sets $\{X, \mathcal{F}\}$ and $\{Y, \mathcal{E}\}$ the instrument $\mathfrak{N} \circ \mathfrak{M}$ in the space $\mathcal{H}$ with the outcome set $\{X \times Y, \mathcal{F} \otimes \mathcal{E}\}$, defined by the relation $\mathfrak{N} \circ \mathfrak{M}(F \times E) = \mathfrak{N}(E) \circ \mathfrak{M}(F)$, $F \in \mathcal{F}$, $E \in \mathcal{E}$, is efficient.*

(2) *For arbitrary efficient instruments $\mathfrak{M}$ and $\mathfrak{N}$ in separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with the outcome sets $\{X, \mathcal{F}\}$ and $\{Y, \mathcal{E}\}$ the instrument $\mathfrak{M} \otimes \mathfrak{N}$ in the space $\mathcal{H} \otimes \mathcal{K}$ with the outcome set $\{X \times Y, \mathcal{F} \otimes \mathcal{E}\}$, defined by the relation $\mathfrak{M} \otimes \mathfrak{N}(F \times E) = \mathfrak{M}(F) \otimes \mathfrak{N}(E)$, $F \in \mathcal{F}$, $E \in \mathcal{E}$, is efficient.*

Proof: It is easy to show that the instruments $\mathfrak{N} \circ \mathfrak{M}$ and $\mathfrak{M} \otimes \mathfrak{N}$ have representation (13) with the spectral measure $P_{\mathfrak{M}} \otimes P_{\mathfrak{N}}$, where $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ are spectral measures corresponding to the instruments $\mathfrak{M}$ and $\mathfrak{N}$. $\qquad\square$

B. A representation for the entropy reduction

To extend the results of the Sec. II to the case of general type measurement consider the construction proposed by Barchielli and Lupieri in Ref. 2. Choose a positive complete measure μ_0 on $(X, \mathcal{F})$ such that μ_ρ is absolutely continuous with respect to μ_0 for all ρ in $\mathfrak{S}(\mathcal{H})$ (this can be done by using the measure μ_{ρ_0}, where ρ_0 is a given full rank state in $\mathfrak{S}(\mathcal{H})$). Let $L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H}))$ be the W*-algebra of μ_0-essentially bounded $\mathfrak{B}(\mathcal{H})$-valued weakly* measurable functions on X with the predual Banach space $L_1(X, \mathcal{F}, \mu_0, \mathfrak{T}(\mathcal{H}))$ of $\mathfrak{T}(\mathcal{H})$-valued Bochner μ_0-integrable functions on X. By Theorem 2 in Ref. 2 with an arbitrary instrument $\mathfrak{M}$, one can associate a channel $\Lambda_{\mathfrak{M}}^* : L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H})) \to \mathfrak{B}(\mathcal{H})$ defined by the relation

$$\mathrm{Tr}\,\Lambda_{\mathfrak{M}}^*(A \otimes f)\rho = \int_X f(x)\mathrm{Tr}\mathfrak{M}(dx)[\rho]A, \quad A \in \mathfrak{B}(\mathcal{H}), \ f \in L_\infty(X, \mathcal{F}, \mu_0), \ \rho \in \mathfrak{S}(\mathcal{H}).$$

The preadjoint channel $\Lambda_{\mathfrak{M}} : \mathfrak{T}(\mathcal{H}) \to L_1(X, \mathcal{F}, \mu_0, \mathfrak{T}(\mathcal{H}))$ produces the *posteriori* family as follows:

$$\rho_x = \begin{cases} (\mathrm{Tr}\sigma(x))^{-1}\sigma(x), & \mathrm{Tr}\sigma(x) \neq 0, \\ \rho_0, & \mathrm{Tr}\sigma(x) = 0, \end{cases} \tag{16}$$

where $\sigma(x)$ is a particular representative of the class $\Lambda_{\mathfrak{M}}(\rho)$, while the function $\mathrm{Tr}\sigma(x)$ is a probability density (the Radon-Nikodym derivative) of the measure μ_ρ with respect to the measure μ_0.

Consider the channel $\Pi_{\mathfrak{M}}^* : L_\infty(X, \mathcal{F}, \mu_0) \to \mathfrak{B}(\mathcal{H})$ defined by the relation

$$\mathrm{Tr}\,\rho\,\Pi_{\mathfrak{M}}^*(f) = \int_X f(x)\mathrm{Tr}\mathfrak{M}(dx)[\rho], \quad f \in L_\infty(X, \mathcal{F}, \mu_0), \ \rho \in \mathfrak{S}(\mathcal{H}). \tag{17}$$

The preadjoint channel $\Pi_{\mathfrak{M}} : \mathfrak{T}(\mathcal{H}) \to L_1(X, \mathcal{F}, \mu_0)$ maps an arbitrary *a priori* state ρ to the probability density of the *posteriori* measure μ_ρ with respect to the measure μ_0, and hence it can be considered as a natural generalization of the channel $\Pi_{\mathfrak{M}}$ defined by (6) (since the last channel is a quantum-quantum modification of the channel mapping an a priory state ρ to the probability distribution $\{\pi_i(\rho)\}_{i \in I}$).

Note that $\Pi_{\mathfrak{M}} = \Theta \circ \Lambda_{\mathfrak{M}}$, where Θ is the preadjoint channel of the channel

$$\Theta^* : L_\infty(X, \mathcal{F}, \mu_0) \ni f \mapsto f \otimes I_{\mathcal{H}} \in L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H})).$$

Since the channel $\Pi_{\mathfrak{M}}$ defined by (17) has no quantum-quantum modification, to extend the results of Section II we have to generalize the notion of quantum mutual information.

Definition 4: Let $\mathcal{A}$ be an arbitrary W-algebra and $\Phi^* : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ be a channel with the preadjoint channel $\Phi : \mathfrak{T}(\mathcal{H}) \to \mathcal{A}_*$. Let ρ be a state in $\mathfrak{S}(\mathcal{H})$. The quantum mutual information of the channel Φ at the state ρ is defined as follows:*

$$I(\rho, \Phi) = H\left(\Phi \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}) \| \Phi \otimes \mathrm{Id}_{\mathcal{K}}(\rho \otimes \varrho)\right),$$

where $\mathcal{K}$ is a Hilbert space isomorphic to $\mathcal{H}$, $\hat{\rho}$ is a purification of the state ρ in the space $\mathcal{H} \otimes \mathcal{K}$, $\varrho = \mathrm{Tr}_{\mathcal{H}} \hat{\rho}$ is a state in $\mathfrak{S}(\mathcal{K})$ isomorphic to ρ and $H(\cdot \| \cdot)$ is the relative entropy for two states in $(\mathcal{A} \otimes \mathfrak{B}(\mathcal{K}))_*$.

Remark 1: It is natural to ask about validity for the above-defined value of the properties of the quantum mutual information of a quantum-quantum infinite dimensional channel presented in Propositions 1 and 4 in Ref. 8. Since the proofs of these propositions can not be directly generalized to the case of a channel considered in Definition 4, the above question is not trivial.

