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Schmidt Number and Partially Entanglement-Breaking Channels in Infinite-Dimensional Quantum Systems

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Abstract—The Schmidt number of a state of an infinite-dimensional composite quantum system is defined and several properties of the corresponding Schmidt classes are considered. It is shown that there are states with given Schmidt number such that any of their countable convex decompositions does not contain pure states of finite Schmidt rank. The classes of infinite-dimensional partially entanglement-breaking channels are considered, and generalizations of several properties of such channels, which were obtained earlier in the finite-dimensional case, are proved. At the same time, it is shown that there are partially entanglement-breaking channels (in particular, entanglement-breaking channels) such that all the operators in any of their Kraus representations are of infinite rank.

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1. INTRODUCTION

The Schmidt rank of a pure state and its “generalization” to mixed states, which is called the *Schmidt number*, are important quantitative characteristics of entanglement in composite quantum systems.

The *Schmidt rank* of a pure state of a composite system AB described by the unit vector $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ (where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert states corresponding to the systems A and B) is defined as the number of nonzero terms in the Schmidt decomposition

$$|\psi\rangle = \sum_i \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle$$

of this vector; it coincides with the rank of partial states $\mathrm{Tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi|$ and $\mathrm{Tr}_{\mathcal{H}_A} |\psi\rangle\langle\psi|$.

The Schmidt number of a mixed state ω of a finite-dimensional composite quantum system AB was defined in [1] as the maximum Schmidt rank in the ensemble of pure states with average state ω minimized over all of such ensembles (see Sec. 3). In [1], it is shown that the Schmidt number does not increase under the action of LOCC-operations and the set of states whose Schmidt number does not exceed k (the Schmidt class of order k) can be characterized in terms of k-positive maps.[1] Various properties of the Schmidt number and Schmidt classes are considered in subsequent papers [2]–[4].

The Schmidt number is significantly used in the definition of partially entanglement-breaking quantum channels [5]. It was recently discovered in [6, Theorem 1] that there is a relationship between this notion and a necessary condition for equality in the law stating that the Holevo quantity of an ensemble of quantum states does not increase under the action of a quantum channel.

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[1] The definitions of LOCC-operations and k-positive maps, as well as of several other notions of noncommutative probability theory, are given in Sec. 2.

The present paper deals with infinite-dimensional generalizations of the notions listed above. A partial motivation of this paper is the author's desire to generalize the above-mentioned result [6] to the case of infinite-dimensional quantum systems and channels.

In Sec. 3, we consider the definition of the Schmidt number for states of an infinite-dimensional composite quantum system. Since the existence of separable not countably decomposable states (see [7]) shows that the finite-dimensional formula for the Schmidt number is not correct, a "continuous" modification of this formula is proposed on the basis of the notion of essential supremum of a function with respect to a given measure. It is shown that this formula gives an adequate definition of the Schmidt number in the sense that the corresponding Schmidt classes (the sets of states with Schmidt number $\leq k$) coincide with the convex closures of sets of pure states of Schmidt rank $\leq k$.

The properties of the Schmidt classes in the infinite-dimensional case are considered in Sec. 4. In particular, the characterization of the Schmidt class of order k in terms of k-positive maps (generalizing Theorem 1 in [1]) is proved. It is shown that an arbitrary state of Schmidt class of order k is the barycenter of some probability measure supported in the set of pure states of Schmidt rank $\leq k$. It is simultaneously proved that there are states with a given Schmidt number such that any of their convex decompositions does not contain pure states of finite Schmidt rank.

The definition and several properties of infinite-dimensional partially entanglement-breaking quantum channels are considered in Sec. 5. It is shown that, in contrast to the finite-dimensional case, the class of partially entanglement-breaking channels of order k does not coincide with the class of channels such that the operators in their Kraus representation are of rank $\leq k$ (the last class is a proper subclass of the first class). Moreover, it is shown that there are partially entanglement-breaking channels (in particular, entanglement-breaking channels) such that all the operators in any of their Kraus representations are of infinite rank.

2. PRELIMINARIES

The following notation is used:

- $\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{K}$ are separable Hilbert spaces;
- $\mathfrak{B}(\mathcal{H})$ is the Banach space of all bounded operators in $\mathcal{H}$;
- $\mathfrak{T}(\mathcal{H})$ is the Banach space of all trace-class operators in $\mathcal{H}$;
- $\mathfrak{T}_+(\mathcal{H})$ is the cone of all positive trace-class operators in $\mathcal{H}$;
- $\mathfrak{S}(\mathcal{H})$ is the subset of the cone $\mathfrak{T}_+(\mathcal{H})$ consisting of operators with unit trace.

The closure, convex hull, convex closure, and the set of extreme points of a subset $\mathcal{A}$ of a topological linear space will be denoted by $\mathrm{cl}(\mathcal{A})$, $\mathrm{co}(\mathcal{A})$, $\overline{\mathrm{co}}(\mathcal{A})$, and $\mathrm{extr}(\mathcal{A})$, respectively [8]–[10].

The operators in $\mathfrak{S}(\mathcal{H})$ are denoted by ρ, σ, ω, $\dots$ and are called *density operators* or *states*, because each density operator uniquely determines a normal state on the algebra $\mathfrak{B}(\mathcal{H})$. The states corresponding to density operators of rank 1 are said to be *pure*. The set of pure states in $\mathfrak{S}(\mathcal{H})$ coincides with $\mathrm{extr}\,\mathfrak{S}(\mathcal{H})$.

For vectors and operators of rank 1 in a Hilbert space, we use the Dirac notation $|\varphi\rangle$, $|\chi\rangle\langle\psi|$, $\dots$ (where the action of the operator $|\chi\rangle\langle\psi|$ on the vector $|\varphi\rangle$ is the vector $\langle\psi,\varphi\rangle\,|\chi\rangle$).

We denote the unit operator in a Hilbert space $\mathcal{H}$ and the identity transformation of the space $\mathfrak{T}(\mathcal{H})$ by $I_{\mathcal{H}}$ and $\mathrm{Id}_{\mathcal{H}}$, respectively.

Let $\mathcal{P}(\mathcal{A})$ be the set of Borel probability measures on a closed subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ equipped with the weak convergence topology [11], [12]. This set can be regarded as a complete separable metric space [12]. The barycenter $\mathbf{b}(\mu)$ of the measure μ in $\mathcal{P}(\mathcal{A})$ is the state in $\overline{\mathrm{co}}(\mathcal{A})$ defined by the Bochner integral

$$\mathbf{b}(\mu) = \int_{\mathcal{A}} \rho\mu\,(d\rho).$$

For an arbitrary subset $\mathcal{B} \subseteq \overline{\mathrm{co}}(\mathcal{A})$, the subset of the set $\mathcal{P}(\mathcal{A})$ consisting of measures with barycenter in $\mathcal{B}$ will be denoted by $\mathcal{P}_{\mathcal{B}}(\mathcal{A})$.

A finite or countable set of states $\{\rho_i\} \subset \mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ with the corresponding probability distribution $\{\pi_i\}$ is traditionally called an *ensemble* and is denoted by $\{\pi_i, \rho_i\}$. An ensemble of states can be considered as an atomic (discrete) measure in $\mathcal{P}(\mathcal{A})$. The barycenter of this measure is the average state $\sum_i \pi_i \rho_i$ of the corresponding ensemble.

