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Generalized compactness in linear spaces and its applications

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Abstract. For a fixed convex domain in a linear metric space the problems of the continuity of convex envelopes (hulls) of continuous concave functions (the CE-property) and of convex envelopes (hulls) of arbitrary continuous functions (the strong CE-property) arise naturally. In the case of compact domains a comprehensive solution was elaborated in the 1970s by Vesterstrom and O'Brien. First Vesterstrom showed that for compact sets the strong CE-property is equivalent to the openness of the barycentre map, while the CE-property is equivalent to the openness of the restriction of this map to the set of maximal measures. Then O'Brien proved that in fact both properties are equivalent to a geometrically obvious 'stability property' of convex compact sets. This yields, in particular, the equivalence of the CE-property to the strong CE-property for convex compact sets. In this paper we give a solution to the following problem: can these results be extended to noncompact convex sets, and, if the answer is positive, to which sets? We show that such an extension does exist. This is an extension to the class of so-called μ-compact sets. Moreover, certain arguments confirm that this could be the maximal class to which such extensions are possible. Then properties of μ-compact sets are analysed in detail, several examples are considered, and applications of the results obtained to quantum information theory are discussed.

Bibliography: 32 titles.

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§ 1. Introduction

Various properties and structure of compact sets in the convex analytic context have been studied thoroughly starting from the middle of the last century. An extensive bibliography is devoted to this theme (see [1]–[3] and references therein). The most important results are well known: the Krein-Milman theorem on the convex hulls of extreme points, Choquet's theory of barycentric decompositions, properties of convex hulls (envelopes) of functions on convex compact sets. Some

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classical results have been extended to noncompact sets in locally convex spaces by Edgar [4], [5] and Bourgin [6], [7]. Such generalizations are interesting not only theoretically, but also very important in applications, for instance, in mathematical physics [8], in quantum information theory [9], and so on. Of course, classical results of convex analysis cannot be extended to all noncompact sets. One has to postulate some special properties of these sets. Choquet's theory, for example, has been generalized to sets possessing the Radon-Nikodym property [4]. In [9] several results on the continuity of convex hulls of functions were extended to a special class of sets called *μ-compact sets*. This class, characterized by the special relation between the topology and the structure of linear operations, is the main subject of this paper.

Problems of continuity of the convex hulls of continuous functions (see definitions in the next section) have been studied in the literature since the 70s of the last century. Under what conditions on the convex compact set $\mathscr{A}$ is the convex hull of any continuous (another assumption: concave continuous) function defined on $\mathscr{A}$ continuous? Vesterstrom [10] showed that a necessary and sufficient condition for this is the openness of the barycentre map. He conjectured the equivalence of the continuity of the convex hull of any continuous concave function (this property was called by Lima [11] the CE-*property*) to the continuity of the convex hull of any continuous function (called in [9] the *strong* CE-*property*). This conjecture was proved by O'Brien [12], who, moreover, showed the equivalence of both CE-properties to the openness of the convex mixing map

$$(x, y, \lambda) \mapsto \lambda x + (1 - \lambda)y$$

(the so-called 'stability property for convex sets' [13]–[15]). The question arises if these results can be extended to noncompact sets $\mathscr{A}$. The first step towards the solution of this problem was made in [9], where so-called μ-compact sets were defined. Some results on CE-properties were generalized from compact sets to the class of μ-compact sets. This, in particular, made it possible to derive several results concerning the entropy characteristics of infinite-dimensional quantum channels and systems.

In this paper we analyse the μ-compactness property in detail, consider several examples that are important in applications, and extend some classical results of convex analysis known earlier for compact sets only, in particular, the Vesterstrom-O'Brien theory, to the class of μ-compact sets.

The class of μ-compact convex sets is defined by the requirement of the weak compactness of the preimages of all compact sets under the barycentre map. This property is not purely topological, it expresses certain relations between topology and the structure of linear operations. This class contains all compact sets, as well as some important noncompact sets, for example, the set of density operators in a separable Hilbert space. These μ-compact sets lack many properties of compact sets, such as the boundedness of continuous functions, the Weierstrass theorem, and so on. Nevertheless, as we shall see, a lot of results of Choquet's theory and of the Vesterstrom-O'Brien theory can be extended to this class. Moreover, we present arguments showing that the class of μ-compact sets is, in some sense, the largest class to which the Vesterstrom-O'Brien theory can be extended.

This paper is organized as follows. In § 2 we derive basic properties of μ-compact sets. By a simple example we show that several results true for μ-compact sets become false after relaxing slightly this assumption to pointwise μ-compactness (this property is defined by the requirement of the weak compactness of sets of measures with fixed barycentre). Further we consider examples of μ-compact sets. We show, in particular, that the bounded part of the positive cone in the space l_p for $p = 1$ is μ-compact, while for $p > 1$ it is not even pointwise μ-compact. The μ-compactness of the set of Borel probability measures on a complete separable metric space is also established. This result makes it possible to show that the convex closure operation respects the μ-compactness property.

In § 3 we complete the generalization of the Vesterstrom-O'Brien theory to the class of μ-compact sets started in [9]. The μ-compact version of the main result from [12] is proved. This establishes the equivalence of the continuity property for convex hulls of concave bounded continuous functions and the continuity property for convex hulls of arbitrary bounded continuous functions. We construct an example confirming our conjecture that μ-compact sets form the largest class of convex metrizable sets for which this extension is possible. In § 4 we apply some of our results to quantum information theory. In § 5 we discuss possible generalizations and formulate several open problems.

§ 2. On μ-compact sets

2.1. Definitions and basic properties. Throughout § 2 and § 3 we assume $\mathscr{A}$ to be a closed bounded subset of a locally convex space. We also assume that its convex closure $\overline{\mathrm{co}}\,\mathscr{A}$ (defined as the closure of the convex hull $\mathrm{co}\,\mathscr{A}$ of $\mathscr{A}$) is a complete separable metric space.[1] We use the following notation:

$\mathrm{extr}\,\mathscr{A}$ is the set of extreme points of $\mathscr{A}$;

$C(\mathscr{A})$ is the set of continuous bounded functions on the set $\mathscr{A}$;

$P(\mathscr{A})$ and $Q(\mathscr{A})$ are the sets of convex and concave continuous bounded functions on the convex set $\mathscr{A}$, respectively;

$\mathrm{co}\, f$ and $\overline{\mathrm{co}}\, f$ are the convex hull and the convex closure of a function f on a convex set; they are defined as the maximal convex and the maximal convex closed (that is, lower semicontinuous) function not exceeding f, respectively (see [3], [16]);

$\mathfrak{P}_n = \bigl\{\{\pi_i\}_{i=1}^n \mid \pi_i \geqslant 0, \sum_{i=1}^n \pi_i = 1\bigr\}$ is the simplex of all probability distributions with $n \leqslant +\infty$ outcomes.

