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SHORT COMMUNICATIONS

A Property of the Output Entropy of a Positive Map of Spaces of Nuclear Operators

M. E. Shirokov*

Steklov Mathematical Institute, Russian Academy of Sciences

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1. INTRODUCTION

In the study of informational characteristics of a quantum communication channel (that is, a completely positive trace-preserving linear map of spaces of nuclear operators), an important role is played by the von Neumann output entropy of this channel, which is regarded as a function on the set of input states (positive operators with trace 1) of the channel under consideration [1]. In quantum statistics, the notions of a quantum operation (a completely positive linear map not increasing trace) and of its output entropy are also used [1].

The output entropy of quantum channels and operations on finite-dimensional quantum systems is a continuous concave function on the compact set of input quantum states; however, its properties change drastically under the passage to the infinite-dimensional case, because the von Neumann entropy is a discontinuous function on the set of states of an infinite-dimensional quantum system taking the value $+\infty$ everywhere except on a subset of first category [2]. It is this fact that causes difficulties in the analysis of infinite-dimensional quantum systems and channels and specific properties of their characteristics (such as, e.g., the discontinuity of channel capacity as a function of the channel).

Nevertheless, there exist nontrivial infinite-dimensional quantum channels and operations with continuous output entropy, which substantially facilitates analyzing their information properties [3]. The objective of this paper is to show that, to prove the *continuity* of the output entropy of a positive linear map of spaces of nuclear operators (in particular, a quantum channel or a quantum operation) on the set of input states, it suffices to prove the *finiteness* of its values on this set.

2. DEFINITIONS AND NOTATION

Suppose that $\mathscr{H}$ is a separable Hilbert space, $\mathfrak{B}(\mathscr{H})$ is the algebra of all bounded operators on $\mathscr{H}$ with operator norm $\|\cdot\|$, and $\mathfrak{T}(\mathscr{H})$ is the separable Banach space of nuclear operators on $\mathscr{H}$ with trace norm $\|\cdot\|_1 = \operatorname{Tr}|\cdot|$ and positive cone $\mathfrak{T}_+(\mathscr{H})$, which contains the closed convex set $\mathfrak{S}(\mathscr{H})$ of positive nuclear operators with trace 1. We denote operators from $\mathfrak{S}(\mathscr{H})$ by the symbols $\rho, \sigma, \dots$ and call them *states*, because any such operator ρ determines the normal state $A \mapsto \operatorname{Tr} A\rho$ on the algebra $\mathfrak{B}(\mathscr{H})$ [1].

The von Neumann entropy $H(\rho) = -\operatorname{Tr}\rho\log\rho$ of a state $\rho \in \mathfrak{S}(\mathscr{H})$ has a natural extension to $\mathfrak{T}_+(\mathscr{H})$ (see [4]), which is defined by

$$H(A) = \operatorname{Tr} AH\left(\frac{A}{\operatorname{Tr} A}\right) = \operatorname{Tr}\eta(A) - \eta(\operatorname{Tr} A), \qquad A \in \mathfrak{T}_+(\mathscr{H}), \quad \eta(x) = -x\log x.$$

The nonnegativity, concavity, and lower semicontinuity of the von Neumann entropy on the set $\mathfrak{S}(\mathscr{H})$ imply the same properties of the function $A \mapsto H(A)$ on the set $\mathfrak{T}_+(\mathscr{H})$; we refer to this function as the *quantum entropy*. The function $\{x_i\} \mapsto H(\{x_i\}) = \sum_i \eta(x_i) - \eta(\sum_i x_i)$ on the positive cone of the Banach space l_1 of summable sequences which coincides with the Shannon entropy on the set $\mathfrak{P}_{+\infty}$ of all probability distributions is called the *classical entropy*.

*E-mail: msh@mi.ras.ru

3. MAIN RESULT

Let $\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$ be a bounded positive linear map. The output entropy $H \circ \Phi$ of this map is a concave lower semicontinuous function on $\mathfrak{S}(\mathscr{H}) \subset \mathfrak{T}(\mathscr{H})$ taking values in $[0, +\infty]$. The following theorem shows that this function cannot be discontinuous if it takes only finite values.