A linear map $\Phi\colon \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}')$ is said to be *k-positive* if, for any k-dimensional Hilbert space $\mathcal{K}$, the map $\Phi^* \otimes \mathrm{Id}^*_{\mathcal{K}}$ of the C^*-algebra $\mathfrak{B}(\mathcal{H}' \otimes \mathcal{K})$ into the C^*-algebra $\mathfrak{B}(\mathcal{H} \otimes \mathcal{K})$ is positive. If a map Φ is k-positive for any k, then it is said to be *completely positive*. A completely positive trace-preserving linear map is called a *quantum channel* [13], [14]. The set of all quantum channels from $\mathfrak{T}(\mathcal{H})$ into $\mathfrak{T}(\mathcal{H}')$ is denoted by $\mathfrak{F}(\mathcal{H}, \mathcal{H}')$.

A state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ is said to be *separable* or *not entangled* if it belongs to the convex closure of the set of all pure state-products in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ (i.e., states of the form $\rho \otimes \sigma$, where $\rho \in \mathfrak{S}(\mathcal{H})$ and $\sigma \in \mathfrak{S}(\mathcal{K})$); otherwise, it is said to be *entangled*.

The key notion in the theory of entanglement is the notion of *LOCC-operation*, i.e., of transformation of states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ that can be reduced to a sequence of local operations (Local Operation) over each of the subsystems and to the exchange of classical information between these subsystems (Classical Communication) [14], [15]. The simplest examples of LOCC-operations are quantum channels of the form $\Phi \otimes \Psi$, where $\Phi \in \mathfrak{F}(\mathcal{H}, \mathcal{H})$ and $\Psi \in \mathfrak{F}(\mathcal{K}, \mathcal{K})$.

3. SCHMIDT NUMBER

The *Schmidt rank* $\mathrm{SR}(\omega)$ of a pure state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ can be defined as the rank of isomorphic states $\mathrm{Tr}_{\mathcal{K}}\,\omega$ and $\mathrm{Tr}_{\mathcal{H}}\,\omega$.

If the spaces $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional, then the Schmidt number of an arbitrary state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ is defined by the expression

$$\mathrm{SN}(\omega) = \inf_{\sum_i \pi_i \omega_i = \omega} \sup_i \mathrm{SR}(\omega_i), \tag{3.1}$$

where the infimum is taken over all ensembles $\{\pi_i, \omega_i\}$ of pure states with average state ω [1]. Using the Carathéodory theorem, one can easily show that, for each positive integer k, the set $\mathfrak{S}_k(\mathcal{H} \otimes \mathcal{K}) = \{\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \mid \mathrm{SN}(\omega) \le k\}$ is closed and coincides with the convex hull of pure states of Schmidt rank $\le k$. This means that the function $\omega \mapsto \mathrm{SN}(\omega)$ is lower semicontinuous on the set $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. Thus, we have the following increasing finite sequence [2]

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \cdots \subset \mathfrak{S}_{n-1} \subset \mathfrak{S}_n = \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$$

of closed subsets, where $\mathfrak{S}_1$ is the set of separable (not entangled) states and $n = \min\{\dim \mathcal{H}, \dim \mathcal{K}\}$.

If the spaces $\mathcal{H}$ and $\mathcal{K}$ are infinite-dimensional, then the right-hand side of (3.1) is well defined, but it does not give an adequate definition of the Schmidt number. This follows from the existence of separable states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ (they are said to be not countably decomposable), which cannot be represented as a countable convex combination of pure state-products [7]. The fact that a separable state ω is not countably decomposable implies that the right-hand side of (3.1) is greater than 1 for each of such states, despite the natural requirement that must be satisfied for the Schmidt number.[3]

In what follows, it will be shown that a reasonable generalization of definition (3.1) to the infinite-dimensional case is given by the formula

$$\mathrm{SN}(\omega) = \inf_{\mu \in \mathcal{P}_{\{\omega\}}(\mathrm{extr}\,\mathfrak{S}(\mathcal{H} \otimes \mathcal{K}))} \operatorname{ess\,sup}_{\mu} \mathrm{SR}(\,\cdot\,), \tag{3.2}$$

where "$\operatorname{ess\,sup}_{\mu}$" is the essential supremum with respect to the measure μ [9, Sec. 13.1]. We note that $\operatorname{ess\,sup}_{\mu} \mathrm{SR}(\,\cdot\,) = \|\mathrm{SR}\|_\infty$ is the norm of the function SR in the space $L^\infty(X, \mu)$, where $X = \mathrm{extr}\,\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$.

[2] From now on, for brevity, we write $\mathfrak{S}_k$ instead of $\mathfrak{S}_k(\mathcal{H} \otimes \mathcal{K})$.

[3] This problem is similar to the problem arising in the infinite-dimensional generalization of the method of the convex roof construction of entanglement monotones [15]): the existence of not countably decomposable separable states results in that the discrete version of this construction is not well defined (see Remark 9 in [16]).

Proposition 1. (A) *The function* $\mathrm{SN}(\omega)$ *defined by* (3.2) *is lower semicontinuous on the set* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. *For each state* $\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, *the infimum in* (3.2) *is attained on a measure in* $\operatorname{extr}\mathcal{P}_{\{\omega\}}(\operatorname{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$.

(B) *For each positive integer* k, *the set* $\mathfrak{S}_k=\{\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})\mid \mathrm{SN}(\omega)\le k\}$, *where* $\mathrm{SN}(\omega)$ *is defined by formula* (3.2), *is closed and convex. It coincides with the convex closure of the set of pure states in* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ *which have Schmidt rank* $\le k$.

(C) *If* ω *is a state of finite rank in* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, *then the values of* $\mathrm{SN}(\omega)$ *determined by formulas* (3.1) *and* (3.2) *coincide.*

Proof. Since the nonnegative function $\omega\mapsto\mathrm{SR}(\omega)$ is lower semicontinuous on the set $\operatorname{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, the first assertion in the proposition follows from Proposition 9 in the appendix.

The second assertion follows from the first one and Lemma 1 in [7].

To prove the third assertion, we note that if the right-hand side of (3.2) is equal to $k<+\infty$, then the right-hand side of (3.1) is also equal to k because of the coincidence of the convex hull and the convex closure of the subset

$$\mathfrak{S}_k^{\omega}=\{\varpi\in\operatorname{extr}\mathfrak{S}(\operatorname{supp}\omega)\mid \mathrm{SR}(\varpi)\le k\}$$

of the finite-dimensional space $\mathfrak{T}(\operatorname{supp}\omega)$ (see [9, Corollary 5.33]). □

The following proposition is a generalization of Proposition 1 in [1] to the infinite-dimensional case.

Proposition 2. *The Schmidt number* (*defined by formula* (3.2)) *of a state of an infinite-dimensional composite quantum system does not increase under the action of LOCC-operations.*

This assertion can be reduced to the assertion of Proposition 1 in [1] by using the following approximation result.