Let $M(\mathscr{A})$ be the set of all Borel probability measures on the set $\mathscr{A}$ with the topology of weak convergence [17], [18].

With an arbitrary measure $\mu \in M(\mathscr{A})$ we associate its barycentre (average) $\mathbf{b}(\mu) \in \overline{\mathrm{co}}\,\mathscr{A}$, which is defined by the Pettis integral (see [17], [19])

$$\mathbf{b}(\mu) = \int_{\mathscr{A}} x\, \mu(dx). \tag{1}$$

[1] This means that the topology on the set $\overline{\mathrm{co}}\,\mathscr{A}$ is defined by a countable subset of the family of seminorms generating the topology of the entire locally convex space, and this set is separable and complete in the metric generated by this subset of seminorms.

Let $M_x(\mathscr{A})$ be the convex closed subset of the set $M(\mathscr{A})$ consisting of measures μ such that $\mathbf{b}(\mu) = x \in \overline{\mathrm{co}}\,\mathscr{A}$.

We denote by $\{\pi_i, x_i\}$ the measure with finitely or countably many atoms $\{x_i\}$ with weights $\{\pi_i\}$. Let $M^{\mathrm{f}}(\mathscr{A})$ and $M^{\mathrm{f}}_x(\mathscr{A})$ be the subsets of the sets $M(\mathscr{A})$ and $M_x(\mathscr{A})$, respectively, that consist of measures with finite support.

The barycentre map

$$M(\mathscr{A}) \ni \mu \mapsto \mathbf{b}(\mu) \in \overline{\mathrm{co}}\,\mathscr{A} \tag{2}$$

is continuous, which can be shown easily by applying Prokhorov's theorem. Therefore, the image of any compact set in $M(\mathscr{A})$ under the map (2) is compact in $\overline{\mathrm{co}}\,\mathscr{A}$. The inverse map $\mathbf{b}^{-1}$ may not possess this property. Generalizing a definition in [9] consider the class of convex sets for which the map $\mathbf{b}^{-1}$ takes compact sets to compact sets.

Definition 1. The set $\mathscr{A}$ is said to be μ-*compact* if the preimage of any compact subset of $\overline{\mathrm{co}}\,\mathscr{A}$ under the barycentre map (2) is a compact subset of the set $M(\mathscr{A})$.

Any compact set is μ-compact. Indeed, the compactness of $\mathscr{A}$ implies the compactness of $M(\mathscr{A})$ [18]. Using Prokhorov's theorem one can derive the following criterion of μ-compactness [9].

Proposition 1. *A convex set $\mathscr{A}$ is μ-compact if and only if for any compact subset $\mathscr{K} \subseteq \mathscr{A}$ and any $\varepsilon > 0$ there is a compact subset $\mathscr{K}_\varepsilon$ of $\mathscr{A}$ such that for any $x \in \mathscr{K}$ and any expansion $x = \sum_{i=1}^n \lambda_i x_i$, where $\{x_i\}_{i=1}^n \subset \mathscr{A}$, $\{\lambda_i\}_{i=1}^n \in \mathfrak{P}_n$, we have $\sum_{i:x_i\in\mathscr{A}\setminus\mathscr{K}_\varepsilon} \lambda_i < \varepsilon$.*

Proposition 1 and the basic properties of the set $M(\mathscr{A})$ yield the following criterion of μ-compactness, which is most convenient for applications.

Proposition 2. *A convex set $\mathscr{A}$ is μ-compact if and only if there is a family $F(\mathscr{A})$ of nonnegative concave functions on $\mathscr{A}$ with the following properties*:

- *the set $\{x \in \mathscr{A} \mid f(x) \leqslant c\}$ is relatively compact for any function $f \in F(\mathscr{A})$ and any $c > 0$;*
- *for any compact set $\mathscr{K} \subseteq \mathscr{A}$ there is a function $f \in F(\mathscr{A})$ such that $\sup_{x\in\mathscr{K}} f(x) < +\infty$.*

Proof. The sufficiency easily follows from Proposition 1 (see [9]). Let us prove the necessity. Let $\mathfrak{V}(\mathscr{A})$ be the set of lower semicontinuous function φ on $\mathscr{A}$ taking values in $[0, +\infty]$ and such that $\{x \in \mathscr{A} \mid \varphi(x) \leqslant c\}$ is compact for any $c \geqslant 0$. From Prokhorov's theorem (see [17], Example 8.6.5) we conclude that a set $M_0 \subseteq M(\mathscr{A})$ is relatively compact if and only if there exists a function $\varphi \in \mathfrak{V}(\mathscr{A})$ such that

$$\sup_{\mu\in M_0} \int_{\mathscr{A}} \varphi(x)\,\mu(dx) < +\infty.$$

Consider the following family of concave nonnegative functions on the set $\mathscr{A}$:

$$f_\varphi(x) = \sup_{\mu\in M_x(\mathscr{A})} \int_{\mathscr{A}} \varphi(y)\,\mu(dy), \qquad \varphi \in \mathfrak{V}(\mathscr{A}).$$

This family possesses the first characteristic property of the family $F(\mathscr{A})$, as follows from the continuity of the barycentre map. The second property follows from the μ-compactness of the set $\mathscr{A}$.

Remark 1. It is interesting that for all convex noncompact, but μ-compact sets considered in § 2.2 there exist families $F(\mathscr{A})$ that consist of affine lower semicontinuous functions.

There exists a criterion of μ-compactness of a convex set in terms of properties of functions defined on this set [20]. More precisely, it is shown that μ-compactness is equivalent to the continuity of the operator of convex closure (that is, the double Fenchel transform) with respect to monotone pointwise converging sequences on the classes of continuous bounded and lower semicontinuous lower bounded functions.