Here, we note only that for an arbitrary channel $\Phi : \mathfrak{T}(\mathcal{H}) \mapsto \mathcal{A}_*$ the function $\rho \mapsto I(\rho, \Phi)$ is nonnegative and lower semicontinuous and that for an arbitrary channel $\Psi : \mathcal{A}_* \to \mathfrak{B}_*$, where $\mathfrak{B}$ is an other W^*-algebra, the inequality $I(\rho, \Psi \circ \Phi) \leq I(\rho, \Phi)$ holds for all ρ (the 1-st chain rule). These properties follow from nonnegativity and lower semicontinuity of the relative entropy, Lemma 2 in Ref. 8 and Uhlmann's monotonicity theorem.[18]

Since we will use Definition 4 with $\mathcal{A} = L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H}))$, we will deal with the relative entropy for states in $(\mathcal{A} \otimes \mathfrak{B}(\mathcal{K}))_* = L_1(X, \mathcal{F}, \mu_0, \mathfrak{T}(\mathcal{H} \otimes \mathcal{K}))$.

The relative entropy for two states σ_1 and σ_2 in $L_1(X, \mathcal{F}, \mu_0, \mathfrak{T}(\mathcal{H}))$ can be expressed as follows:

$$H(\sigma_1 \| \sigma_2) = \int_X \mathrm{Tr}\left(\sigma_1(x)\left(\log\sigma_1(x) - \log\sigma_2(x)\right)\right)\mu_0(dx)$$

$$= \int_X H_q(\sigma_1(x) \| \sigma_2(x))\mu_0(dx) = \int_X H_q\left(\frac{\sigma_1(x)}{\mathrm{Tr}\sigma_1(x)} \,\bigg\|\, \frac{\sigma_2(x)}{\mathrm{Tr}\sigma_2(x)}\right)\mu_1(dx) + H_c(\mu_1 \| \mu_2),$$

(18)

where $\mu_1(dx) = \mathrm{Tr}\sigma_1(x)\mu_0(dx)$ (see [2, formula (4)]). In this expression H_q denotes the quantum relative entropy for two positive trace class operators defined by (1), while H_c denotes the classical relative entropy for two probability measures, that is $H_c(\mu_1 \| \mu_2) = \int_X \log\frac{\mathrm{Tr}\sigma_1(x)}{\mathrm{Tr}\sigma_2(x)}\mu_1(dx)$.

For an arbitrary instrument $\mathfrak{M}$ equality (7) does not hold, but one can prove the following estimation.

Proposition 3. Let $\mathfrak{M}$ be an arbitrary instrument in a Hilbert space $\mathcal{H}$ with the outcome set $\{X, \mathcal{F}\}$ and ρ be a state in $\mathfrak{S}(\mathcal{H})$ with finite entropy. Let $\hat{\rho}$ be a purification of the state ρ in the space $\mathcal{H} \otimes \mathcal{K}$. Then

$$|ER(\rho, \mathfrak{M}) - I(\rho, \Pi_{\mathfrak{M}})| \leq \int_X H(\hat{\rho}_x)\mu_\rho(dx),$$

where $\Pi_{\mathfrak{M}}$ is the quantum-classical channel defined by (17), $\mu_\rho(\cdot) = \mathrm{Tr}\mathfrak{M}(\cdot)[\rho]$ and $\{\hat{\rho}_x\}$ is the family of posteriori states corresponding to the instrument $\widehat{\mathfrak{M}}(\cdot) = \mathfrak{M}(\cdot) \otimes \mathrm{Id}_{\mathcal{K}}$ and the a priori state $\hat{\rho}$.

Proof: Consider the channels

$$\Lambda_{\mathfrak{M}} : \mathfrak{T}(\mathcal{H}) \to L_1(X, \mathcal{F}, \mu_0, \mathfrak{T}(\mathcal{H})) \quad \text{and} \quad \Lambda_{\widehat{\mathfrak{M}}} : \mathfrak{T}(\mathcal{H} \otimes \mathcal{K}) \to L_1(X, \mathcal{F}, \mu_0, \mathfrak{T}(\mathcal{H} \otimes \mathcal{K}))$$

produced by the Barchielli-Lupieri construction described before (since $\hat{\mu}_\omega(\cdot) = \mathrm{Tr}\widehat{\mathfrak{M}}(\cdot)[\omega] = \mathrm{Tr}\mathfrak{M}(\cdot)[\omega_{\mathcal{H}}] = \mu_{\omega_{\mathcal{H}}}(\cdot)$ for any $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, we can use the same measure μ_0 in the both cases). By noting that $L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H} \otimes \mathcal{K})) \cong L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H})) \otimes \mathfrak{B}(\mathcal{K})$ it is easy to show that $\Lambda_{\widehat{\mathfrak{M}}} = \Lambda_{\mathfrak{M}} \otimes \mathrm{Id}_{\mathcal{K}}$.

Let $\{\hat{\rho}_x\}$ be the family of *posteriori* states obtained via a given representative of the class $\Lambda_{\widehat{\mathfrak{M}}}(\hat{\rho})$ by the rule similar to (16). It is easy to see that $\{\rho_x \doteq \mathrm{Tr}_{\mathcal{K}}\hat{\rho}_x\}$ is the family of *posteriori* states for the instrument $\mathfrak{M}$ corresponding to the *a priori* state ρ. Since the instrument $\widehat{\mathfrak{M}}$ is localized in the

space $\mathcal{H}$ we have $\int_X \mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x \mu_\rho(dx) = \mathrm{Tr}_{\mathcal{H}} \hat{\rho} = \varrho \cong \rho$. By using expression (18) we obtain

$$I(\rho, \Pi_{\mathfrak{M}}) \doteq H\left(\Pi_{\mathfrak{M}} \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}) \,\|\, \Pi_{\mathfrak{M}}(\rho) \otimes \varrho\right) = H\left(\Theta \otimes \mathrm{Id}_{\mathcal{K}}(\Lambda_{\widehat{\mathfrak{M}}}(\hat{\rho})) \,\|\, \Pi_{\mathfrak{M}}(\rho) \otimes \varrho\right)$$

$$= \int_X H_q(\mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x \,\|\, \varrho) \mu_\rho(dx) = -\int_X H(\mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x) \mu_\rho(dx) + \int_X \mathrm{Tr}(\mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x(-\log \varrho)) \mu_\rho(dx)$$

$$= -\int_X H(\mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x) \mu_\rho(dx) + \mathrm{Tr}\varrho\,(-\log \varrho)$$

$$= \left[H(\rho) - \int_X H(\rho_x) \mu_\rho(dx)\right] + \int_X (H(\mathrm{Tr}_{\mathcal{K}} \hat{\rho}_x) - H(\mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x)) \mu_\rho(dx).$$

By the triangle inequality, the absolute value of the last term in this expression is majorized by $\int_X H(\hat{\rho}_x) \mu_\rho(dx)$. □

Propositions 2 and 3 imply the following generalization of equality (7).

Corollary 1: Let $\mathfrak{M}$ be an efficient instrument in a Hilbert space $\mathcal{H}$. Then $ER(\rho, \mathfrak{M}) = I(\rho, \Pi_{\mathfrak{M}})$ for any state ρ in $\mathfrak{S}(\mathcal{H})$ with finite entropy.