Lemma 1. *Let* $\{P_n\}$ *and* $\{Q_n\}$ *be increasing sequences of projection operators of finite rank that strongly converge to* $I_{\mathcal{H}}$ *and to* $I_{\mathcal{K}}$, *respectively. For an arbitrary state* $\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, *let*

$$\omega_n=(\operatorname{Tr}P_n\otimes Q_n\cdot\omega)^{-1}P_n\otimes Q_n\cdot\omega\cdot P_n\otimes Q_n.$$

Then

$$\lim_{n\to+\infty}\mathrm{SN}(\omega_n)=\mathrm{SN}(\omega).$$

If $\mathrm{SN}(\omega)<+\infty$, *then there is* n_0 *such that* $\mathrm{SN}(\omega_n)=\mathrm{SN}(\omega)$ *for all* $n\ge n_0$.

Proof. Since the Schmidt number is lower semicontinuous (Proposition 1, A), it suffices to show that

$$\mathrm{SN}(\omega_n)\le\mathrm{SN}(\omega)\qquad\forall\, n. \tag{3.3}$$

Since the state ω lies in the convex closure of the set $\mathfrak{S}^p_{\mathrm{SN}(\omega)}$ of pure states of Schmidt rank $\le\mathrm{SN}(\omega)$ (Proposition 1, B), it follows that there is a sequence $\{\omega_m\}$ in the convex hull of the set $\mathfrak{S}^p_{\mathrm{SN}(\omega)}$ converging to the state ω and satisfying the condition $\lim_{m\to+\infty}\mathrm{SN}(\omega_m)=\mathrm{SN}(\omega)$. For each m, inequality (3.3) with $\omega=\omega_m$ can be verified directly. Because of the lower semicontinuity of the Schmidt number, passing to the limit as $m\to+\infty$, we obtain (3.3). □

4. SEVERAL PROPERTIES OF THE SCHMIDT CLASSES $\mathfrak{S}_k$

For $\dim\mathcal{H} = \dim\mathcal{K} = +\infty$, we have an infinite increasing sequence

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \cdots \subset \mathfrak{S}_{n-1} \subset \mathfrak{S}_n \subset \cdots$$

of closed subsets of the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, where $\mathfrak{S}_1$ is the set of separable (not entangled) states.

Let $\mathfrak{S}_k^p$ be the set of all pure states in $\mathfrak{S}_k$.

Proposition 3. (A) *An arbitrary state in $\mathfrak{S}_k$ is the barycenter of a measure in $\mathcal{P}(\mathfrak{S}_k^p)$.*

(B) *There are states ω in $\mathfrak{S}_k \setminus \mathfrak{S}_{k-1}$ such that the operator $\omega - \lambda\sigma$ is not positive for any $\lambda > 0$ and any pure state σ of finite Schmidt rank.*[4] *For each such state ω,*

$$\omega = \sum_i \pi_i\omega_i, \qquad \{\omega_i\} \subset \operatorname{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}) \quad \Longrightarrow \quad \mathrm{SR}(\omega_i) = +\infty \quad \forall\, i.$$

(C) *An arbitrary pure state in $\mathfrak{S}_k \setminus \mathfrak{S}_{k-1}$ can be approximated by a sequence of states in $\mathfrak{S}_k \setminus \mathfrak{S}_{k-1}$ with the property mentioned in assertion* (B).

Proof. The first assertion readily follows from Proposition 1, and the second is confirmed by the example considered in Appendix 6.2 (after Proposition 10).

The third assertion can be proved by using the construction in the above-mentioned example in Appendix 6.2 and taking account of the facts that the functions with nonzero Fourier coefficients form a dense subset in $L^2([0,2\pi))$ and that an arbitrary set $\{|\psi_i\rangle\}_{i=1}^k$ of orthogonal unit vectors in a separable Hilbert space $\mathcal{H}$ is the image of the set of vectors $\{|\varphi_i\rangle\otimes|i\rangle\}_{i=1}^k \subset L^2([0,2\pi))\otimes\mathcal{K}$ under a unitary map from $L^2([0,2\pi))\otimes\mathcal{K}$ into $\mathcal{H}$, where $\{|i\rangle\}_{i=1}^k$ is an orthonormal basis of the space $\mathcal{K}$. $\square$

Let us consider a characterization of the set $\mathfrak{S}_k$ in terms of k-positive maps (which is an infinite-dimensional generalization of Theorem 1 in [1]).

Proposition 4. *A state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ lies in $\mathfrak{S}_k$ if and only if the operator $\Lambda_k\otimes\mathrm{Id}_{\mathcal{K}}(\omega)$ is positive for any k-positive linear transformation Λ_k of the space $\mathfrak{T}(\mathcal{H})$.*

Proof. Let $\omega_0 \in \mathfrak{S}_k$. By Proposition 3, there is a measure μ_0 in $\mathcal{P}(\mathfrak{S}_k^p)$ such that $\omega_0 = \int \omega\mu_0\,(d\omega)$. Since $\Lambda_k\otimes\mathrm{Id}_{\mathcal{K}}(\omega) \geq 0$ for any state $\omega \in \mathfrak{S}_k^p$ by the definition of k-positiveness (see Sec. 2), we have

$$\Lambda_k\otimes\mathrm{Id}_{\mathcal{K}}(\omega_0) = \int \Lambda_k\otimes\mathrm{Id}_{\mathcal{K}}(\omega)\mu_0\,(d\omega) \geq 0.$$

The converse assertion can be derived from the corresponding finite-dimensional result ([1, Theorem 1]) by using an approximation method and Lemma 1.

Let $\omega_0 \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K}) \setminus \mathfrak{S}_k$, i.e., let $\mathrm{SN}(\omega_0) > k$. By Lemma 1, there are projection operators $P \in \mathfrak{B}(\mathcal{H})$ and $Q \in \mathfrak{B}(\mathcal{K})$ of the same finite rank such that the state

$$\omega_* = (\mathrm{Tr}\, P\otimes Q\cdot\omega_0)^{-1} P\otimes Q\cdot\omega_0\cdot P\otimes Q$$

does not lie in the set $\mathfrak{S}_k$. Let $\mathcal{H}_* = P(\mathcal{H})$ and $\mathcal{K}_* = Q(\mathcal{K})$. By Theorem 1 in [1], there is a k-positive map $\Lambda_k\colon \mathfrak{T}(\mathcal{H}_*) \to \mathfrak{T}(\mathcal{H}_*)$ such that the operator $\Lambda_k\otimes\mathrm{Id}_{\mathcal{K}_*}(\omega_*)$ is not positive. We consider a k-positive map $\Lambda_k\circ\Pi$, where $\Pi(\,\cdot\,) = P(\,\cdot\,)P$. Then the operator $(\Lambda_k\circ\Pi)\otimes\mathrm{Id}_{\mathcal{K}}(\omega_0)$ is not positive, because otherwise the operator

$$I_{\mathcal{H}}\otimes Q\cdot(\Lambda_k\circ\Pi)\otimes\mathrm{Id}_{\mathcal{K}}(\omega_0)\cdot I_{\mathcal{H}}\otimes Q = (\mathrm{Tr}\, P\otimes Q\cdot\omega_0)\,\Lambda_k\otimes\mathrm{Id}_{\mathcal{K}_*}(\omega_*)$$

would be positive, and this contradicts the choice of Λ_k. $\square$

The compactness criterion for a subset of the cone $\mathfrak{T}_+(\mathcal{H}\otimes\mathcal{K})$ (see the proposition in the appendix in [17]) can be used to prove the infinite-dimensional generalization of Proposition 1 in [2].

