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SHORT COMMUNICATIONS

On the Gain of Entanglement Assistance in the Classical Capacity of Quantum Gaussian Channels

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Quantum channels are trace-preserving completely positive linear maps between Banach spaces of trace-class operators (Schatten classes of order 1); these are noncommutative analogs of Markov operators in classical probability theory. They also play the role of dynamical maps in quantum theory [1, Chap. 6].

The main characteristics determining the information properties of a quantum channel include its classical entanglement-assisted and unassisted capacities. The classical (unassisted) capacity $C(\Phi)$ of a channel Φ determines the limit rate of classical information transmission through Φ with any block coding at the input and the corresponding measurement at the output, and the classical entanglement-assisted channel capacity $C_{\mathrm{ea}}(\Phi)$ supposes, in addition, the presence of an entangled state between the input and the output of the channel Φ (a detailed description of transmission protocols can be found in [1, Chap. 8]). Since entanglement is an additional resource, it follows that $C_{\mathrm{ea}}(\Phi) \geq C(\Phi)$ for any channel Φ.

Let $\mathcal{H}$ be a separable Hilbert space. By $\mathfrak{T}(\mathcal{H})$ we denote the Banach space of all trace-class operators on $\mathcal{H}$ and by $\mathfrak{S}(\mathcal{H})$, the subset of $\mathfrak{T}(\mathcal{H})$ consisting of all positive operators with trace 1; we refer to such operators as *quantum states* and denote them by Greek letters $\rho, \sigma, \dots$. We denote a set of quantum states $\{\rho_i\}$ with probability distributions $\{\pi_i\}$ by $\{\pi_i, \rho_i\}$ and call it an *ensemble of states*; the state $\overline{\rho} = \sum_i \pi_i \rho_i$ is called the *average state* of the ensemble $\{\pi_i, \rho_i\}$.

A *quantum channel* is a trace-preserving completely positive linear map $\Phi\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ [1, Chap. 6]. Let $H(\rho)$ be the von Neumann entropy of a state ρ, and let $H(\rho\|\sigma)$ be the quantum relative entropy of states ρ and σ. For a given channel Φ and any ensemble $\{\pi_i, \rho_i\}$ of input quantum states, the output χ-quantity is determined by the expression

$$\chi_\Phi(\{\pi_i, \rho_i\}) \doteq \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\overline{\rho})) = H(\Phi(\overline{\rho})) - \sum_i \pi_i H(\Phi(\rho_i)),$$

where the second relation holds under the condition $H(\Phi(\overline{\rho})) < +\infty$.

If a channel Φ is finite-dimensional (i.e., $\dim \mathcal{H}_A, \dim \mathcal{H}_B < +\infty$), then the Holevo–Schumacher–Westmoreland (HSW) theorem implies

$$C(\Phi) = \lim_{n\to+\infty} n^{-1} C_\chi(\Phi^{\otimes n}),$$

where C_χ is the χ-capacity of the channel, which is defined by

$$C_\chi(\Phi) = \sup_{\{\pi_i, \rho_i\}} \chi_\Phi(\{\pi_i, \rho_i\});$$

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here the supremum is taken over all ensembles of quantum states.

The Bennett–Shor–Smolin–Thapliyal (BSST) theorem [2] gives the following expression for the classical entanglement-assisted capacity of a finite-dimensional channel:

$$C_{\mathrm{ea}}(\Phi) = \sup_{\rho} I(\Phi, \rho),$$

where $I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\Phi, \rho)$ is the quantum mutual information of Φ in the state $\rho \in \mathfrak{S}(\mathcal{H}_A)$ (here $H(\Phi, \rho)$ is the exchange entropy of the channel Φ in the state ρ).