Continuous affine maps do not necessarily respect the μ-compactness property. Nevertheless we have the following simple consequence of Propositions 1 and 2.

Proposition 3. *Let $\mathscr{A}$ and $\mathscr{B}$ be convex sets*[2] *and φ a continuous affine map from $\mathscr{A}$ into $\mathscr{B}$ such that for any compact set $\mathscr{C} \subseteq \mathscr{B}$ its preimage $\varphi^{-1}(\mathscr{C})$ is compact in $\mathscr{A}$. Then*

1) *the μ-compactness of $\mathscr{B}$ implies the μ-compactness of $\mathscr{A}$;*
2) *if φ is surjective, then the μ-compactness of $\mathscr{A}$ implies the μ-compactness of $\mathscr{B}$.*

The operations of intersection, taking the convex closure, and Cartesian product respect μ-compactness.

Proposition 4. 1) *A closed subset of any μ-compact set is μ-compact.*

2) *The convex closure of any μ-compact set is μ-compact.*

3) *The Cartesian product of a finite or countable family of μ-compact sets is μ-compact* (*in the topology of coordinatewise convergence*).

Proof. 1) This follows directly from Definition 1.

2) Combining the μ-compactness of the set $\mathscr{A}$ and Proposition 2 in [9] (its proof for our class of sets is literally the same as in that paper) we obtain that the barycentre map $\mu \mapsto \mathbf{b}(\mu)$ is a continuous affine surjection from $M(\mathscr{A})$ into $\overline{\mathrm{co}}\,\mathscr{A}$ satisfying the assumptions of Proposition 3. Applying the second part of that proposition and Corollary 4 (see the next subsection) we conclude that the set $\overline{\mathrm{co}}\,\mathscr{A}$ is μ-compact.

3) By assertion 2) it suffices to consider the case of convex μ-compact sets. Assume that for every $n \in \mathbb{N}$ the set $\mathscr{A}^n$ is μ-compact. We shall show that the set $\mathscr{A} = \bigotimes_{n\in\mathbb{N}} \mathscr{A}^n$ is μ-compact in the topology of coordinatewise convergence. For an arbitrary compact set $\mathscr{K} \subset \mathscr{A}$ and each $n \in \mathbb{N}$ let $\mathscr{K}^n$ be the projection of $\mathscr{K}$ onto $\mathscr{A}^n$. The set $\mathscr{K}^n$ consists of points $x^n \in \mathscr{A}^n$ which are the corresponding coordinates of some point $x \in \mathscr{K}$. This set is compact. Since $\mathscr{A}^n$ is μ-compact, it follows from Proposition 1 that for any $\varepsilon > 0$ there exists a corresponding compact set $\mathscr{K}^n_\varepsilon \subset \mathscr{A}^n$. Since $\mathscr{K} \subseteq \bigotimes_{n\in\mathbb{N}} \mathscr{K}^n$, we have $\mathscr{K}_\varepsilon = \bigotimes_{n\in\mathbb{N}} \mathscr{K}^n_{\varepsilon 2^{-n}}$. It is easy to check that this set satisfies the assumptions of Proposition 1, therefore the set $\mathscr{A}$ is μ-compact.

[2] It is assumed that the set $\mathscr{B}$ possesses all the properties mentioned in the beginning of § 2.1.

The first assertion of Proposition 4 implies that the intersection of μ-compact sets is μ-compact. However, their union and Minkowski sum are not μ-compact in general (Remark 4).

Remark 2. By the second assertion of Proposition 4, to prove the μ-compactness of a convex set $\mathscr{A}$ it suffices to show the μ-compactness of any subset $\mathscr{B}$ of it such that $\mathscr{A} = \overline{\mathrm{co}}\,\mathscr{B}$. Note that the sets of measures $M(\mathscr{A})$ and $M(\mathscr{B})$ (which are involved in the definition of μ-compactness of the sets $\mathscr{A}$ and $\mathscr{B}$) can be completely different. For example, if $\mathscr{A}$ is a simplex, then $\mathscr{B}$ can be a countable family $\{e_i\}$ of isolated extreme points of the set $\mathscr{A}$. Hence $M(\mathscr{B})$ is isomorphic to the set $\mathfrak{P}_{+\infty}$ of all probability distributions with countably many outcomes. The criterion of μ-compactness for the set $\mathscr{B}$, and therefore, for $\mathscr{A}$, can be formulated as follows: for any compact set $\mathscr{K} \subset \mathscr{A}$ and any $\varepsilon > 0$ there exists n such that the inclusion $\sum_{i=1}^{+\infty} \pi_i e_i \in \mathscr{K}$, $\{\pi_i\} \in \mathfrak{P}_{+\infty}$ implies $\sum_{i=n+1}^{+\infty} \pi_i < \varepsilon$.

The second assertion of Proposition 4 together with Proposition 5 below lead to the following observation. Let $\mathscr{A}$ be a μ-compact convex set in the initial topology τ and let τ' be a stronger topology on $\mathscr{A}$ that coincides with τ on the set $\overline{\mathrm{extr}\,\mathscr{A}}$; then the set $\mathscr{A}$ is μ-compact in the topology τ'.

Proposition 3 implies that all isomorphisms in the category of μ-compact convex sets are affine homeomorphisms. The affineness assumption cannot be omitted, as is seen from the following example. Suppose that $\mathscr{A}$ is a convex set that is not μ-compact. The first assertion of Proposition 4 and Corollary 4 from the next subsection yield that the subset of $M(\mathscr{A})$ that consists of Dirac (single-atomic) measures is a μ-compact set, which is, moreover, homeomorphic to the set $\mathscr{A}$. This observation shows that the μ-compactness property, by contrast to compactness, is not purely topological. It is defined by a combination of the topology and the structure of the operation of convex mixing.

Proposition 3 gives the following condition for the μ-compactness of families of maps. This condition is used in the next section.