[4] It is assumed that $\mathfrak{S}_0 = \varnothing$, and hence $\mathfrak{S}_1 \setminus \mathfrak{S}_0 = \mathfrak{S}_1$ is a set of separable states.

Proposition 5. *An arbitrary state $\omega_k \in \mathfrak{S}_k$ can be represented as*

$$\omega_k = (1-p)\omega_{k-1} + p\delta, \qquad p \in [0,1], \tag{4.1}$$

where $\omega_{k-1} \in \mathfrak{S}_{k-1}$ and δ is a state with Schmidt number $\geq k$ such that the operator $\delta - \lambda\sigma$ is not positive for any $\lambda > 0$ and any $\sigma \in \mathfrak{S}_{k-1}$.

In the set of such decompositions, there is a decomposition with minimal p.

In [2], a state with the property of the state δ is called a *k-edge* state. In contrast to the finite-dimensional case, to prove that δ is a k-edge state, it is not sufficient to show that the operator $\delta - \lambda\sigma$ is not positive for any $\lambda > 0$ and any $\sigma \in \mathfrak{S}^{p}_{k-1}$. This follows from Proposition 3, B.

Proof. Let

$$\mathcal{M} = \{0\} \cup \{A \in \mathfrak{T}_+(\mathcal{H}\otimes\mathcal{K}) \mid A \leq \omega_k, (\operatorname{Tr} A)^{-1}A \in \mathfrak{S}_{k-1}\}$$

be a closed subset of the cone $\mathfrak{T}_+(\mathcal{H}\otimes\mathcal{K})$.

We assume that $\mathcal{M} \neq \{0\}$. It follows from the above-mentioned compactness criterion for subsets of the cone $\mathfrak{T}_+(\mathcal{H}\otimes\mathcal{K})$ that the set $\mathcal{M}$ is compact. Therefore, there is an operator $A_0 \in \mathcal{M}$ such that $\operatorname{Tr} A_0 = \sup_{A\in\mathcal{M}} \operatorname{Tr} A$.

Introducing the notation $p = 1 - \operatorname{Tr} A_0$, $\omega_{k-1} = (\operatorname{Tr} A_0)^{-1}A_0$ and $\delta = p^{-1}(\omega_k - A_0)$, we obtain the decomposition (4.1) with minimal p.

If $\mathcal{M} = \{0\}$, then the only method for obtaining (4.1) is to set $p = 1$ and $\delta = \omega_k$. ◻

5. PARTIALLY ENTANGLEMENT-BREAKING CHANNELS

The notion of partially entanglement-breaking channel of order k (or of k-partially entanglement-breaking channel) in the finite-dimensional case was introduced in [5] as a natural generalization of the notion of entanglement-breaking channel (which is partially entanglement-breaking channel of order 1). According to the definition given in [5], a channel $\Phi\colon \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}')$ is said to be *partially entanglement-breaking of order k* if, for any Hilbert space $\mathcal{K}$, the Schmidt number of the state $\Phi\otimes \operatorname{Id}_{\mathcal{K}}(\omega) \in \mathfrak{S}(\mathcal{H}'\otimes\mathcal{K})$ does not exceed k for any state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.

The definition of the Schmidt number introduced in Sec. 3 can be used to generalize this definition of partially entanglement-breaking channels of order k directly to the infinite-dimensional case.

Following tradition, a partially entanglement-breaking channel of order k will be briefly called a *k-PEB channel.*

Let $\mathfrak{P}_k(\mathcal{H},\mathcal{H}')$ be the class of k-PEB channels from $\mathfrak{T}(\mathcal{H})$ into $\mathfrak{T}(\mathcal{H}')$. Since the set $\mathfrak{S}_k(\mathcal{H}'\otimes\mathcal{K})$ is closed and convex, it follows that $\mathfrak{P}_k(\mathcal{H},\mathcal{H}')$ is a closed convex subset of the set $\mathfrak{F}(\mathcal{H},\mathcal{H}')$ of all channels from $\mathfrak{T}(\mathcal{H})$ into $\mathfrak{T}(\mathcal{H}')$ equipped with the strong convergence topology [17].

The following charactrization of k-PEB channels (generalizing the corresponding result in [5], [18]) can easily be derived from Proposition 4.

Proposition 6. *A channel Φ is a k-PEB channel if and only if the map $\Lambda_k \circ \Phi$ is completely positive for any k-positive map Λ_k.*

By definition, the relation $\Phi \in \mathfrak{P}_k(\mathcal{H},\mathcal{H}')$ means that

$$\Phi\otimes \operatorname{Id}_{\mathcal{K}}(\omega) \in \mathfrak{S}_k(\mathcal{H}'\otimes\mathcal{K}) \qquad \text{for any} \quad \omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K}).$$

Because of the following proposition, it suffices to verify this inclusion only for one pure state.

Proposition 7. *Let $\Phi\colon \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}')$ be a quantum channel. If there is a pure state $|\psi\rangle\langle\psi|$ in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ with partial states $\operatorname{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi| \cong \operatorname{Tr}_{\mathcal{H}}|\psi\rangle\langle\psi|$ of full rank such that*

$$\Phi\otimes \operatorname{Id}_{\mathcal{K}}(|\psi\rangle\langle\psi|) \in \mathfrak{S}_k(\mathcal{H}'\otimes\mathcal{K}),$$

then the channel Φ is a k-PEB channel.

Proof. Let

$$|\psi\rangle = \sum_{i=1}^{+\infty} \mu_i |i\rangle \otimes |i\rangle,$$

where $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H} \cong \mathcal{K}$ and $\mu_i > 0$ for all i. Let

$$P_n = \sum_{i=1}^{n} |i\rangle\langle i|$$

be the projection operator in $\mathfrak{B}(\mathcal{K})$.

By Proposition 2, we have

$$\Phi \otimes \mathrm{Id}_{\mathcal{K}}(|\psi_n\rangle\langle\psi_n|) = c_n I_{\mathcal{H}} \otimes P_n \cdot \Phi \otimes \mathrm{Id}_{\mathcal{K}}(|\psi\rangle\langle\psi|) \cdot I_{\mathcal{H}} \otimes P_n \in \mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K}),$$

where

$$|\psi_n\rangle = c_n \sum_{i=1}^{n} \mu_i |i\rangle \otimes |i\rangle, \qquad c_n = \Big[\sum_{i=1}^{n} \mu_i^2\Big]^{-1/2}.$$