Theorem 1. *For a bounded linear positive map Φ from $\mathfrak{T}(\mathscr{H})$ to $\mathfrak{T}(\mathscr{H}')$, the following conditions are equivalent*:

i) *the function $\rho \mapsto H(\Phi(\rho))$ takes only finite values on the set $\mathfrak{S}(\mathscr{H})$*;

ii) *the function $\rho \mapsto H(\Phi(\rho))$ is continuous and bounded on the set $\mathfrak{S}(\mathscr{H})$*;

iii) *there exists an orthonormal basis $\{|i\rangle\}_{i=1}^{+\infty}$ in the space $\mathscr{H}'$ such that the function*
$$\rho \mapsto H(\{\langle i|\Phi(\rho)|i\rangle\}_{i=1}^{+\infty})$$
is continuous and bounded on the set $\mathfrak{S}(\mathscr{H})$;

iv) *there exists an orthonormal basis $\{|i\rangle\}_{i=1}^{+\infty}$ in the space $\mathscr{H}'$ and a sequence of nonnegative numbers $\{h_i\}_{i=1}^{+\infty}$ for which*
$$\left\|\sum_{i=1}^{+\infty} h_i \Phi^*(|i\rangle\langle i|)\right\| < +\infty \qquad \text{and} \qquad \sum_{i=1}^{+\infty} e^{-h_i} < +\infty,$$
where Φ^* is the map from $\mathfrak{B}(\mathscr{H}')$ to $\mathfrak{B}(\mathscr{H})$ dual to Φ.[1]

The proof of this theorem uses some special properties of the von Neumann entropy (see [2], [4]) and results on the χ-capacity of the set of quantum states obtained in [5]. This proof shows that the role of the set $\mathfrak{S}(\mathscr{H})$ in Theorem 1 can be played by any bounded convex subset $\mathscr{A}$ of the cone $\mathfrak{T}_+(\mathscr{H})$ with the property that $\sup_{A\in\mathscr{A}} \lim_{n\to+\infty} \operatorname{Tr} AB_n < +\infty$ implies $\sup_n \|B_n\| < +\infty$ for any increasing sequence $\{B_n\}$ of positive operators from $\mathfrak{B}(\mathscr{H})$.

Remark 1. Theorem 1 *does not assert* that condition (i), i.e., the finiteness of the quantum entropy on the set $\Phi(\mathfrak{S}(\mathscr{H}))$, implies the continuity of quantum entropy on this set, because the continuity of the function $\rho \mapsto H(\Phi(\rho))$ on the noncompact set $\mathfrak{S}(\mathscr{H})$ does not imply that of the function $A \mapsto H(A)$ on the set $\Phi(\mathfrak{S}(\mathscr{H}))$. This is shown by the following example.

Let $\mathscr{A}$ be a convex closed subset of $\mathfrak{S}(\mathscr{H}')$ on which the von Neumann entropy is discontinuous but bounded (see examples in [5]), and let $\{\sigma_n\}_{n=1}^{+\infty}$ be a sequence of states from $\mathscr{A}$ converging to a state σ_0 from $\mathscr{A}$ for which $\lim_{n\to+\infty} H(\sigma_n) \neq H(\sigma_0)$. Consider the map
$$\Phi\colon \rho \mapsto \sum_{n=0}^{+\infty} \langle n|\rho|n\rangle \sigma_n,$$
where $\{|n\rangle\}_{n=0}^{+\infty}$ is an orthonormal basis in the space $\mathscr{H}'$. According to Theorem 1, the function $\rho \mapsto H(\Phi(\rho))$ is continuous on the set $\mathfrak{S}(\mathscr{H})$, but the function $A \mapsto H(A)$ is discontinuous on the set $\Phi(\mathfrak{S}(\mathscr{H}))$, which contains the sequence $\{\sigma_n\}_{n=1}^{+\infty}$.