Corollary 1. *Let $\mathfrak{F}(\mathscr{X},\mathscr{Y})$ be a locally convex space with the topology τ of maps from the set $\mathscr{X}$ into a locally convex space $\mathscr{Y}$. Also let $\mathfrak{F}_0$ be a convex closed bounded subset of the space $\mathfrak{F}(\mathscr{X},\mathscr{Y})$ that consists of maps taking values in a convex μ-compact set $\mathscr{A} \subset \mathscr{Y}$. Moreover, assume that $\mathfrak{F}_0$ is a complete separable metric space for which there is an element $x_0 \in \mathscr{X}$ such that*

1) $\{\tau - \lim_{n\to+\infty} \Phi_n = \Phi_0\} \Longrightarrow \{\lim_{n\to+\infty} \Phi_n(x_0) = \Phi_0(x_0)\} \quad \forall\,\{\Phi_n\} \subset \mathfrak{F}_0$;
2) *the set $\{\Phi \in \mathfrak{F}_0 \mid \Phi(x_0) \in \mathscr{C}\}$ is relatively compact in the topology τ for any compact set $\mathscr{C} \subseteq \mathscr{A}$.*

Then the set $\mathfrak{F}_0$ is μ-compact.

Proof. The continuous affine map

$$\mathfrak{F}_0 \ni \Phi \mapsto \Phi(x_0) \in \mathscr{A}$$

satisfies the assumptions of Proposition 3. The first part of this proposition implies the μ-compactness of the set $\mathfrak{F}_0$.

For a further analysis of μ-compactness we need a weaker version of this property.

Definition 2. The set $\mathscr{A}$ is said to be *pointwise μ-compact* if for any $x \in \overline{\mathrm{co}}\,\mathscr{A}$ the set $M_x(\mathscr{A})$ is a compact subset of the set $M(\mathscr{A})$.

Clearly, pointwise μ-compactness follows from μ-compactness. However, as we shall see in Proposition 13, these two properties are not equivalent.

μ-Compact sets do not possess many properties of compact sets, such as the uniform continuity and boundedness of continuous functions, the Weierstrass theorem on extremal values of continuous functions, and so on. It turns out, however, that μ-compact sets inherit some important properties of compact sets. This allows us to extend many results of Choquet's theory and the Vesterstrom-O'Brien theory to these sets (see the next section).

If $\mathscr{A}$ is a convex set, then we can introduce on the set $M(\mathscr{A})$ the following partial ordering (called the Choquet ordering; see [1], [6]). We shall write $\mu \succ \nu$ if and only if

$$\int_{\mathscr{A}} f(y)\,\mu(dy) \geqslant \int_{\mathscr{A}} f(y)\,\nu(dy)$$

for each function $f \in P(\mathscr{A})$. A measure $\mu \in M(\mathscr{A})$ is called *maximal* if $\nu \succ \mu$ implies $\nu = \mu$ for any $\nu \in M(\mathscr{A})$. We observe that if μ and ν are measures in $M(\mathscr{A})$ such that $\mu \succ \nu$, then $\mathbf{b}(\mu) = \mathbf{b}(\nu)$: this follows from the fact that the set of continuous bounded affine functions on the set $\mathscr{A}$ separates its points. The following result is a straightforward consequence of Definition 2.

Lemma 1. *Let $\mathscr{A}$ be a convex pointwise μ-compact set. Any subset of the set $M(\mathscr{A})$ that is linearly ordered by the relation $\prec$ has the least upper bound.*

Proof. Any subset of the set $M(\mathscr{A})$ that is linearly ordered by the relation $\prec$, which is actually a net $\{\mu_\lambda\}_{\lambda\in\Lambda}$, lies in $M_x(\mathscr{A})$ for some $x \in \mathscr{A}$ and consequently, is relatively compact. Hence there is a subnet $\{\mu_{\lambda_\pi}\}_{\pi\in\Pi}$ that converges to some measure $\mu_0 \in M_x(\mathscr{A})$. One can easily verify that $\mu_\lambda \prec \mu_0$ for all $\lambda \in \Lambda$.

Combining Lemma 1 with Theorem 2.4 in [5] we see that the class of convex pointwise μ-compact sets is a proper subclass of the class of convex sets with the Radon-Nikodym property. This class is studied in an extensive literature (see [4]–[6]).

Applying Lemma 1 and Zorn's lemma one can easily prove the following generalizations of the Krein-Milman theorem and of Choquet's theorem to pointwise μ-compact sets. This follows actually from the Radon-Nikodym property of these sets [5].

Proposition 5. *Let $\mathscr{A}$ be a pointwise μ-compact convex set. Then $\overline{\mathrm{co}}(\mathrm{extr}\,\mathscr{A}) = \mathscr{A}$ and $\mathbf{b}(M(\overline{\mathrm{extr}\,\mathscr{A}})) = \mathscr{A}$.*

Proof. Since the set of finitely supported measures is dense in $M(\overline{\mathrm{extr}\,\mathscr{A}})$ (see [18], Theorem 6.3), the first assertion follows from the second.

Let $x_0 \in \mathscr{A}$. Lemma 1 combined with Zorn's lemma yields the existence of a maximal measure μ_* in M_{x_0}. From the properties of the Choquet ordering it follows that the measure μ_* is also maximal in $M(\mathscr{A})$; hence it lies in $M(\overline{\mathrm{extr}\,\mathscr{A}})$ (see the proof of Theorem 5.2 in [4]). Thus, $x_0 \in \mathbf{b}(M(\overline{\mathrm{extr}\,\mathscr{A}}))$.

Another property inherited by μ-compact sets from compact ones is the following representation of the convex closure of a lower semicontinuous function ([9], Proposition 3). Its proof is easily generalized to our class of sets.

Proposition 6. *If f is a lower semicontinuous lower bounded function on a convex μ-compact set $\mathscr{A}$, then its convex closure can be described by the expression*

$$\overline{\mathrm{co}}\, f(x) = \inf_{\mu \in M_x(\mathscr{A})} \int_{\mathscr{A}} f(y)\,\mu(dy) \qquad \forall\, x \in \mathscr{A}, \tag{3}$$

where the infimum is attained on some measure μ_x^f in $M_x(\mathscr{A})$.

This representation is a crucial point for most results on the convex closures of functions defined on μ-compact sets.

If f is a continuous bounded function on a convex μ-compact set $\mathscr{A}$, then combining the continuity of the functional

$$M(\mathscr{A}) \ni \mu \mapsto \int_{\mathscr{A}} f(x)\,\mu(dx)$$

and [9], Lemma 1 we see that the infimum in (3) can be taken over finitely-supported measures.