Corollary 1 makes it possible to consider the entropy reduction of an efficient quantum measurement not only for *a priori* states with finite entropy and motivates the following extended version of Definition 1.

Definition 5: The entropy reduction of an efficient instrument $\mathfrak{M}$ in a Hilbert space $\mathcal{H}$ at an arbitrary a priori state $\rho \in \mathfrak{S}(\mathcal{H})$ is defined as follows:

$$ER(\rho, \mathfrak{M}) \doteq I(\rho, \Pi_{\mathfrak{M}}),$$

where $\Pi_{\mathfrak{M}}$ is the quantum-classical channel defined by (17).

The following theorem is an extended version of Theorem 1.

Theorem 2: *Let $\mathfrak{M}$ be an arbitrary efficient instrument in a Hilbert space $\mathcal{H}$. The function $\rho \mapsto ER(\rho, \mathfrak{M})$ is nonnegative concave and lower semicontinuous on the set $\mathfrak{S}(\mathcal{H})$. It has the following properties:*

(1) $\{ER(\rho, \mathfrak{M}) = 0\} \Leftrightarrow \{\rho_x \cong \rho \ for \ \mu_\rho\text{-almost all } x\}$, *where $\{\rho_x\}$ and μ_ρ are respectively the family of posteriori states and the posteriori probability measure corresponding to the a priori state ρ;*

(2) *continuity on any subset of $\mathfrak{S}(\mathcal{H})$ on which the von Neumann entropy is continuous:*

$$\lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty \ \Rightarrow \ \lim_{n \to +\infty} ER(\rho_n, \mathfrak{M}) = ER(\rho_0, \mathfrak{M}) < +\infty$$

for any sequence $\{\rho_n\}$ of states converging to a state ρ_0;

(3) *monotonicity: for arbitrary efficient instruments $\mathfrak{M}$ and $\mathfrak{N}$ in a separable Hilbert space $\mathcal{H}$ the inequality*

$$ER(\rho, \mathfrak{N} \circ \mathfrak{M}) \geq ER(\rho, \mathfrak{M}) \tag{19}$$

holds for any $\rho \in \mathfrak{S}(\mathcal{H})$, where $\mathfrak{N} \circ \mathfrak{M}$ is the instrument defined in Lemma 1;

(4) *subadditivity: for arbitrary efficient instruments $\mathfrak{M}$ and $\mathfrak{N}$ in separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ the inequality*

$$ER(\omega, \mathfrak{M} \otimes \mathfrak{N}) \leq ER(\omega_{\mathcal{H}}, \mathfrak{M}) + ER(\omega_{\mathcal{K}}, \mathfrak{N}) \tag{20}$$

holds for any $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, where $\mathfrak{N} \otimes \mathfrak{M}$ is the instrument defined in Lemma 1.

Proof: By Definition 5 lower semicontinuity of the function $\rho \mapsto ER(\rho, \mathfrak{M})$ follows from the second part of Remark 1.

Concavity of the function $\rho \mapsto ER(\rho, \mathfrak{M})$ on the convex subset of states with finite entropy follows from inequality (56) in Ref. 2. Concavity of this function on the set $\mathfrak{S}(\mathcal{H})$ can be proved by using lower semicontinuity of this function and Lemma 2 below (see the proof of Proposition 1 in Ref. 8).

Below, we will use the notations introduced in the proof of Proposition 3.

(1) Note that

$$ER(\rho, \mathfrak{M}) \doteq I(\rho, \Pi_{\mathfrak{M}}) \doteq H\left(\Theta \otimes \mathrm{Id}_{\mathcal{K}}(\Lambda_{\widehat{\mathfrak{M}}}(\hat{\rho})) \,\|\, \Theta(\Lambda_{\mathfrak{M}}(\rho)) \otimes \varrho\right) = 0$$

means

$$\Theta \otimes \mathrm{Id}_{\mathcal{K}}(\Lambda_{\widehat{\mathfrak{M}}}(\hat{\rho})) = \Theta(\Lambda_{\mathfrak{M}}(\rho)) \otimes \varrho$$

by the well known property of the relative entropy. This equality holds if and only if

$$\mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x = \varrho \quad \text{for } \mu_\rho\text{-almost all } x. \tag{21}$$

"$\Leftarrow$" Let $\varrho_x = \mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x$. Since the instrument $\widehat{\mathfrak{M}}$ is irreducible (by Proposition 2), we have $\rho_x \cong \varrho_x$ for μ_ρ-almost all x, and hence $\varrho_x \cong \varrho = \mathrm{Tr}_{\mathcal{H}} \hat{\rho}$ for μ_ρ-almost all x. Since the instrument $\widehat{\mathfrak{M}}$ is localized in the space $\mathcal{H}$, we have $\varrho = \int_X \varrho_x \mu_\rho(dx)$. Thus, Lemma 6 in the Appendix implies (21).

"$\Rightarrow$" Since the instrument $\widehat{\mathfrak{M}}$ is irreducible (by Proposition 2), it follows from (21) that $\rho_x = \mathrm{Tr}_{\mathcal{K}} \hat{\rho}_x \cong \mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x = \varrho \cong \rho$ for μ_ρ-almost all x.

(2) This property follows from identity (24) in Lemma 3 below, since the both summands in the left side of this identity are lower semicontinuous functions on the set $\mathfrak{S}(\mathcal{H})$ by the second part of Remark 1.

(3) This follows from the 1-st chain rule for the generalized quantum mutual information mentioned in the second part of Remark 1. Indeed, let μ_0 be a measure on $\{X \times Y, \mathcal{F} \otimes \mathcal{E}\}$ chosen in accordance with the Barchielli-Lupieri construction for the instrument $\mathfrak{N} \circ \mathfrak{M}$ and ν_0 be a measure on $\{X, \mathcal{F}\}$ such that $\nu_0(F) = \mu_0(F \times Y)$ for any $F \in \mathcal{F}$. Since $\mathfrak{N}(Y)$ is a trace preserving map, we have $\mathrm{Tr}\,\mathfrak{N} \circ \mathfrak{M}(F \times Y)[\rho] = \mathrm{Tr}\,\mathfrak{N}(Y)[\mathfrak{M}(F)[\rho]] = \mathrm{Tr}\,\mathfrak{M}(F)[\rho]$ for any $F \in \mathcal{F}$ and $\rho \in \mathfrak{S}(\mathcal{H})$. Hence, the measure ν_0 can be used in the Barchielli-Lupieri construction for the instrument $\mathfrak{M}$. Let Ξ be a channel from $L_1(X \times Y, \mathcal{F} \otimes \mathcal{E}, \mu_0)$ to $L_1(X, \mathcal{F}, \nu_0)$ preadjoint to the channel

$$L_\infty(X, \mathcal{F}, \nu_0) \ni f \mapsto \Xi^*(f) = f \otimes \mathbf{1}_y \in L_\infty(X \times Y, \mathcal{F} \otimes \mathcal{E}, \mu_0),$$

where $f \otimes \mathbf{1}_y(x, y) \doteq f(x)\mathbf{1}(y)$. Then $\Pi_{\mathfrak{M}} = \Xi \circ \Pi_{\mathfrak{N} \circ \mathfrak{M}}$. This follows from the relation

$$\mathrm{Tr}\,\rho\,\Pi_{\mathfrak{N} \circ \mathfrak{M}}^* \circ \Xi^*(f) = \mathrm{Tr}\,\rho\,\Pi_{\mathfrak{N} \circ \mathfrak{M}}^*(f \otimes \mathbf{1}_y) \doteq \int_{X \times Y} f(x)\mathrm{Tr}\,\mathfrak{N}(dy)[\mathfrak{M}(dx)[\rho]]$$

$$= \int_X f(x)\mathrm{Tr}\,\mathfrak{M}(dx)[\rho] \doteq \mathrm{Tr}\,\rho\,\Pi_{\mathfrak{M}}^*(f), \quad \rho \in \mathfrak{S}(\mathcal{H}), \ f \in L_\infty(X, \mathcal{F}, \nu_0),$$

which can be proved easily by noting that $\mathfrak{N}(Y)$ is a trace preserving map.