Let $\mathcal{H}_n = \mathrm{lin}(\{|i\rangle\}_{i=1}^n)$ and $\mathcal{K}_n = \mathrm{lin}(\{|i\rangle\}_{i=1}^n)$ be n-dimensional subspaces of the spaces $\mathcal{H}$ and $\mathcal{K}$. An arbitrary vector $|\varphi\rangle$ in $\mathcal{H}_n \otimes \mathcal{K}_n$ can be represented as

$$|\varphi\rangle = \sum_{i,j=1}^{n} \gamma_{ij} |i\rangle \otimes |j\rangle = \sum_{i=1}^{n} \mu_i |i\rangle \otimes A|i\rangle, \qquad \text{where} \quad A = \sum_{i,j=1}^{n} (\mu_i)^{-1} \gamma_{ij} |j\rangle\langle i|$$

is an operator in $\mathfrak{B}(\mathcal{K}_n)$. Therefore,

$$|\varphi\rangle\langle\varphi| = I_{\mathcal{H}_n} \otimes A \cdot |\psi_n\rangle\langle\psi_n| \cdot I_{\mathcal{H}_n} \otimes A^*,$$

and hence

$$\Phi \otimes \mathrm{Id}_{\mathcal{K}}(|\varphi\rangle\langle\varphi|) = I_{\mathcal{H}} \otimes A \cdot \Phi \otimes \mathrm{Id}_{\mathcal{K}}(|\psi_n\rangle\langle\psi_n|) \cdot I_{\mathcal{H}} \otimes A^* \in \mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K}).$$

This means that the restriction of the channel Φ to the set $\mathfrak{S}(\mathcal{H}_n)$ is a k-PEB channel. By Lemma 2 given below, the channel Φ is a k-PEB channel. $\square$

Lemma 2. *Let $\{\mathcal{H}_n\}$ be an increasing sequence of subspaces of the space $\mathcal{H}$ satisfying the relation $\mathrm{cl}\big(\bigcup_n \mathcal{H}_n\big) = \mathcal{H}$. If the restriction of the channel Φ to the set $\mathfrak{S}(\mathcal{H}_n)$ is a k-PEB channel for each n, then the channel Φ is a k-PEB channel.*

Proof. Since an arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ can be approximated by a sequence $\{\omega_n\}$ such that $\mathrm{supp}\, \mathrm{Tr}_{\mathcal{K}}\, \omega_n \subset \mathcal{H}_n$ (see Lemma 1), this assertion follows from the fact that the set $\mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K})$ is closed. $\square$

Let $|\psi\rangle\langle\psi|$ be a pure state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, and let it have partial states

$$\mathrm{Tr}_{\mathcal{K}}\, |\psi\rangle\langle\psi| \cong \mathrm{Tr}_{\mathcal{H}}\, |\psi\rangle\langle\psi| = \sigma$$

of full rank. Consider the one-to-one Choi–Jamiolkowski correspondence

$$\mathfrak{F}(\mathcal{H}, \mathcal{H}') \ni \Phi \quad \longleftrightarrow \quad \Phi \otimes \mathrm{Id}_{\mathcal{K}}(|\psi\rangle\langle\psi|) \in \mathfrak{C}_\sigma \doteq \{\omega \in \mathfrak{S}(\mathcal{H}' \otimes \mathcal{K}) \mid \mathrm{Tr}_{\mathcal{H}'}\, \omega = \sigma\},$$

which is a topological isomorphism if the set $\mathfrak{F}(\mathcal{H}, \mathcal{H}')$ of all channels is equipped with the strong convergence topology [16, Proposition 3]. Proposition 7 implies the following observation.

Corollary 1. *The restriction of the Choi–Jamiolkowski isomorphism to the class $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}')$ is an isomorphism between this class and $\mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K}) \cap \mathfrak{C}_\sigma$, which is a closed subset of the set $\mathfrak{S}(\mathcal{H}' \otimes \mathcal{K})$.*

For a given isomorphism, the set $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}') \setminus \mathfrak{P}_{k-1}(\mathcal{H}, \mathcal{H}')$ corresponds to the set

$$(\mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K}) \setminus \mathfrak{S}_{k-1}(\mathcal{H}' \otimes \mathcal{K})) \cap \mathfrak{C}_\sigma, \qquad k = 2, 3, \dots .$$

In [5], it was proved that the finite-dimensional channel Φ is a k-PEB channel if and only if it has the Kraus representation

$$\Phi(\,\cdot\,) = \sum_i V_i(\,\cdot\,)V_i^* \tag{5.1}$$

such that $\operatorname{rank} V_i \le k$ for all i (this is a natural generalization of the well-known characterization of finite-dimensional entanglement-breaking channels proved in [18]). In the infinite-dimensional case, the class of k-PEB channels is significantly wider than the class of channels with the Kraus representation mentioned above.

Proposition 8. (A) *If a channel Φ has the Kraus representation* (5.1) *such that* $\operatorname{rank} V_i \le k$ *for all i, then it belongs to the class $\mathfrak{P}_k(\mathcal{H},\mathcal{H}')$.*

(B) *There are channels Φ in $\mathfrak{P}_k(\mathcal{H},\mathcal{H}') \setminus \mathfrak{P}_{k-1}(\mathcal{H},\mathcal{H}')$ that have the following property*[5]:

$$\Phi(\,\cdot\,) = \sum_i V_i(\,\cdot\,)V_i^* \quad\Longrightarrow\quad \operatorname{rank} V_i = +\infty \quad \forall\, i.$$

Proof. The first assertion is obvious, because, for any pure state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, the expression

$$\Phi \otimes \mathrm{Id}_{\mathcal{K}}(\omega) = \sum_i V_i \otimes I_{\mathcal{K}} \cdot \omega \cdot V_i^* \otimes I_{\mathcal{K}}$$

leads to the decomposition of the state $\Phi \otimes \mathrm{Id}_{\mathcal{K}}(\omega)$ into a convex combination of pure states of Schmidt rank $\le k$.

To prove the second assertion, in the set $\mathfrak{S}_k(\mathcal{H}'\otimes\mathcal{K}) \setminus \mathfrak{S}_{k-1}(\mathcal{H}'\otimes\mathcal{K})$, we choose any state ω with the property of assertion (B) in Proposition 3. We can assume that $\mathrm{Tr}_{\mathcal{H}'}\,\omega$ is a state of full rank in $\mathfrak{S}(\mathcal{K})$. Let $|\psi\rangle\langle\psi|$ be purification of this state in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. By Corollary 1, the channel Φ_ω associated with the state ω by the Choi–Jamiolkowski isomorphism, which was induced by the state $|\psi\rangle\langle\psi|$, lies in $\mathfrak{P}_k(\mathcal{H},\mathcal{H}') \setminus \mathfrak{P}_{k-1}(\mathcal{H},\mathcal{H}')$. If we assume that

$$\Phi_\omega(\,\cdot\,) = \sum_i V_i(\,\cdot\,)V_i^*$$

and $\operatorname{rank} V_{i_0} < +\infty$ for some i_0, then we see that the result contradicts the basic property of the state ω, because $V_{i_0} \otimes \mathrm{Id}_{\mathcal{K}}|\psi\rangle \ne 0$ (otherwise, $V_{i_0}(\mathrm{Tr}_{\mathcal{K}}\,|\psi\rangle\langle\psi|)(V_{i_0})^* = 0$, which contradicts the assumption that $\mathrm{Tr}_{\mathcal{K}}\,|\psi\rangle\langle\psi|$ is a state of full rank). $\square$

Corollary 2. *There are entanglement-breaking channels Φ such that the operators V_i in any Kraus representation* (5.1) *are of infinite rank.*

Corollary 2 shows that infinite-dimensional entanglement-breaking channels can have a structure which is more complicated than that of entanglement-breaking channels between finite-dimensional systems [18].