Determining the capacity of an infinite-dimensional quantum channel requires imposing certain constraints on the states used as codes, e.g., a constraint on the mean energy of these states. We take a positive self-adjoint operator F on $\mathcal{H}_A$ and subject the input states $\rho^{(n)}$ of the channel $\Phi^{\otimes n}$ to linear constraints of the form[1]

$$\operatorname{Tr} \rho^{(n)} F^{(n)} \leq nE, \tag{1}$$

where

$$F^{(n)} = F \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes F. \tag{2}$$

Another special feature of the infinite-dimensional case is the necessity of using generalized ensembles of quantum states. Such an ensemble can be defined as the Borel probability measure μ on the set $\mathfrak{S}(\mathcal{H}_A)$ of quantum states; its average state is the barycenter $\overline{\rho}(\mu)$ of the measure μ. The output χ-quantity of a generalized ensemble μ for a channel Φ is defined as

$$\chi_{\Phi}(\mu) \doteq \int H(\Phi(\rho) \| \Phi(\overline{\rho}(\mu)))\, \mu(d\rho) = H(\Phi(\overline{\rho}(\mu))) - \int H(\Phi(\rho))\, \mu(d\rho),$$

where the second equality holds under the condition $H(\Phi(\overline{\rho}(\mu))) < +\infty$ [1, Chap. 10].

The operational definition of the classical entanglement-assisted and unassisted capacities of a quantum channel with linear constraints can be found in [3], where the corresponding generalizations of the HSW and BSST theorems were also proved under certain conditions on the form of the constraints. A maximally complete generalization of the BSST theorem for an infinite-dimensional channel with linear constraints is given in [4].

By the HSW theorem for a channel with linear constraints [3, Proposition 3], the classical entanglement-assisted capacity of a channel Φ with constraint (1) is determined by the regularized expression

$$C(\Phi, F, E) = \lim_{n \to +\infty} n^{-1} C_{\chi}(\Phi^{\otimes n}, F^{(n)}, nE),$$

in which the operator $F^{(n)}$ is defined by (2) and

$$C_{\chi}(\Phi, F, E) = \sup_{\mu : \operatorname{Tr} \overline{\rho}(\mu) F \leq E} \chi_{\Phi}(\mu). \tag{3}$$

By the BSST theorem for a channel with linear constraints [4, Theorem 1], the classical capacity of a channel Φ with constraint (1) is determined by the expression

$$C_{\mathrm{ea}}(\Phi, F, E) = \sup_{\rho : \operatorname{Tr} \rho F \leq E} I(\rho, \Phi),$$

where $I(\rho, \Phi)$ is the quantum mutual information (appropriately generalized to the infinite-dimensional case).

For any quantum channel Φ, we have

$$C_{\mathrm{ea}}(\Phi, F, E) \geq C(\Phi, F, E), \tag{4}$$

which follows straightforwardly from the operational definitions of the channel capacities under consideration. The strict inequality in (4) means that using the entangled state between the input and the

[1] The quantity $\operatorname{Tr} \rho^{(n)} F^{(n)}$ (finite or infinite) is defined as $\sup_t \operatorname{Tr} \rho^{(n)} P_t F^{(n)} P_t$, where P_t is the spectral projection operator $F^{(n)}$ corresponding to the interval $[0, t]$.

output increases the limit speed of information transmission over the channel Φ and gives a gain in the size of an optimal code; this gain increases exponentially as $n \to \infty$. There naturally arises the question about conditions under which (4) becomes an equality (a strict inequality). For an unconstrained finite-dimensional channel, in [5], a criterion for the equality $C_{\mathrm{ea}}(\Phi) = C_{\chi}(\Phi)$ was obtained, which is formally stronger than the equality $C_{\mathrm{ea}}(\Phi) = C(\Phi)$ (these equalities are equivalent if the χ-capacity of the channel Φ is additive).

In this paper, we obtain sufficient conditions under which (4) is a strict inequality for bosonic Gaussian channels, which play a central role in the theory of quantum information systems with continuous variables.

Let $\mathcal{H}_X$ $(X = A, B, \dots)$ be the space of an irreducible representation of *canonical commutation relations*

$$W_X(z)W_X(z') = \exp\left[-\frac{i}{2}\Delta_X(z, z')\right]W_X(z' + z),$$

where (Z_X, Δ_X) is a symplectic space and the $W_X(z)$ are the Weyl operators [1, Chap. 11]. By s_X we denote the number of modes of the system X, which is determined by $2s_X = \dim Z_X$.