Corollary 2. *An arbitrary continuous bounded function f on a convex μ-compact set $\mathscr{A}$ possesses a lower semicontinuous (closed) convex hull, that is,*

$$\mathrm{co}\, f(x) = \inf_{\{\pi_i, x_i\} \in M_x^{\mathrm{f}}(\mathscr{A})} \sum_i \pi_i f(x_i) = \overline{\mathrm{co}}\, f(x) \qquad \forall\, x \in \mathscr{A}. \tag{4}$$

This is a μ-compact generalization of Corollary I.3.6 in [2]. The μ-compactness assumption about the set $\mathscr{A}$ in Proposition 6 and Corollary 2 cannot be weakened to pointwise μ-compactness.

Proposition 7. *If a convex set $\mathscr{A}$ is not μ-compact, but only pointwise μ-compact, then there exists a continuous bounded function f on $\mathscr{A}$ whose convex hull $\mathrm{co}\, f$ is not lower semicontinuous. This means that the representation (3) fails for the convex closure $\overline{\mathrm{co}}\, f$ of the function f.*

Proof. Since the set $\mathscr{A}$ is not μ-compact, there exists a non-compact sequence of measures $\{\mu_n\} \subset M(\mathscr{A})$ such that the corresponding sequence $\{x_n = \mathbf{b}(\mu_n)\} \subset \mathscr{A}$ converges. By Prokhorov's theorem the sequence $\{\mu_n\}$ is not tight. As shown in the proof of Theorem 8.6.2 in [17], this guarantees the existence of $\varepsilon > 0$ and $\delta > 0$ such that for any compact set $\mathscr{K} \subset \mathscr{A}$ and any positive integer N there is $n > N$ such that $\mu_n(U_\delta(\mathscr{K})) < 1 - \varepsilon$, where $U_\delta(\mathscr{K})$ is the closed δ-neighbourhood of the compact set $\mathscr{K}$. Since finitely supported measures are dense in the set of all measures with fixed barycentre ([9], Lemma 1), applying Corollary 8.2.9 from [17] we may assume that the sequence $\{\mu_n\}$ contains only finitely supported measures.

Let x_0 be the limit of the sequence $\{x_n\}$. By the pointwise μ-compactness of the set $\mathscr{A}$, for ε defined above there exists a convex compact set $\mathscr{K}_\varepsilon$ such that $\mu(\mathscr{K}_\varepsilon) \geqslant 1 - \varepsilon/2$ for any measure $\mu \in M_{x_0}(\mathscr{A})$.

Let f be a bounded continuous function on $\mathscr{A}$, such that $f(x) = 1$ for all $x \in \mathscr{K}_\varepsilon$ and $f(x) \leqslant 0$ for every $x \in \mathscr{A} \setminus U_\delta(\mathscr{K}_\varepsilon)$. Then it follows from the properties of the sequences $\{x_n\}$ and $\{\mu_n\}$ that for any positive integer N there is $n > N$ for which $\operatorname{co} f(x_n) < 1 - \varepsilon$. On the other hand, the properties of the set $M_{x_0}(\mathscr{A})$ imply that $\operatorname{co} f(x_0) \geqslant 1 - \varepsilon/2$. Thus, the function $\operatorname{co} f$ is not lower semicontinuous.

Remark 3. The condition of pointwise μ-compactness cannot be omitted in the proof of Proposition 7. This shows that μ-compact sets do not form the maximal class of convex sets for which Corollary 2 holds. However, without pointwise μ-compactness Corollary 2 may fail (see Example 1 in the next subsection).

To construct an example showing that the condition of pointwise μ-compactness is essential in Proposition 7 we consider the unit ball $\mathscr{B}$ in a separable Hilbert space. Clearly, it is not a pointwise μ-compact set. We claim that for any continuous bounded function f on $\mathscr{B}$ its convex hull $\operatorname{co} f$ is continuous. Since a bounded convex function is continuous at interior points of its domain, it is sufficient to prove the continuity of $\operatorname{co} f$ at the boundary of $\mathscr{B}$, that is, at the unit sphere.

Let x be a point of the unit sphere. For arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $\|x - y\| < 2\delta$. From Lemma 2 stated below we conclude that for any z such that $\|z - x\| < \delta$ and any $\mu \in M_z(\mathscr{B})$ the ball of radius 2δ with centre at x has measure at least $r(\delta, z)$. Hence

$$\left| f(x) - \int_{\mathscr{B}} f(t)\, d\mu \right|$$

does not exceed $\varepsilon r(\delta, z) + N(1 - r(\delta, z))$, where $N = \sup_{t \in \mathscr{B}} |f(t)|$. Therefore,

$$|f(x) - \operatorname{co} f(z)| \leqslant \varepsilon r(\delta, z) + N(1 - r(\delta, z)).$$

Since $\|z\| \to 1$ as $z \to x$, it follows that $r(\delta, z) \to 1$, and so $\operatorname{co} f(z) \to f(x)$. It remains to note that $\operatorname{co} f(x) = f(x)$ because x is an extreme point of $\mathscr{B}$.

Lemma 2. *Let $\mathscr{B}$ be the unit ball of the Hilbert space and let $\delta > 0$. For an arbitrary point $z \in \mathscr{B}$ such that $\|z\| > 1 - \delta$ and any measure $\mu \in M_z(\mathscr{B})$ the measure of a ball of radius δ with centre at z is at least*

$$r(\delta, z) = \frac{\delta^2 - (1 - \|z\|^2)}{\delta^2 - (1 - \|z\|)^2}.$$

Proof. Let $\bar{z} = z/\|z\|$ and

$$\mathscr{B}_0 = \left\{ y \in \mathscr{B},\ (y, \bar{z}) < \frac{1 + \|z\|^2 - \delta^2}{2\|z\|} \right\},$$
$$\mathscr{B}_1 = \left\{ y \in \mathscr{B},\ (y, \bar{z}) \geqslant \frac{1 + \|z\|^2 - \delta^2}{2\|z\|} \right\}.$$

By direct calculation we show that if $\|y\| = 1$ and $\|y - z\| > \delta$, then $y \in \mathscr{B}_0$. Consequently, all points in the ball $\mathscr{B}$ lying at a distance more than δ from z belong to the set $\mathscr{B}_0$. We denote by c_i the barycentre of the measure μ on the set $\mathscr{B}_i$, $i = 0, 1$.