6. APPENDIX

6.1. On a Property of the Set $\mathfrak{S}(\mathcal{H})$

Here we consider a consequence of the compactness criterion for subsets of probability measures on the set $\mathfrak{S}(\mathcal{H})$ (this criterion was considered in detail in [16, Sec. 1]), which states that a *subset $\mathcal{P}$ of the set $\mathcal{P}(\mathfrak{S}(\mathcal{H}))$ is compact* (*in the weak convergence topology*) *if and only if $\{\mathbf{b}(\mu) \mid \mu \in \mathcal{P}\}$ is a compact subset of the set $\mathfrak{S}(\mathcal{H})$.*

[5] It is assumed that $\mathfrak{P}_0(\mathcal{H},\mathcal{H}') = \varnothing$, and hence $\mathfrak{P}_1(\mathcal{H},\mathcal{H}') \setminus \mathfrak{P}_0(\mathcal{H},\mathcal{H}') = \mathfrak{P}_1(\mathcal{H},\mathcal{H}')$ is a class of entanglement-breaking channels.

Proposition 9. *Let f be a nonnegative lower semicontinuous function on a closed subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$. The function*

$$F(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}(\mathcal{A})} \operatorname{ess\,sup}_{\mu} f(\,\cdot\,) \tag{6.1}$$

is lower semicontinuous on the set $\overline{\mathrm{co}}(\mathcal{A})$.[6] *For each state $\rho \in \overline{\mathrm{co}}(\mathcal{A})$, the infimum in* (6.1) *is attained on a certain measure in* $\operatorname{extr} \mathcal{P}_{\{\rho\}}(\mathcal{A})$.

For each $c \geq 0$, the set $\{\rho \in \overline{\mathrm{co}}(\mathcal{A}) \mid F(\rho) \leq c\}$ coincides with the convex closure of the set $\{\rho \in \mathcal{A} \mid f(\rho) \leq c\}$.

Proof. The function $F(\rho)$ is well defined on the set $\overline{\mathrm{co}}(\mathcal{A})$ by Lemma 1 in [7].

Let us show that the functional

$$\mathcal{P}(\mathcal{A}) \ni \mu \mapsto \widehat{f}(\mu) = \operatorname{ess\,sup}_{\mu} f(\,\cdot\,) \tag{6.2}$$

is concave and lower semicontinuous. Because, for a given measure μ in $\mathcal{P}(\mathcal{A})$, the μ-essential supremum of the function f (coinciding with the norm $\|f\|_\infty$ of the space $L^\infty(\mathcal{A},\mu)$) is the least upper bound of the increasing family of norms $\|f\|_p$ of the spaces $L^p(\mathcal{A},\mu)$, $p \in [1,+\infty)$, the concavity and lower semicontinuity of the functional (6.2) follows from the concavity and lower semicontinuity of the functional

$$\mathcal{P}(\mathcal{A}) \ni \mu \mapsto \|f\|_p = \sqrt[p]{\int_{\mathcal{A}} [f(\rho)]^p \mu\,(d\rho)}$$

(the lower semicontinuity of this functional follows from the basic properties of weak convergence of probability measures, see [11, Chap. I, Sec. 2]).

Since the functional (6.2) is concave and lower semicontinuous and the set $\mathcal{P}_{\{\rho\}}(\mathcal{A})$ is compact (which follows from the above compactness criterion), the infimum in the definition of the quantity $F(\rho)$ for each ρ in $\overline{\mathrm{co}}(\mathcal{A})$ is attained on a certain measure in $\operatorname{extr} \mathcal{P}_{\{\rho\}}(\mathcal{A})$.

We assume that the function (6.1) is not lower semicontinuous. Then there is a sequence $\{\rho_n\} \subset \overline{\mathrm{co}}(\mathcal{A})$ converging to the state $\rho_0 \in \overline{\mathrm{co}}(\mathcal{A})$ such that

$$\exists \lim_{n\to+\infty} F(\rho_n) < F(\rho_0). \tag{6.3}$$

As was shown above, for each $n = 1, 2, \dots$, there is a measure μ_n in $\mathcal{P}_{\{\rho_n\}}(\mathcal{A})$ such that $F(\rho_n) = \widehat{f}(\mu_n)$. Since the sequence $\{\rho_n\}$ is a compact set, it follows from the above compactness criterion that there is a subsequence $\{\mu_{n_k}\}$ converging to some measure μ_0. Because the map $\mu \mapsto \mathbf{b}(\mu)$ is continuous, the measure μ_0 lies in $\mathcal{P}_{\{\rho_0\}}(\mathcal{A})$. The lower semicontinuity of the functional (6.2) implies

$$F(\rho_0) \leq \widehat{f}(\mu_0) \leq \liminf_{k\to+\infty} \widehat{f}(\mu_{n_k}) = \lim_{k\to+\infty} F(\rho_{n_k}),$$

which contradicts (6.3).

The last assertion of the proposition is a consequence of the preceding assertions and Lemma 1 in [7]. $\square$

[6] The symbol "$\operatorname{ess\,sup}_\mu$" denotes the essential supremum with respect to the measure μ [9, Sec. 13.1].

6.2. On the Existence of a State with a Given Schmidt Number Such That None of Its Countable Convex Decompositions Contains Pure States of a Finite Schmidt Rank

First, we show that the separable not countably decomposable state constructed in [7] has in fact a stronger property, i.e., any countable convex decomposition of this state does not contain pure states of finite Schmidt rank (the property of being not countably decomposable means that it cannot be decomposed in a convex combination of pure state-products, i.e., states of Schmidt rank $= 1$). Further, we use this observation to construct a state with a given Schmidt number such that none of its countable convex decompositions contains pure states of finite Schmidt rank.

We use the notation introduced [7] to present the construction of such a separable state. We consider a one-dimensional group of rotations G identifying it with the interval $[0, 2\pi)$ with addition $\operatorname{mod} 2\pi$. Let $\mathcal{H} = L^2([0, 2\pi))$ with normalized Lebesgue measure $dx/2\pi$, and let $\{|k\rangle; k \in \mathbf{Z}\}$ be an orthonormal trigonometric basis in $\mathcal{H}$ such that

$$\langle k|\psi\rangle = \int_0^{2\pi} e^{-ixk}\, \psi(x)\, \frac{dx}{2\pi}.$$

The unitary representation $x \to V_x$ of the group G, where $V_x = \sum_{-\infty}^{+\infty} e^{ixk}|k\rangle\langle k|$, such that $(V_u\psi)(x) = \psi(x+u)$ is considered.

For arbitrary unit vectors $|\varphi_j\rangle \in \mathcal{H}_j \simeq L^2([0, 2\pi))$, $j = 1, 2$, we consider the separable state

$$\rho_{12} = \int_0^{2\pi} V_x^{(1)}|\varphi_1\rangle\langle\varphi_1|(V_x^{(1)})^* \otimes V_x^{(2)}|\varphi_2\rangle\langle\varphi_2|(V_x^{(2)})^*\, \frac{dx}{2\pi}. \tag{6.4}$$

The following proposition sharpens the statement of Theorem 3 in [7]. Its proof is a natural generalization of the proof of that theorem.