(4) Let μ_0 and ν_0 be measures on $\{X, \mathcal{F}\}$ and on $\{Y, \mathcal{E}\}$ chosen in accordance with the Barchielli-Lupieri construction for the instruments $\mathfrak{M}$ and $\mathfrak{N}$ correspondingly. Then for the instrument $\mathfrak{M} \otimes \mathfrak{N}$, one can take the measure $\mu_0 \otimes \nu_0$. By noting that

$$L_\infty(X \times Y, \mathcal{F} \otimes \mathcal{E}, \mu_0 \otimes \nu_0, \mathfrak{B}(\mathcal{H} \otimes \mathcal{K})) = L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H})) \otimes L_\infty(Y, \mathcal{E}, \nu_0, \mathfrak{B}(\mathcal{K}))$$

it is easy to show that $\Lambda_{\mathfrak{M} \otimes \mathfrak{N}}^* = \Lambda_{\mathfrak{M}}^* \otimes \Lambda_{\mathfrak{N}}^*$.

Let ω be a state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that $H(\omega_{\mathcal{H}})$ and $H(\omega_{\mathcal{K}})$ are finite. Then inequality (20) for the state ω follows from the inequality

$$H(\omega_{\mathcal{H}}) + H(\omega_{\mathcal{K}}) - H(\omega) = H(\omega \,\|\, \omega_{\mathcal{H}} \otimes \omega_{\mathcal{K}}) \geq H(\Lambda_{\mathfrak{M}\otimes\mathfrak{N}}(\omega) \,\|\, \Lambda_{\mathfrak{M}}(\omega_{\mathcal{H}}) \otimes \Lambda_{\mathfrak{N}}(\omega_{\mathcal{K}}))$$

$$\geq \int_{X \times Y} H_q(\omega_{xy} \,\|\, (\omega_{\mathcal{H}})_x \otimes (\omega_{\mathcal{K}})_y)\mu_\omega(dxdy) = -\int_{X \times Y} H(\omega_{xy})\mu_\omega(dxdy)$$

$$+ \int_{X \times Y} \mathrm{Tr}\, \omega_{xy}\left((-\log(\omega_{\mathcal{H}})_x) \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes (-\log(\omega_{\mathcal{K}})_y)\right)\mu_\omega(dxdy)$$

$$= -\int_{X \times Y} H(\omega_{xy})\mu_\omega(dxdy) + \int_X H((\omega_{\mathcal{H}})_x)\mu_{\omega_{\mathcal{H}}}(dx) + \int_Y H((\omega_{\mathcal{K}})_y)\mu_{\omega_{\mathcal{K}}}(dy),$$

obtain by using monotonicity of the relative entropy, expression (18) and the equalities

$$\int_{X \times Y} \mathrm{Tr}\, \omega_{xy}((-\log(\omega_{\mathcal{H}})_x) \otimes I_{\mathcal{K}})\mu_\omega(dxdy) = \int_X H((\omega_{\mathcal{H}})_x)\mu_{\omega_{\mathcal{H}}}(dx),$$

$$\int_{X \times Y} \mathrm{Tr}\, \omega_{xy}(I_{\mathcal{H}} \otimes (-\log(\omega_{\mathcal{K}})_y))\mu_\omega(dxdy) = \int_Y H((\omega_{\mathcal{K}})_y)\mu_{\omega_{\mathcal{K}}}(dy). \tag{22}$$

Prove the first of the above equalities. Let f be a continuous bounded function on $\mathbb{R}$. Then $f((\omega_{\mathcal{H}})_x) \in L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H}))$ and

$$\int_{X \times Y} \mathrm{Tr}\, \omega_{xy}(f((\omega_{\mathcal{H}})_x) \otimes I_{\mathcal{K}})\mu_\omega(dxdy) = \left\langle \Lambda_{\mathfrak{M}\otimes\mathfrak{N}}(\omega),\ f((\omega_{\mathcal{H}})_x) \otimes (I_{\mathcal{K}} \otimes \mathbf{1}_y) \right\rangle$$

$$= \mathrm{Tr}\, \omega \Lambda^*_{\mathfrak{M}\otimes\mathfrak{N}}(f((\omega_{\mathcal{H}})_x) \otimes (I_{\mathcal{K}} \otimes \mathbf{1}_y)) = \mathrm{Tr}\, \omega_{\mathcal{H}} \Lambda^*_{\mathfrak{M}}(f((\omega_{\mathcal{H}})_x))$$

$$= \langle \Lambda_{\mathfrak{M}}(\omega_{\mathcal{H}}),\ f((\omega_{\mathcal{H}})_x) \rangle = \int_X \mathrm{Tr}(\omega_{\mathcal{H}})_x f((\omega_{\mathcal{H}})_x)\mu_{\omega_{\mathcal{H}}}(dx).$$

Hence, the first equality in (22) can be proved by using approximation of the function $-\log x$ on $[0, 1]$ by an increasing sequence of continuous bounded functions and the monotone convergence theorem.

Let ω^0 be an arbitrary state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. Let $\{P_n\}$ and $\{Q_n\}$ be increasing sequences of finite rank spectral projectors of the states $\omega^0_{\mathcal{H}}$ and $\omega^0_{\mathcal{K}}$ strongly converging to the operators $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ correspondingly. Consider the sequence of states

$$\omega^n = \left(\mathrm{Tr}\left((P_n \otimes Q_n) \cdot \omega^0\right)\right)^{-1} (P_n \otimes Q_n) \cdot \omega^0 \cdot (P_n \otimes Q_n),$$

converging to the state ω^0. A direct verification shows that

$$\lambda_n \omega^n_{\mathcal{H}} \leq \omega^0_{\mathcal{H}} \quad \text{and} \quad \lambda_n \omega^n_{\mathcal{K}} \leq \omega^0_{\mathcal{K}}, \quad \text{where} \quad \lambda_n = \mathrm{Tr}\left((P_n \otimes Q_n) \cdot \omega^0\right).$$

Hence, concavity and lower semicontinuity of the entropy reduction imply

$$\lim_{n \to +\infty} ER(\omega^n_{\mathcal{H}}, \mathfrak{M}) = ER(\omega^0_{\mathcal{H}}, \mathfrak{M}) \quad \text{and} \quad \lim_{n \to +\infty} ER(\omega^n_{\mathcal{K}}, \mathfrak{N}) = ER(\omega^0_{\mathcal{K}}, \mathfrak{N})$$

(this can be shown by using the arguments from the proof of Ref. 23, (Lemma 6).]).