We have $z=\mu(\mathscr{B}_0)c_0+\mu(\mathscr{B}_1)c_1$. It is obvious that $(c_0,\bar{z})\leqslant(1+\|z\|^2-\delta^2)/(2\|z\|)$ and $(c_1,\bar{z})\leqslant 1$. Consequently,

$$\|z\|=(z,\bar{z})<(1-\mu(\mathscr{B}_1))\frac{1+\|z\|^2-\delta^2}{2\|z\|}+\mu(\mathscr{B}_1),$$

which gives the required inequality for $\mathscr{B}_1$ and therefore also for the measure of the ball of radius δ with centre at z since this ball contains $\mathscr{B}_1$.

One of the most important conclusions from Proposition 6 is that any lower semicontinuous function f bounded below defined on a convex μ-compact set $\mathscr{A}$ coincides with its convex closure $\overline{\mathrm{co}}\, f$ on the set of extreme points $\operatorname{extr}\mathscr{A}$. Furthermore, this proposition enables us to obtain the following representation for the set $\operatorname{extr}\mathscr{A}$, which will help us in § 3.

Proposition 8. *Let $\mathscr{A}$ be a μ-compact convex set. Then*

$$\operatorname{extr}\mathscr{A}=\bigcap_{f\in Q(\mathscr{A})}\mathscr{B}_f, \qquad \text{where } \mathscr{B}_f=\{x\in\mathscr{A}\mid f(x)=\overline{\mathrm{co}}\, f(x)\}. \tag{5}$$

Proof. The inclusion $\operatorname{extr}\mathscr{A}\subseteq\mathscr{B}_f$ for any function $f\in Q(\mathscr{A})$ follows from (4). Suppose $x_0\in\mathscr{A}\setminus\operatorname{extr}\mathscr{A}$. Then there are two distinct points x_1 and x_2 in $\mathscr{A}$ such that $x_0=\frac{1}{2}x_1+\frac{1}{2}x_2$. To prove that x_0 is not in $\bigcap_{f\in Q(\mathscr{A})}\mathscr{B}_f$ we can find a function f in $Q(\mathscr{A})$ such that $f(x_0)>\frac{1}{2}f(x_1)+\frac{1}{2}f(x_2)$. The function $-a^2(\,\cdot\,)$ will do, where a is an affine continuous bounded function on $\mathscr{A}$ such that $a(x_1)\neq a(x_2)$.

Example 1 in the next subsection shows the importance of the assumption of μ-compactness in Proposition 8.

Arguing as in the proof of Proposition 7 one can easily produce the following necessary condition for the representation (5), the condition of local μ-compactness in a neighbourhood of the set $\operatorname{extr}\mathscr{A}$: for any extreme point x_0 of the set $\mathscr{A}$ and any sequence $\{x_n\}\subset\mathscr{A}$ converging to x_0 the set $\mathbf{b}^{-1}(\{x_n\})$ is compact in $M(\mathscr{A})$.

2.2. Examples. Any compact set is obviously μ-compact. In this section we consider several most important examples of μ-compact, but not compact sets.

Proposition 9. *The bounded part[3] of the positive cone in the space l_1 is μ-compact.*

Proof. It is sufficient to take the family of functions of the form $l_1\ni\{x_i\}\mapsto\sum_i h_ix_i$, where $\{h_i\}$ is an increasing unbounded sequence of positive numbers, and then to apply Proposition 2 taking into account the compactness criterion for subsets of the space l_1.

Corollary 3. *The set $\mathfrak{P}_{+\infty}$ of probability distributions with countably many outcomes is a μ-compact subset of the space l_1.*

Remark 4. Let $\mathscr{A}_1=\{x\in l_1\mid x\geqslant 0,\ \|x\|_1\leqslant 1\}$. Proposition 9 implies that both $\mathscr{A}_1$ and $-\mathscr{A}_1$ are μ-compact in the metric l_1. However, neither their convex

[3] Here and in what follows by the bounded part of a positive cone in an ordered Banach space we mean the intersection of this cone with a unit ball.

hull nor their Minkowski sum is μ-compact. These are actually not even pointwise μ-compact because they contain a unit ball of the space l_1, which is not pointwise μ-compact (this is easy to show).

Proposition 10. *An arbitrary weakly closed bounded (in variation) set of Borel measures on a complete separable metric space is μ-compact in the weak-convergence topology.*

Proof. Let X be a complete separable metric space, M a weakly closed bounded set of Borel measures on the space X and $\mathfrak{V}(X)$ the set of all lower semicontinuous functions φ on X taking values in $[0,+\infty]$ such that the set $\{x \in X \mid \varphi(x) \leqslant c\}$ is compact for all $c \geqslant 0$. Prokhorov's theorem (see [17], Example 8.6.5) implies that the set $M_0 \subseteq M$ is relatively compact if and only if there exists a function $\varphi \in \mathfrak{V}(X)$ for which

$$\sup_{\mu \in M_0} \int_X \varphi(x)\, \mu(dx) < +\infty.$$

Hence the family of affine lower semicontinuous functions

$$f_\varphi(\mu) = \int_X \varphi(x)\, \mu(dx), \qquad \varphi \in \mathfrak{V}(X),$$

on the set M satisfies all the assumptions of Proposition 2.

Corollary 4. *The set of all Borel probability measures on a complete separable metric space is μ-compact in the topology of weak convergence.*

Using Propositions 3 and 9 as well as Proposition 15 stated below one can prove the following result on the properties of the cone of positive operators in the Shatten class of order p, that is, in the Banach space of all operators acting in a separable Hilbert space $\mathscr{H}$ such that $\operatorname{Tr}|A|^p < +\infty$, with the norm $\|A\|_p = (\operatorname{Tr}|A|^p)^{1/p}$.

Proposition 11. *The bounded part of the positive cone in the Shatten class of order p is μ-compact precisely for $p = 1$.*

Proof. In the case $p = 1$, applying the compactness criterion for subsets of the positive cone $\mathfrak{T}_+(\mathscr{H})$ of the space of trace class operators $\mathfrak{T}(\mathscr{H})$ ([21], Appendix) we obtain that the map taking an operator $A \in \mathfrak{T}_+(\mathscr{H})$ to the sequence of its diagonal elements (in some fixed basis of the space $\mathscr{H}$) satisfies the assumptions of the first part of Proposition 3 (with the bounded parts of the positive cones of the spaces $\mathfrak{T}(\mathscr{H})$ and l_1 taking the roles of $\mathscr{A}$ and $\mathscr{B}$, respectively). Combining that proposition with Proposition 9 we obtain the μ-compactness of the bounded part of the cone $\mathfrak{T}_+(\mathscr{H})$.