Proposition 10. *Assume that ρ_{12} is the separable state defined in* (6.4). *If all Fourier coefficients* (*coordinates in the basis $\{|k\rangle\}$*) *of the vectors $|\varphi_j\rangle$ are nonzero, then the operator*

$$\rho_{12} - \lambda\sigma$$

is not positive for any $\lambda > 0$ and a pure state σ of finite Schmidt rank.

In particular, this means that any countable convex decomposition of the state ρ_{12} does not contain pure states of finite Schmidt rank.

Proof. We assume that there is a vector $|\psi\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ of Schmidt rank n such that

$$\rho_{12} \geq |\psi\rangle\langle\psi|. \tag{6.5}$$

Assume that

$$|\psi\rangle = \sum_{i=1}^{n} |\alpha_i^1\rangle \otimes |\alpha_i^2\rangle,$$

where $\{|\alpha_i^j\rangle\}_{i=1}^n$, $j = 1, 2$, are sets of orthogonal vectors. It follows from inequality (6.5) that

$$\int_0^{2\pi} \bigl|\langle\lambda_1|V_x^{(1)}|\varphi_1\rangle\bigr|^2 \bigl|\langle\lambda_2|V_x^{(2)}|\varphi_2\rangle\bigr|^2\, \frac{dx}{2\pi} \geq \biggl|\sum_{i=1}^{n}\langle\lambda_1|\alpha_i^1\rangle\langle\lambda_2|\alpha_i^2\rangle\biggr|^2 \tag{6.6}$$

for any $\lambda_j \in L^2([0, 2\pi))$, $j = 1, 2$.

We consider the linear mappings

$$L^2([0, 2\pi)) \ni \lambda \mapsto \Phi_j(\lambda) = \{\langle\alpha_i^j|\lambda\rangle\}_{i=1}^n \in \mathbb{C}^n,$$

$$L^2([0, 2\pi)) \ni \lambda \mapsto \Psi_j(\lambda) = \overline{\langle\lambda|V_x^{(j)}|\varphi_j\rangle} = \sum_{k=-\infty}^{+\infty} \langle\varphi_j|k\rangle\langle k|\lambda\rangle e^{-ikx}, \qquad j = 1, 2.$$

In the space $L^2([0,2\pi))$, let $\mathcal{H}_0$ be a dense subset consisting of trigonometric polynomials (functions with finitely many nonzero Fourier coefficients). Since $\langle\varphi_j|k\rangle \neq 0$ for all k, the maps Ψ_j, $j=1,2$, are linear isomorphisms in $\mathcal{H}_0$. Therefore, it follows from (6.6) that

$$|\langle A_1(\xi), \Xi(A_2(\eta))\rangle_{\mathbb{C}^n}|^2 \leq \int_0^{2\pi} |\xi(x)\eta(x)|^2 \,\frac{dx}{2\pi}, \qquad \xi,\eta \in \mathcal{H}_0, \tag{6.7}$$

where $A_j(\,\cdot\,) = \Phi_j(\Psi_j^{-1}(\,\cdot\,))$, $j=1,2$, are linear mappings from $\mathcal{H}_0$ into $\mathbb{C}^n$, and Ξ is the complex conjugation in $\mathbb{C}^n$.

Since $\{\Phi_2(\lambda) \mid \lambda \in L^2([0,2\pi))\} = \mathbb{C}^n$, we have $\{\Phi_2(\lambda) \mid \lambda \in \mathcal{H}_0\} = \mathbb{C}^n$, and hence

$$\{A_2(\xi) \mid \xi \in \mathcal{H}_0\} = \mathbb{C}^n.$$

Therefore, there is a set $|\eta_1\rangle, \dots, |\eta_n\rangle$ of vectors in the basis $\{|k\rangle\}$ such that the vectors

$$A_2(\eta_1), \dots, A_2(\eta_n)$$

form a basis in $\mathbb{C}^n$. Since $|\eta_i(x)| = 1$, it follows from (6.7) that

$$|\langle A_1(\xi), \Xi(A_2(\eta_i))\rangle_{\mathbb{C}^n}|^2 \leq \int_0^{2\pi} |\xi(x)|^2 \,\frac{dx}{2\pi} = \|\xi\|^2, \quad i=1,\dots,n, \qquad \xi \in \mathcal{H}_0.$$

Therefore, the map A_1 is bounded on $\mathcal{H}_0$ and can be extended to a bounded linear map A_1 from $L^2([0,2\pi))$ into $\mathbb{C}^n$.

Similar considerations show that it is possible to extend the map A_2 to a bounded linear map A_2 from $L^2([0,2\pi))$ into $\mathbb{C}^n$.

Since the anti-linear operator $B = A_1^* \circ \Xi \circ A_2$ in the space $L^2([0,2\pi))$ is of rank $\leq n$, it can be represented as

$$B(\,\cdot\,) = \sum_{i=1}^{n} \langle\,\cdot\,|\beta_i^2\rangle|\beta_i^1\rangle,$$

where $\{|\beta_i^j\rangle\}$, $j=1,2$, are sets of vectors in $L^2([0,2\pi))$ and the set $\{|\beta_i^1\rangle\}$ consists of linearly independent vectors.

Therefore, (6.7) can be rewritten as

$$\left|\sum_{i=1}^{n}\langle\xi|\beta_i^1\rangle\langle\eta|\beta_i^2\rangle\right|^2 \leq \int_0^{2\pi} |\xi(x)\eta(x)|^2 \,\frac{dx}{2\pi}. \tag{6.8}$$

It follows from Lemma 3 below that, for an arbitrary $\varepsilon > 0$, one can find a subset $\mathcal{A} \subset [0,2\pi)$ with Lebesgue measure $< \varepsilon$ such that the functions $\beta_1^1, \beta_2^1, \dots, \beta_n^1$ are linearly independent on $\mathcal{A}$. Therefore, for each i, one can find a function ξ supported in $\mathcal{A}$ such that $\langle\xi|\beta_i^1\rangle \neq 0$ but $\langle\xi|\beta_j^1\rangle = 0$ for all $j \neq i$. For this function ξ and an arbitrary function η supported in $[0,2\pi) \setminus \mathcal{A}$, the right-hand side of (6.8) is zero, and hence $\langle\eta|\beta_i^2\rangle = 0$; thus, $\beta_i^2(x) = 0$ almost everywhere in $[0,2\pi) \setminus \mathcal{A}$. Therefore, the measure of the support of the function β_i^2 does not exceed ε, and hence $\beta_i^2(x) = 0$ almost everywhere in $[0,2\pi)$. Thus, we have $B = 0$, which implies $|\psi\rangle = 0$. $\square$

In the following lemma, the linear independence of measurable functions $f_1, \dots, f_n$ on a measurable subset $\mathcal{A} \subset \mathbb{R}$ means that any nontrivial linear combination of these functions is not equal to zero almost everywhere on $\mathcal{A}$.