The continuity of the function $\rho \mapsto H(\Phi(\rho))$ on the set $\mathfrak{S}(\mathscr{H})$ is equivalent to that of the function $A \mapsto H(A)$ on any set of the form $\Phi(\mathscr{C})$, where $\mathscr{C}$ is a compact subset of $\mathfrak{S}(\mathscr{H})$.

Remark 2. The main assertion of Theorem 1 (the implication (i) ⇒ (ii)) is based on special properties of the von Neumann entropy; it cannot be proved by using only general properties of entropy-like functions, such as concavity, lower semicontinuity, and so on. The simplest example is the Renyi output entropy of order $p = 0$ of the map Φ, that is, the function $\rho \mapsto \log \operatorname{rank}(\Phi(\rho))$.

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[1] The map Φ^* is defined by the relation $\operatorname{Tr} \Phi^*(A)\rho = \operatorname{Tr} A\Phi(\rho)$ for $\rho \in \mathfrak{S}(\mathscr{H})$.

Remark 3. Condition (iii) in Theorem 1 formally strengthens condition (ii), because the continuity (boundedness) of the quantum entropy on the set $\mathscr{A} \subset \mathfrak{T}_+(\mathscr{H})$ always follows from the continuity (respectively, boundedness) of the classical entropy on the set $\{\{\langle i|A|i\rangle\}_{i=1}^{+\infty} \mid A \in \mathscr{A}\}$ for at least one basis $\{|i\rangle\}_{i=1}^{+\infty}$ of the space $\mathscr{H}$ [5, Proposition 5], while the converse is false.

Condition (iv) in Theorem 1 can be regarded as a criterion for the continuity of the output entropy of a positive map in terms of the dual map.

Example 1. The simplest positive linear map from $\mathfrak{T}(\mathscr{H})$ to $\mathfrak{T}(\mathscr{H}')$ has the form $\Phi_V(\,\cdot\,) = V(\,\cdot\,)V^*$, where V is a bounded linear operator from $\mathscr{H}$ to $\mathscr{H}'$. Using the implication (iv) $\Rightarrow$ (ii) of Theorem 1 and results of [5], we can show that the function $\rho \mapsto H(V\rho V^*)$ is continuous on the set $\mathfrak{S}(\mathscr{H})$ if and only if the operator V is compact and the sequence $\{s_i\}$ of its singular numbers (the eigenvalues of the operator V^*V) satisfies the condition $\sum_{i=1}^{+\infty} e^{-\lambda/s_i} < +\infty$ for some $\lambda > 0$ (we assume that $e^{-\lambda/0} = 0$).

Theorem 1 makes it possible to substantially simplify the proofs of the continuity of the output entropy of the quantum channels considered in [3] and obtain general conditions for the continuity of the output entropy of quantum channels and operations in terms of their Kraus representation [1].

In conclusion, we give a commutative version of Theorem 1, which may be useful in studying the Shannon output entropy of Markov and sub-Markov operators.

Corollary 1. *For the matrix $\|\phi_{ij}\|$ of a bounded positive linear transformation of the space l_1, the following conditions are equivalent*:

i) *the function $\{\pi_i\} \mapsto H(\{\sum_j \phi_{ij}\pi_j\}_{i=1}^{+\infty})$ takes only finite values on the set $\mathfrak{P}_{+\infty}$*;

ii) *the function $\{\pi_i\} \mapsto H(\{\sum_j \phi_{ij}\pi_j\}_{i=1}^{+\infty})$ is continuous and bounded on the set $\mathfrak{P}_{+\infty}$*;

iii) *there exists a sequence of nonnegative numbers $\{h_i\}_{i=1}^{+\infty}$ for which $\sup_j \sum_{i=1}^{+\infty} h_i\phi_{ij} < +\infty$ and $\sum_{i=1}^{+\infty} e^{-h_i} < +\infty$.*

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