The cone of positive operators in the Shatten class of order $p > 1$ contains a subcone of commuting operators. That subcone is affinely homeomorphic to $\mathscr{A}_p$ (the bounded part of the positive cone in the space l_p), which is not μ-compact (Proposition 15).

Proposition 11 yields, in particular, the μ-compactness of the set $\mathfrak{S}(\mathscr{H})$ of quantum states, density operators in a separable Hilbert space $\mathscr{H}$. This fact was originally proved in [22]. Corollary 1 and Lemma 5 from the appendix imply the following result.

Proposition 12. *Let $\mathfrak{L}_+(\mathscr{H},\mathscr{H}')$ be the cone of positive linear continuous maps from the Banach space $\mathfrak{T}(\mathscr{H})$ of trace class operators in a separable Hilbert space $\mathscr{H}$ into a similar space $\mathfrak{T}(\mathscr{H}')$. Then the bounded* (*in operator norm*) *part of this cone is μ-compact in the strong operator topology.*

Proposition 12 implies the μ-compactness of the set of quantum operations and of quantum channels in the topology of strong convergence [21].

Proposition 12 also yields the μ-compactness (in the strong operator topology) of the bounded part of the cone of linear continuous positive operators in l_1. Indeed, this cone is naturally identified with a subset of the cone $\mathfrak{L}_+(\mathscr{H},\mathscr{H}')$. This shows the μ-compactness of the set of sub-Markov operators, in particular, Markov operators.

Now consider some 'negative' examples. Proposition 13 gives an example of pointwise μ-compact sets (Definition 2) that are not μ-compact. We are going to see that in a Hilbert space there are no μ-compact sets that are not compact (Proposition 14). Then we give several examples of sets that are not even pointwise μ-compact.

By definition the pointwise μ-compactness property of a set survives weakening the topology. The next proposition shows that μ-compactness does not possess this property. The bounded part of the positive cone in l_1 loses μ-compactness after weakening the topology.

Proposition 13. *For any $p>1$ the simplex*

$$\Delta_p=\biggl\{x\in l_p\biggm| x\geqslant 0,\ \sum_{i=1}^{+\infty}x^i\leqslant 1\biggr\}$$

in the space l_p is pointwise μ-compact, but not μ-compact.

Proof. The pointwise μ-compactness of the set Δ_p follows from the remark above. Let us show that Δ_p is not μ-compact in the space l_p. Consider an increasing sequence of natural numbers $\{n_r\}_{r\in\mathbb{N}}$ and a sequence of nonnegative numbers $\{z_i\}_{i\in\mathbb{N}}$ such that $\sum_{i=n_r}^{n_{r+1}-1}z_i=1$ and $\sum_{i=n_r}^{n_{r+1}-1}(z_i)^p\leqslant 1/r$ for each $r\geqslant 1$. The set $\mathscr{K}=\bigl\{y\in\Delta_p \mid \forall\, r\in\mathbb{N}\ \sum_{i=n_r}^{+\infty}(y^i)^p\leqslant 1/r\bigr\}$ is compact. If the set Δ_p is μ-compact, then by Proposition 1 for any $\varepsilon>0$ there is the corresponding compact set $\mathscr{K}_\varepsilon$. The set $\mathscr{K}_\varepsilon$ can contain only a finite number of vectors of the canonical basis $\{e_i\}$, otherwise it is not compact. Let N be such that $e_i\notin\mathscr{K}_\varepsilon$ for $i>N$. We take an arbitrary r for which $n_r>N$ and denote by $x\in\Delta_p$ the vector with coordinates $x^i=z^i$ for $n_r\leqslant i<n_{r+1}$ and $x^i=0$ for all other i. Clearly, $x\in\mathscr{K}$. Since $x=\sum_{i=n_r}^{n_{r+1}-1}x^ie_i$ and $\sum_{i=n_r}^{n_{r+1}-1}x^i=1$, but $e_i\notin\mathscr{K}_\varepsilon$ for all $i=n_r,\dots,n_{r+1}-1$, we arrive at a contradiction with the μ-compactness criterion from Proposition 1.

In particular, Proposition 13 shows that a Hilbert space contains noncompact pointwise μ-compact sets. It appears, however, that it does not contain μ-compact sets that are not compact.

Proposition 14. *There are no μ-compact subsets of a Hilbert space that are not compact.*

Proof. We describe the idea of the proof omitting details, which can be easily reconstructed by the reader. Let $\mathscr{A}$ be a bounded convex closed subset of a Hilbert space. Without loss of generality it can be assumed that its diameter is 1. If $\mathscr{A}$ is not compact, then there exists a sequence of its elements $\{a_k\}_{k\in\mathbb{N}}$ such that all their norms and all pairwise distances between them exceed some value $\varepsilon > 0$. Because of the weak compactness of $\mathscr{A}$ it can be assumed, maybe after passing to a subsequence, that the sequence $\{a_k\}$ converges weakly to some element a. Since the set $\mathscr{A}$ is convex and closed, we have $a \in \mathscr{A}$. The sequence $\{b_k = a_k - a\}_{k\in\mathbb{N}}$ converges weakly to zero. Hence by passing to subsequences one can achieve a rapid convergence to zero for the scalar products: $(b_i, b_j) \leqslant \varepsilon^2 2^{-i-j}$ for every $i \neq j$. Since $\varepsilon \leqslant \|b_i\|_2 \leqslant 1$ for all $i \in \mathbb{N}$, invoking the Cauchy-Schwarz inequality we conclude that any sequence $x \in l_2$ satisfies the inequalities

$$\frac{\varepsilon}{2}\,\|x\|_2 \leqslant \Big\|\sum_i x^i b_i\Big\|_2 \leqslant 2\|x\|_2.$$

This means that the system of elements $\{b_k\}$ possesses the Riesz basis property. Then there is a continuous linear operator that is continuously invertible and takes the system $\{b_k\}$ to an orthonormal system [23]. This operator maps the convex hull of the points a and $\{a_k\}_{k\in\mathbb{N}}$ onto the set Δ_2 from Proposition 3, which is not μ-compact. Hence this convex hull is not μ-compact, therefore nor is the set $\mathscr{A}$.