A bosonic Gaussian channel $\Phi\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is determined by the action of the adjoint map $\Phi^*\colon \mathfrak{B}(\mathcal{H}_B) \to \mathfrak{B}(\mathcal{H}_A)$ on the Weyl operators:

$$\Phi^*(W_B(z)) = W_A(Kz)\exp\left[ilz - \frac{1}{2}z^\top \alpha z\right], \qquad z \in Z_B,$$

where K is a linear operator $Z_B \to Z_A$, l is a row $(2s_B)$-vector, and α is a real symmetric $(2s_B) \times (2s_B)$ matrix satisfying the inequality

$$\alpha \geq \pm\frac{i}{2}\left[\Delta_B - K^\top \Delta_A K\right].$$

For a long time, one of the main open problems in quantum information theory was the conjecture about *Gaussian minimizers*, according to which the output entropy of a Gaussian channel Φ, i.e., the function $\rho \mapsto H(\Phi(\rho))$, attains its minimum at a pure Gaussian state.

Recently, the Gaussian minimizer conjecture has been proved for a large class of Gaussian channels with the property of gauge covariance or contravariance [6]. Among the many consequences of this result, we mention the following theorem.

Theorem 1. *Let Φ be a Gaussian nontrivial* ($K \neq 0$) *gauge-covariant or gauge-contravariant channel, and let $F = \sum_{ij} \epsilon_{ij} a_i^\dagger a_j$ be a gauge-invariant oscillatory energy operator* (*here* $[\epsilon_{ij}]$ *is a positive definite matrix*). *Then*

$$C_{\mathrm{ea}}(\Phi, F, E) > C(\Phi, F, E). \tag{5}$$

Proof. As shown in [6] and [7], it follows from the truth of the Gaussian minimizer conjecture for a channel Φ that

(1) $C(\Phi, F, E) = C_{\chi}(\Phi, F, E)$;

(2) there exists an optimal ensemble-measure μ at which the supremum in (3) is attained, and the barycenter of this measure is a nondegenerate Gaussian state.

Combining these assertions, observing that the finiteness of $\operatorname{Tr} F\rho$ for an operator F of the form specified above implies the finiteness of the entropy $H(\rho)$, and applying Theorem 2 from [4], we conclude that the equality $C_{\mathrm{ea}}(\Phi, F, E) = C(\Phi, F, E)$ can hold only if Φ is a classical-quantum channel of discrete type. By Proposition 5 from [4], this means that $K = 0$, i.e., Φ is a completely depolarizing channel, for which $C_{\mathrm{ea}}(\Phi, F, E) = C(\Phi, F, E) = 0$. $\square$

Remark 1. Theorem 1 is valid for any quadratic energy operator F, provided that condition (18) from [8] holds in the form of a strict operator inequality, because this ensures the satisfiability of conditions (1) and (2) specified above.[2]

Remark 2. The assumptions of Theorem 1 ensuring the strictness of inequality (5) are essential: there exist nontrivial Gaussian channels with linear constraints for which

$$C_{\mathrm{ea}}(\Phi, F, E) = C(\Phi, F, E) = C_\chi(\Phi, F, E) > 0.$$

The simplest one-mode ($s = 1$) example of such a channel Φ and an operator F is given in [9]. An important special feature of this example is the noncompactness of the solution set of the inequality $\operatorname{Tr} F\rho \le E$, which implies the absence of an optimal measure for the channel Φ with constraint (1).

Remark 3. Under the assumptions of Theorem 1, the gain from entanglement assistance is estimated from below as

$$C_{\mathrm{ea}}(\Phi, F, E) - C(\Phi, F, E) \ge H(\overline{\rho}(\mu_{\mathrm{opt}})) + H(\Phi(\rho_0)) - H(\Phi, \overline{\rho}(\mu_{\mathrm{opt}})),$$

where $\overline{\rho}(\mu_{\mathrm{opt}})$ is the average state of an optimal ensemble (determined by using an optimization procedure described in [7]) and ρ_0 is the pure vacuum state.

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[2] If $\operatorname{Ran} K = Z_A$, then, by virtue of Proposition 5 from [4], the requirement that the barycenter in condition (2) is nondegenerate is not necessary for proving Theorem 1, and it suffices to require the fulfillment of condition (18) from [8] in its usual form.