Since inequality (20) holds with $\omega = \omega^n$ for all n, these limit relations and lower semicontinuity of the entropy reduction show that inequality (20) holds for the state ω^0. □

Lemma 2: Let $\mathfrak{M}$ *be an efficient instrument in a Hilbert space* $\mathcal{H}$ *and* ρ_0 *be a state in* $\mathfrak{S}(\mathcal{H})$ *with the spectral representation* $\rho_0 = \sum_{i=1}^{+\infty} \lambda_i |e_i\rangle\langle e_i|$. *Let* $\rho_n = c_n^{-1} \sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i|$, *where* $c_n = \sum_{i=1}^{n} \lambda_i$ *for each n, then*

$$\lim_{n \to +\infty} ER(\rho_n, \mathfrak{M}) = ER(\rho_0, \mathfrak{M}).$$

Proof: Let $\mathcal{K} \cong \mathcal{H}$ and $P_n = \sum_{i=1}^{n} |e_i\rangle\langle e_i|$ be a projector in $\mathcal{K}$, $n = 1, 2 \ldots$ Consider the value

$$I_n = H(\Pi_{\mathfrak{M}} \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}_n) \| \Pi_{\mathfrak{M}}(\rho_0) \otimes \varrho_n) = H(c_n^{-1} \Pi_{\mathfrak{M}} \otimes \Psi_n(\hat{\rho}_0)) \| c_n^{-1} \Pi_{\mathfrak{M}}(\rho_0) \otimes \Psi_n(\varrho_0))),$$

where

$$\hat{\rho}_0 = \sum_{i,j=1}^{+\infty} \sqrt{\lambda_i \lambda_j} |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j|, \quad \hat{\rho}_n = c_n^{-1} I_{\mathcal{H}} \otimes P_n \cdot \hat{\rho}_0 \cdot I_{\mathcal{H}} \otimes P_n,$$

$\varrho_0 = \mathrm{Tr}_{\mathcal{H}}\hat{\rho}_0$, $\varrho_n = \mathrm{Tr}_{\mathcal{H}}\hat{\rho}_n$ and $\Psi_n(\cdot) = P_n(\cdot)P_n$ is a map from $\mathfrak{T}(\mathcal{K})$ to itself. We will show that

$$\lim_{n \to +\infty} I_n = H\left(\Pi_{\mathfrak{M}} \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}_0) \| \Pi_{\mathfrak{M}}(\rho_0) \otimes \varrho_0\right) = ER(\rho_0, \mathfrak{M}). \tag{23}$$

Let $\sigma_1(x)$ and $\sigma_2(x)$ be representatives of the classes $\Pi_{\mathfrak{M}} \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}_0)$ and $\Pi_{\mathfrak{M}}(\rho_0) \otimes \varrho_0$ correspondingly. Then $c_n^{-1} P_n \sigma_1(x) P_n$ and $c_n^{-1} P_n \sigma_2(x) P_n$ are respectively representatives of the classes $c_n^{-1} \Pi_{\mathfrak{M}} \otimes \Psi_n(\hat{\rho}_0)$ and $c_n^{-1} \Pi_{\mathfrak{M}}(\rho_0) \otimes \Psi_n(\varrho_0)$. Expression (18) implies

$$I_n = \int_X c_n^{-1} H_q(P_n \sigma_1(x) P_n \| P_n \sigma_2(x) P_n) \mu_0(dx),$$

$$H\left(\Pi_{\mathfrak{M}} \otimes \mathrm{Id}_{\mathcal{K}}(\hat{\rho}_0) \| \Pi_{\mathfrak{M}}(\rho_0) \otimes \varrho_0\right) = \int_X H_q(\sigma_1(x) \| \sigma_2(x)) \mu_0(dx)$$

Hence, (23) follows from Lemma 4 in Ref. 13 and the monotone convergence theorem.

By using expression (18) we obtain

$$0 \leq I_n - ER(\rho_n, \mathfrak{M}) = H_c(\mu_{\rho_n} \| \mu_{\rho_0}) = \int_X \log \frac{\mu_{\rho_n}(dx)}{\mu_{\rho_0}(dx)} \mu_{\rho_n}(dx) \leq -\log c_n$$

since $c_n \mu_{\rho_n}(F) \leq \mu_{\rho_0}(F)$ for all $F \in \mathcal{F}$. Hence, $\lim_n (I_n - ER(\rho_n, \mathfrak{M})) = 0$. This and (23) imply the assertion of the lemma. $\square$

Lemma 3: Let ρ be a state in $\mathfrak{S}(\mathcal{H})$ such that $H(\rho) < +\infty$. Then

$$I(\rho, \Pi_{\mathfrak{M}}) + I(\rho, \Lambda_{\mathfrak{M}}) = 2H(\rho). \tag{24}$$

Proof: Since the instrument $\widehat{\mathfrak{M}}$ is irreducible (by Proposition 2) and is localized in the space $\mathcal{H}$ we have $H(\hat{\rho}_x) = 0$ for μ_ρ-almost all $x \in X$ and $\int_X \mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x \mu_\rho(dx) = \mathrm{Tr}_{\mathcal{H}} \hat{\rho} = \varrho \cong \rho$. By using expression (18), we obtain

$$I(\rho, \Lambda_{\mathfrak{M}}) \doteq H\left(\Lambda_{\widehat{\mathfrak{M}}}(\hat{\rho}) \| \Lambda_{\mathfrak{M}}(\rho) \otimes \varrho\right) = \int_X H_q(\hat{\rho}_x \| \rho_x \otimes \varrho) \mu_\rho(dx)$$

$$= -\int_X H(\hat{\rho}_x) \mu_\rho(dx) + \int_X \mathrm{Tr}_{\mathcal{K}} \hat{\rho}_x(-\log \rho_x) \mu_\rho(dx) + \int_X \mathrm{Tr}_{\mathcal{H}} \hat{\rho}_x(-\log \varrho) \mu_\rho(dx)$$

$$= \int_X \mathrm{Tr}\rho_x(-\log \rho_x) \mu_\rho(dx) + \mathrm{Tr}\varrho(-\log \varrho) = \int_X H(\rho_x) \mu_\rho(dx) + H(\rho).$$

This expression and Corollary 1 imply (24). $\square$

Remark 2: By proving concavity of the function $\rho \mapsto I(\rho, \Lambda_{\mathfrak{M}})$ and by using Lemma 6 in Ref. 23, one can show validity of equality (24) for any ρ in $\mathfrak{S}(\mathcal{H})$. By comparing this equality with the assertion of Theorem 1 in Ref. 8, we see that the channel $\Lambda_{\mathfrak{M}}$ plays the role of the complementary channel to the channel $\Pi_{\mathfrak{M}}$. Strictly speaking, this holds in the discrete case when the quantum-classical channel $\Pi_{\mathfrak{M}}$ can be considered as a quantum-quantum channel (see Section II).