Lemma 3. *Let $f_1, \dots, f_n$ be linearly independent measurable functions on $[a,b]$. For any arbitrary $\varepsilon > 0$, there is a subset $\mathcal{A} \subset [a,b]$ with Lebesgue measure $\mu(\mathcal{A}) < \varepsilon$ such that the functions $f_1, \dots, f_n$ are linearly independent on $\mathcal{A}$.*

Proof. For $n = 1, 2$, the assertion of the lemma is obvious. We assume that it holds for a given n and we will show that it can be satisfied for $n + 1$.

By assumption, for any $\varepsilon > 0$ and each set of functions $\{f_i\} \setminus f_j$, $j = 1, \ldots, n+1$, there is a subset $\mathcal{A}_j^\varepsilon \subset [a, b]$ with $\mu(\mathcal{A}_j^\varepsilon) < \varepsilon$ such that the functions in this set are linearly independent on $\mathcal{A}_j^\varepsilon$.

If the assertion of the lemma does not hold for $n+1$, then there is an $\varepsilon_* > 0$ such that the functions $f_1, \ldots, f_{n+1}$ are linearly dependent on any subset $\mathcal{A} \subset [a, b]$ with $\mu(\mathcal{A}) < \varepsilon_*$.

Let $\varepsilon < \varepsilon_*/2(n+1)$, and let

$$\mathcal{A}^\varepsilon = \bigcup_{j=1}^{n+1} \mathcal{A}_j^\varepsilon.$$

We choose a finite set $\{\mathcal{B}_k\}$ of nonintersecting subsets of the set $[a, b] \setminus \mathcal{A}^\varepsilon$ such that $\mu(\mathcal{B}_k) < \varepsilon_*/2$ and $\bigcup_k \mathcal{B}_k = [a, b] \setminus \mathcal{A}^\varepsilon$.

For each k, we let $\mathcal{C}_k = \mathcal{A}^\varepsilon \cup \mathcal{B}_k$. Since $\mu(\mathcal{C}_k) < \varepsilon_*$, there is a set $\{\lambda_i^k\}_{i=1}^{n+1}$ of complex numbers such that

$$\sum_{i=1}^{n+1} \lambda_i^k f_i(x) = 0 \quad \text{almost everywhere on } \mathcal{C}_k, \qquad \sum_{i=1}^{n+1} |\lambda_i^k| > 0. \tag{6.9}$$

Since $\mathcal{A}_j^\varepsilon \subset \mathcal{C}_k$ for all j, it is easy to see that $\lambda_i^k \neq 0$ for all i. Therefore, we can assume that $\lambda_{n+1}^k = 1$. Since the functions $f_1, \ldots, f_{n+1}$ are linearly independent on $[a, b] = \bigcup_k \mathcal{C}_k$, there are k_1 and k_2 such that

$$\{\lambda_i^{k_1}\}_{i=1}^{n+1} \neq \{\lambda_i^{k_2}\}_{i=1}^{n+1}.$$

It follows from (6.9) that

$$\sum_{i=1}^{n} (\lambda_i^{k_1} - \lambda_i^{k_2}) f_i(x) = 0 \quad \text{almost everywhere on } \mathcal{A}_{n+1}^\varepsilon \subseteq \mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}, \qquad \sum_{i=1}^{n} |\lambda_i^{k_1} - \lambda_i^{k_2}| > 0,$$

which contradicts the construction of the set $\mathcal{A}_{n+1}^\varepsilon$. □

6.3. Example of the State ω with $\mathrm{SN}(\omega) = k \in \mathbb{N}$ Such That the Operator $\omega - \lambda\sigma$ is Not Positive for Any Pure State σ of a Finite Schmidt Rank and Any $\lambda > 0$

Let $\{|\varphi_1^i\rangle\}_{i=1}^k$ and $\{|\varphi_2^i\rangle\}_{i=1}^k$ be sets of orthogonal unit vectors in

$$\mathcal{H}_1 = L^2([0, 2\pi)) \qquad \text{and} \qquad \mathcal{H}_2 = L^2([0, 2\pi)),$$

respectively, which have nonzero Fourier coefficients. Let $\mathcal{K}$ be a k-dimensional Hilbert space with orthonormal basis $\{|i\rangle\}_{i=1}^k$. For each positive integer n, we consider the state

$$\rho_{123}^n = \int_0^{2\pi/n} V_x^{(1)} \otimes V_x^{(2)} \otimes I_{\mathcal{K}} \cdot |\Omega\rangle\langle\Omega| \cdot (V_x^{(1)})^* \otimes (V_x^{(2)})^* \otimes I_{\mathcal{K}} \frac{n\,dx}{2\pi} \tag{6.10}$$

in $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$, where

$$|\Omega\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\varphi_1^i\rangle \otimes |\varphi_2^i\rangle \otimes |i\rangle$$

is the unit vector in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K}$.

Further (speaking about the Schmidt rank and the Schmidt number), we assume that the space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K}$ is the tensor product of the spaces $\mathcal{H}_1$ and $\mathcal{H}_2 \otimes \mathcal{K}$.

Since the state ρ_{123}^n lies in the convex closure of a family of local unitary "translations" of the state $|\Omega\rangle\langle\Omega|$ such that $\mathrm{SR}(|\Omega\rangle\langle\Omega|) = k$, we have $\mathrm{SN}(\rho_{123}^n) \leq k$ for all n. Since the sequence $\{\rho_{123}^n\}$ converges to the state $|\Omega\rangle\langle\Omega|$, it follows from the lower semicontinuity of the Schmidt number (Proposition 1, A) that $\mathrm{SN}(\rho_{123}^n) = k$ for a sufficiently large n.

We assume that

$$\rho_{123}^{n} \geq \lambda|\Psi\rangle\langle\Psi| \tag{6.11}$$

for some $\lambda > 0$, where $|\Psi\rangle\langle\Psi|$ is a pure state in $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$ of finite Schmidt rank. Let $P_i = I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2} \otimes |i\rangle\langle i|$. Since

$$\sum_{i=1}^{k} P_i = I_{\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{K}},$$

there is an i_0 such that $P_{i_0}|\Psi\rangle \neq 0$. We can assume that $i_0 = 1$. Therefore, we have $P_1|\Psi\rangle = \nu|\psi\rangle \otimes |1\rangle$, where $\nu > 0$ and $|\psi\rangle$ is the unit vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Since $P_1\rho_{123}^{n}P_1 = k^{-1}\rho_{12}^{n} \otimes |1\rangle\langle 1|$, where

$$\rho_{12}^{n} = \int_0^{2\pi/n} V_x^{(1)}|\varphi_1^1\rangle\langle\varphi_1^1|(V_x^{(1)})^* \otimes V_x^{(2)}|\varphi_2^1\rangle\langle\varphi_2^1|(V_x^{(2)})^* \frac{n\,dx}{2\pi},$$

it follows from (6.11) that $\rho_{12}^{n} \geq k\lambda\nu|\psi\rangle\langle\psi|$. Since $P_1(\,\cdot\,)P_1$ and $\mathrm{Tr}_{\mathcal{K}}(\,\cdot\,)$ are local operations, the state $|\psi\rangle\langle\psi|$ is of finite Schmidt rank. Therefore, Proposition 10 shows that $\lambda = 0$.

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