The following proposition is a generalization of Proposition 1.

Proposition 4: Let $\{\mathfrak{M}_n\}$ be a sequence of efficient quantum instruments with the same outcome space $\{X, \mathcal{F}\}$ converging to the instrument $\mathfrak{M}_0$ in the following sense

$$\|\cdot\|_1\text{-}\lim_{n \to +\infty} \mathfrak{M}_n(F)[\rho] = \mathfrak{M}_0(F)[\rho] \quad \forall F \in \mathcal{F}, \ \forall \rho \in \mathfrak{S}(\mathcal{H}). \tag{25}$$

Then for an arbitrary sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$ converging to a state ρ_0 the following relation holds

$$\liminf_{n\to+\infty} ER(\rho_n, \mathfrak{M}_n) \geq ER(\rho_0, \mathfrak{M}_0).$$

If, in addition, $\lim_{n\to+\infty} H(\rho_n) = H(\rho_0) < +\infty$ then

$$\lim_{n\to+\infty} ER(\rho_n, \mathfrak{M}_n) = ER(\rho_0, \mathfrak{M}_0).$$

Proof: By Lemma 2 in Ref. 8 there exists a sequence $\{\hat{\rho}_n\}$ of purifications of the states $\{\rho_n\}$ converging to a purification $\hat{\rho}_0$ of the state ρ_0. It is clear that we can choose a positive complete measure μ_0 on $(X, \mathcal{F})$ which can be used in the Barchielli-Lupieri construction for each instrument from the sequence $\{\mathfrak{M}_n\}$. By using Lemma 4 below it is easy to show that $p.w.\text{-}\lim_{n\to+\infty} \Lambda_{\mathfrak{M}_n}(\rho_n) = \Lambda_{\mathfrak{M}_0}(\rho_0)$ and $p.w.\text{-}\lim_{n\to+\infty} \Lambda_{\widehat{\mathfrak{M}}_n}(\hat{\rho}_n) = \Lambda_{\widehat{\mathfrak{M}}_0}(\hat{\rho}_0)$.

The first assertion follows from lower semicontinuity of the relative entropy with respect to pointwise convergence of states Ref. [18, (Corollary 5.12)].

The second assertion follows from identity (24) in Lemma 3, since by the above arguments we have

$$\liminf_{n\to+\infty} I(\rho_n, \Pi_{\mathfrak{M}_n}) \geq I(\rho_0, \Pi_{\mathfrak{M}_0}) \quad \text{and} \quad \liminf_{n\to+\infty} I(\rho_n, \Lambda_{\mathfrak{M}_n}) \geq I(\rho_0, \Lambda_{\mathfrak{M}_0}).$$

$\square$

Proposition 4 implies the following "continuous" version of the third assertion of Corollary 2 in Ref. 8. Let $\langle H \rangle_{\mathfrak{M},\rho} \doteq \int_X H(\rho_x)\mu_\rho(dx)$ be the mean entropy of *posteriori* states corresponding to a quantum instrument $\mathfrak{M}$ and an a priori state ρ.

Corollary 2: Let $\{\mathfrak{M}_n\}$ be a sequence of efficient quantum instruments with the same outcome space $\{X, \mathcal{F}\}$ converging to the instrument $\mathfrak{M}_0$ in the sense of (25) and $\{\rho_n\}$ be a sequence in $\mathfrak{S}(\mathcal{H})$ converging to a state ρ_0 such that $\lim_{n\to+\infty} H(\rho_n) = H(\rho_0) < +\infty$. Then

$$\lim_{n\to+\infty} \langle H \rangle_{\mathfrak{M}_n, \rho_n} = \langle H \rangle_{\mathfrak{M}_0, \rho_0}.$$

Lemma 4: Convergence of the sequence $\{\mathfrak{M}_n\}$ to the instrument $\mathfrak{M}_0$ defined by (25) means that

$$p.w.\text{-}\lim_{n\to+\infty} \Lambda_{\mathfrak{M}_n}(\rho) = \Lambda_{\mathfrak{M}_0}(\rho) \quad \forall \rho \in \mathfrak{S}(\mathcal{H}), \tag{26}$$

that is $\lim_{n\to+\infty} \langle \Lambda_{\mathfrak{M}_n}(\rho), \widehat{A} \rangle = \langle \Lambda_{\mathfrak{M}_0}(\rho), \widehat{A} \rangle$ for any $\widehat{A} \in L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H}))$.

Proof: Note first that $\lim_{n\to+\infty} \langle \Lambda_{\mathfrak{M}_n}(\rho), \chi_F \otimes A \rangle = \langle \Lambda_{\mathfrak{M}_0}(\rho), \chi_F \otimes A \rangle$, where χ_F is the indicator function of the set $F \in \mathcal{F}$ and $A \in \mathfrak{B}(\mathcal{H})$, means that $\lim_{n\to+\infty} \text{Tr}\mathfrak{M}_n(F)[\rho]A = \text{Tr}\mathfrak{M}_0(F)[\rho]A$. Thus, (26) implies (25).

To prove the converse implication assume that $\mathcal{H}_0$ is a finite-dimensional subspace of $\mathcal{H}$. Since $L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H}_0))$ coincides with the C^*-tensor product of $L_\infty(X, \mathcal{F}, \mu_0)$ and $\mathfrak{B}(\mathcal{H}_0)$, arbitrary $\widehat{A}_0 \in L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H}_0))$ can be approximated in the norm topology by a sequence $\{\widehat{A}_m\}$ belonging to the linear span of the set $\{\chi_F \otimes A \mid F \in \mathcal{F}, A \in \mathfrak{B}(\mathcal{H}_0)\}$. As mentioned before (25) implies $\lim_{n\to+\infty} \langle \Lambda_{\mathfrak{M}_n}(\rho), \widehat{A}_m \rangle = \langle \Lambda_{\mathfrak{M}_0}(\rho), \widehat{A}_m \rangle$ for each m. By using the standard argumentation we conclude that $\lim_{n\to+\infty} \langle \Lambda_{\mathfrak{M}_n}(\rho), \widehat{A}_0 \rangle = \langle \Lambda_{\mathfrak{M}_0}(\rho), \widehat{A}_0 \rangle$.

Note that (25) implies that the set $\{\mathfrak{M}_n(X)[\rho]\}_{n\geq 0}$ of states in $\mathfrak{S}(\mathcal{H})$ is compact. By the compactness criterion for subsets of $\mathfrak{S}(\mathcal{H})$ (see Lemma 10 in Ref. 23) for arbitrary $\varepsilon > 0$ there exists a finite dimensional projector P_ε such that $\text{Tr}\mathfrak{M}_n(X)[\rho]P_\varepsilon^\perp < \varepsilon$ for all $n = 0, 1, 2, \ldots$, where $P_\varepsilon^\perp = I_\mathcal{H} - P_\varepsilon$. This means that

$$\int_X \text{Tr}P_\varepsilon^\perp \sigma_n(x)\mu_0(dx) < \varepsilon \quad \text{for all } n = 0, 1, 2, \ldots, \tag{27}$$

where $\sigma_n(x)$ is a representative of the class $\Lambda_{\mathfrak{M}_n}(\rho)$.

Let $\varepsilon > 0$ be arbitrary and $A(x)$ be a representative of a class $\widehat{A} \in L_\infty(X, \mathcal{F}, \mu_0, \mathfrak{B}(\mathcal{H}))$. Then

$$\langle \Lambda_{\mathfrak{M}_n}(\rho), \widehat{A} \rangle = \int_X \mathrm{Tr} A(x) \sigma_n(x) \mu_0(dx) = \int_X \mathrm{Tr} P_\varepsilon A(x) P_\varepsilon \sigma_n(x) \mu_0(dx)$$

$$+ \int_X \mathrm{Tr} P_\varepsilon^\perp A(x) P_\varepsilon \sigma_n(x) \mu_0(dx) + \int_X \mathrm{Tr} P_\varepsilon A(x) P_\varepsilon^\perp \sigma_n(x) \mu_0(dx) + \int_X \mathrm{Tr} P_\varepsilon^\perp A(x) P_\varepsilon^\perp \sigma_n(x) \mu_0(dx).$$

By means of (27) it is easy to show that the last three terms in this expression are less than $\varepsilon \|\widehat{A}\|$ for all $n = 0, 1, 2, \ldots$ As proved before (25) implies that the first term tends to $\int_X \mathrm{Tr} P_\varepsilon A(x) P_\varepsilon \sigma_0(x) \mu_0(dx)$, since $P_\varepsilon A(x) P_\varepsilon \in \mathfrak{B}(P_\varepsilon(\mathcal{H}))$. Thus, we conclude that (26) holds. □

ACKNOWLEDGMENTS

APPENDIX A: A CHARACTERIZATION OF A QUANTUM INSTRUMENT WHICH IS IRREDUCIBLE BUT NOT EFFICIENT

In Theorem 1 of Ref. 10, the representation of a quantum instrument analogous to the Kraus representation of a completely positive map is obtained. By this theorem for an arbitrary instrument $\mathfrak{M}$ in a Hilbert space $\mathcal{H}$ with the outcome set $\{X, \mathcal{F}\}$ there exist a positive σ-finite measure μ on $\{X, \mathcal{F}\}$, a dense domain $\mathcal{D} \subset \mathcal{H}$ and a countable family of functions $x \mapsto V_k(x)$ defined for μ-almost all x, such that $V_k(x)$ is a linear operator from $\mathcal{D}$ to $\mathcal{H}$, satisfying

$$\langle \varphi | \mathfrak{M}^*(F)[A] \varphi \rangle = \int_F \sum_k \langle V_k(x) \varphi | A V_k(x) \varphi \rangle \mu(dx),$$

for any $\varphi \in \mathcal{D}$, $F \in \mathcal{F}$, $A \in \mathfrak{B}(\mathcal{H})$. We may assume that for each x the all nonzero operators from the family $V_k(x)$ are not proportional to each other, since if $V_{k'}(x_0) = \lambda V_k(x_0)$ for some x_0 and $\lambda \in \mathbb{C}$ then we may replace $V_k(x_0)$ by $\sqrt{1 + |\lambda|^2} V_k(x_0)$ and consider that $V_{k'}(x_0) = 0$.

The instrument $\mathfrak{M}$ is efficient if and only if $V_k(x) = 0$ for $k > 1$ and μ-almost all x. This follows from the proof of Theorem 1 in Ref. 10 and the proof of the implication (iv) $\Rightarrow$ (i) in Proposition 2.

So, if the instrument $\mathfrak{M}$ is not efficient then there exists a subset F_s of X such that $\mathfrak{M}(F_s) \neq 0$ and $V_2(x) \neq 0$ for all $x \in F_s$ (we assume that if $V_k(x) = 0$ then $V_{k'}(x) = 0$ for all $k' > k$).

Proposition 5: If the instrument $\mathfrak{M}$ is irreducible but not efficient then for μ-almost all x in F_s the all nonzero operators $V_k(x)$ have the same one dimensional range (depending on x), that is $V_k(x)|\varphi\rangle = \omega_k(x)[\varphi]|\psi_x\rangle$ for all $\varphi \in \mathcal{D}$, where $\omega_k(x)$ is a linear functional defined on $\mathcal{D}$ (not necessary bounded) and ψ_x is a unit vector in $\mathcal{H}$ (not depending on φ). This means that

$$\langle \varphi | \mathfrak{M}^*(F)[A] \varphi \rangle = \int_F \langle \psi_x | A \psi_x \rangle \sum_k |\omega_k(x)[\varphi]|^2 \mu(dx), \tag{A1}$$

for any $\varphi \in \mathcal{D}$, $A \in \mathfrak{B}(\mathcal{H})$ and $F \in \mathcal{F}$ such that $F \subseteq F_s$.

It follows that for an arbitrary a priori state ρ the posteriori state is $\rho_x = |\psi_x\rangle\langle\psi_x|$ for μ_ρ-almost all $x \in F_s$ (that is, ρ_x does not depend on ρ).

This assertion can be obtained by using the arguments from the proof of the implication (ii) $\Rightarrow$ (iv) in Proposition 2 with $\dim \mathcal{K} = 1$. The only difference appears at the point where Lemma 5 is used, since in this case we can not exclude the possibility of $\operatorname{rank} V_k(x) = 1$. It is this possibility that prevents to prove that any irreducible instrument is efficient.

Corollary 3: If $\mathfrak{M}$ is an instrument not taking values within the set of nonzero entanglement breaking quantum operations (see Ref. [25]) then $\mathfrak{M}$ is efficient if and only if $\mathfrak{M}$ is irreducible.

Proof: Let $\mathfrak{M}$ be a instrument in a Hilbert space $\mathcal{H}$ with the outcome set $\{X, \mathcal{F}\}$, which is irreducible but not efficient. By Proposition 5 there exist a positive σ-finite measure μ on $\{X, \mathcal{F}\}$, a subset F of X such that $\mathfrak{M}(F) \neq 0$, a dense domain $\mathcal{D} \subseteq \mathcal{H}$, a family $\{x \mapsto \omega_k(x)\}_k$ of functions on F, such that $\omega_k(x)$ is a linear functional defined on $\mathcal{D}$ for each $x \in F$, and a function $x \mapsto \psi_x$ on F such that ψ_x is a unit vector in $\mathcal{H}$ for each $x \in F$, for which relation (A1) holds. By using the polarization identity, one can show that

$$\mathfrak{M}(F)[|\varphi_1\rangle\langle\varphi_2|] = \int_F |\psi_x\rangle\langle\psi_x| \sum_k \omega_k(x)[\varphi_1] \overline{\omega_k(x)[\varphi_2]} \mu(dx), \tag{A2}$$

where $\int$ denotes the Bochner integral.

Let $\omega = \sum_{i,j=1}^m |\varphi_i\rangle\langle\varphi_j| \otimes |\phi_i\rangle\langle\phi_j|$ be a pure state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, where $\{\varphi_i\}_{i=1}^m \subset \mathcal{D}$ and $\{\phi_i\}_{i=1}^m \subset \mathcal{K}$. Then (A2) implies

$$\mathfrak{M}(F) \otimes \operatorname{Id}_{\mathcal{K}}(\omega) = \int_F |\psi_x\rangle\langle\psi_x| \otimes \sum_k \sum_{i,j=1}^m \omega_k(x)[\varphi_i] \overline{\omega_k(x)[\varphi_j]} |\phi_i\rangle\langle\phi_j| \mu(dx)$$

$$= \int_F |\psi_x\rangle\langle\psi_x| \otimes \sum_k |\eta_x^k\rangle\langle\eta_x^k| \mu(dx),$$

where $|\eta_x^k\rangle = \sum_{i=1}^m \omega_k(x)[\varphi_i]|\phi_i\rangle$ is a vector in $\mathcal{K}$. It follows that the operator $\mathfrak{M}(F) \otimes \operatorname{Id}_{\mathcal{K}}(\omega)$ is separable. Since an arbitrary pure state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ can be approximated by a sequence of above-considered states, the operator $\mathfrak{M}(F) \otimes \operatorname{Id}_{\mathcal{K}}(\omega)$ is separable for any state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. Thus, the operation $\mathfrak{M}(F)$ is entanglement-breaking. $\qquad\square$

Example 1: Let P be a spectral projector valued measure on a measurable space $\{X, \mathcal{F}\}$ and $|\psi_0\rangle$ be a fixed unit vector in $\mathcal{H}$. Then the instrument $\mathfrak{M}(F)[\rho] = [\operatorname{Tr} P(F)\rho]|\psi_0\rangle\langle\psi_0|$ is obviously irreducible, but it is efficient if and only if the spectral measure P has (uniform) multiplicity one, since it is easy to see that the spectral measure from representation (13) coincides with P. Note that the channel $\mathfrak{M}(X): \rho \mapsto |\psi_0\rangle\langle\psi_0|$ is entanglement-breaking.

APPENDIX B: TWO AUXILIARY LEMMAS

Lemma 5: Let $\mathcal{L}_1 = \operatorname{lin}(\{\varphi_i\}_{i\in\mathbb{N}})$ be a linear space and $\mathcal{L}_1^0$ be a countable subset of $\mathcal{L}_1$ consisting of finite linear combinations of the vectors $\varphi_1, \varphi_2, \ldots$ with rational coefficients. Let $\{A_k\}$ be a finite or countable family of nonzero linear operators from $\mathcal{L}_1$ to a linear space $\mathcal{L}_2$ such that the set $\{A_k(\varphi)\} \subset \mathcal{L}_2$ consists of collinear vectors for any $\varphi \in \mathcal{L}_1^0$. If at least one operator in the family $\{A_k\}$ has rank > 1 then $A_k = \lambda_k A_1$ for all k, where $\{\lambda_k\}$ is a set of nonzero scalars.

Proof: Suppose $\operatorname{rank} A_1 > 1$ and k is arbitrary. By the condition $A_k(\varphi) = \lambda_k^\varphi A_1(\varphi)$ for all $\varphi \in \mathcal{L}_1^0 \setminus \ker A_1$. We will show that λ_k^φ does not depend on φ.

Since $\operatorname{rank} A_1 > 1$, without loss of generality we may assume that the vectors $A_1(\varphi_1)$ and $A_1(\varphi_2)$ are not collinear. Let $\psi = c_1\varphi_1 + c_2\varphi_2$, where c_1 and c_2 are nonzero rational coefficients. By linearity we have

$$A_k(\psi) = \lambda_k^\psi A_1(\psi) = \lambda_k^\psi (c_1 A_1(\varphi_1) + c_2 A_1(\varphi_2))$$

and

$$A_k(\psi) = c_1 A_k(\varphi_1) + c_2 A_k(\varphi_2) = c_1 \lambda_k^{\varphi_1} A_1(\varphi_1) + c_2 \lambda_k^{\varphi_2} A_1(\varphi_2).$$

Hence, $\lambda_k^{\varphi_1} = \lambda_k^{\varphi_2}$. Let φ be an arbitrary vector in $\mathcal{L}_1^0 \setminus \ker A_1$. Then the vector $A_1(\varphi)$ is not collinear with $A_1(\varphi_i)$, where either $i = 1$ or $i = 2$. By repeating the above arguments for the pair (φ, φ_i) instead of (φ_1, φ_2) we obtain $\lambda_k^{\varphi} = \lambda_k^{\varphi_i}$. Thus, $\lambda_k^{\varphi} = \lambda_k$ for all $\varphi \in \mathcal{L}_1^0 \setminus \ker A_1$.

If $\lambda_k = 0$ then $A_k(\varphi) = 0$ for all $\varphi \in \mathcal{L}_1^0 \setminus \ker A_1$. This implies $A_k = 0$ contradicting to the assumption. Indeed, if $\varphi_i \in \ker A_1$ for some i then $\varphi_i = (\varphi_i + \varphi_1) - \varphi_1$, and hence $A_k(\varphi_i) = A_k(\varphi_i + \varphi_1) - A_k(\varphi_1) = 0$.

Thus, we have

$$A_k(\varphi) = \lambda_k A_1(\varphi) \text{ for all } \varphi \in \mathcal{L}_1^0 \setminus \ker A_1, \text{ where } \lambda_k \neq 0. \tag{B1}$$

Hence, the vectors $A_k(\varphi_1)$ and $A_k(\varphi_2)$ are not collinear. By repeating the above arguments with A_k instead of A_1 and A_1 instead of A_k we obtain

$$A_1(\varphi) = \lambda_k' A_k(\varphi) \text{ for all } \varphi \in \mathcal{L}_1^0 \setminus \ker A_k, \text{ where } \lambda_k' \neq 0. \tag{B2}$$

It follows from (B1) and (B2) that $A_k(\varphi_i) = \lambda_k A_1(\varphi_i)$ for all $i \in \mathbb{N}$, and hence $A_k = \lambda_k A_1$. $\square$

Lemma 6: Let $\{\pi_\alpha, \rho_\alpha\}$ *be a countable or continuous ensemble of states in* $\mathfrak{S}(\mathcal{H})$ *such that* $\rho_\alpha \cong \bar{\rho}$ *for all* α, *where* $\bar{\rho}$ *is the average state of this ensemble. Then* $\rho_\alpha = \bar{\rho}$ *for all* α.

The assertion of this lemma follows from existence of a finite strictly convex function[22] on the set $\mathfrak{S}(\mathcal{H})$ depending only on the spectrum of a state. As the simplest example, one can consider the function $f(\rho) = \mathrm{Tr}\rho^2$.

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[24] In Ref. 21, the term *quasicomplete* is used. The term *irreducible* seems to be more reasonable, it appeared in Ref. 19 and is used in the subsequent papers.

[25] A quantum operation Φ is called entanglement-breaking if for an arbitrary state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, where $\mathcal{K}$ is a separable Hilbert space, the operator $\Phi \otimes \mathrm{Id}_{\mathcal{K}}(\omega)$ belongs to the convex closure of the product-operators $A \otimes B$, $A \in \mathfrak{T}_+(\mathcal{H})$, $B \in \mathfrak{T}_+(\mathcal{K})$.