Now we consider two important examples of not pointwise μ-compact sets.

Proposition 15. *For any $p > 1$ the set $\mathscr{A}_p = \{x \in l_p \mid x \geqslant 0,\ \|x\|_p \leqslant 1\}$ (the bounded part of the positive cone in the space l_p) is not pointwise μ-compact.*

Proof. We shall show that if $x \in \mathscr{A}_p$ is such that $\|x\|_p < 1/3$ and $\sum_i x_i = +\infty$, then it has no compact set $\mathscr{K}_\varepsilon$ for $\varepsilon = 1/3$ (see Proposition 1). If such a compact set exists, then it can contain only finitely many elements of the canonical basis $\{e_i\}$. Take a sufficiently large N such that $e_i \notin \mathscr{K}_\varepsilon$ for every $i > N$. Since the series $\sum_i x_i$ diverges, we see that there exists r for which $s = \sum_{i=N+1}^{N+r} x^i \in (1/3, 2/3)$. Let $\bar{x} = x - \sum_{i=N+1}^{N+r} x^i e_i$ (the components of the vector $\bar{x}$ from the $(N+1)$st to the $(N+r)$th are zeros, and the others coincide with the corresponding components of x). Then

$$x = (1-s)\left(\frac{1}{1-s}\,\bar{x}\right) + \sum_{i=N+1}^{N+r} x^i e_i.$$

Since $1/(1-s) < 3$ and $\|\bar{x}\|_p < 1/3$, it follows that $\bar{x}/(1-s) \in \mathscr{A}_p$. All the points e_i in this barycentric combination lie outside $\mathscr{K}_\varepsilon$, but their total weight s exceeds $1/3$, which is a contradiction.

The next example complements Proposition 7 and demonstrates that sets that are not pointwise μ-compact do not necessarily satisfy relations (3) and (4) for continuous bounded concave functions. This violates representation (5).

Example 1. Let f be a continuous function on the bounded part $\mathscr{A}_p$ of the positive cone of the set l_p for $p > 1$ that takes value 1 at zero and vanishes at all the vectors of the canonical basis $\{e_n\}$ of the space l_p. For example, the function $f(\,\cdot\,) = 1 - \|\cdot\|_p$

will do. Since the zero vector of the space l_p is a limit point of the set of all convex combinations of the vectors $\{e_n\}$, the convex closure of the function f is identically zero on the set $\mathscr{A}_p$. Therefore, $\overline{\mathrm{co}}\, f(0) \neq f(0)$. This example shows also that for the set $\mathscr{A}_p$ such that $\mathscr{A}_p = \overline{\mathrm{co}}(\mathrm{extr}\, \mathscr{A}_p)$ and the map (2) is open (as follows from a theorem in [15] in combination with the strict convexity of the space l_p for $p > 1$ and the proof of Theorem 1 in [9]) the assertion of Corollary 2 in [9] does not hold, that is, a continuous function on the closed set $\mathrm{extr}\, \mathscr{A}_p$ need not have a convex continuous (or even lower semicontinuous) extension to $\mathscr{A}_p$.

A *Hilbert cube* is the set

$$\mathscr{H}_a = \{x \in l_2 \mid |x^i| \leqslant a^i,\ i \in \mathbb{N}\},$$

where $a = \{a^i\}_{i\in\mathbb{N}}$ is an arbitrary sequence of positive numbers. We claim that the following alternative holds for any Hilbert cube.

Proposition 16. *If* $\|a\|_2 < \infty$, *then the set* $\mathscr{H}_a$ *is compact*; *otherwise, if* $\|a\|_2 = +\infty$, *then it is not even pointwise* μ*-compact.*

Proof. The first assertion is well known. It follows, for instance, from the compactness criterion for l_2. If $\|a\|_2 = +\infty$, then we split the sequence a into blocks so that each block contains a finite number of consecutive elements the sum of whose squares exceeds 1. Suppose that the nth block consists of elements a^k, $a^{k+1}, \dots, a^{k+m}$. We write

$$b_n = \sum_{i=k}^{k+m} a^i e_i$$

for every n and we denote by $\mathscr{L}$ the closure of the linear span of the elements b_n, $n \in \mathbb{N}$. The set $\mathscr{L}$ is a Hilbert space with orthogonal basis $\{b_n\}$. The unit ball of the space $\mathscr{L}$ lies in $\mathscr{H}_a$. Since a ball in the Hilbert space is not pointwise μ-compact, it follows that the set $\mathscr{H}_a$ is not so either.

§ 3. The CE-properties of μ-compact convex sets

In the 1970s various properties of convex compact sets, in particular, the continuity properties of the convex hulls of continuous functions were intensively studied. Vesterstrom proved in [10] a relation between the continuity of the convex hull[4] of an arbitrary continuous function and the openness of the barycentre map. He also conjectured the equivalence between the continuity of the convex hull of any continuous concave function (called the CE-*property* in [11]) and the continuity of the convex hull of any continuous function (called the *strong* CE-*property* in [9]). This conjecture was proved by O'Brien [12], who showed the equivalence of these properties to the openness of the convex mixture map $(x, y, \lambda) \mapsto \lambda x + (1-\lambda)y$. In many subsequent papers the latter property was studied for convex sets that are not necessarily compact, and was called the *stability* property. Convex sets having this property were called *stable* convex sets [13]. Relations between the stability property and several other properties of convex sets were also revealed [14], [15].

[4]By Corollary I.3.6 in [2] the convex hull of any continuous function on a compact set coincides with the convex closure of this function.

In this section we generalize the Vesterstrom-O'Brien theory to the class of μ-compact convex sets. The first partial result in this direction was obtained in [9], where the μ-compact version of Theorem 3.1 in [10] was proved. The following theorem is the μ-compact generalization of the main result in [12].

Theorem 1. *For a convex μ-compact set $\mathscr{A}$ the following properties are equivalent*:

(i) *the map* $\mathscr{A}\times\mathscr{A}\ni(x,y)\mapsto\frac{x+y}{2}\in\mathscr{A}$ *is open* (*the stability property* [13]);

(ii) *the map* $M(\mathscr{A})\ni\mu\mapsto\mathbf{b}(\mu)\in\mathscr{A}$ *is open*;

(iii) *the map* $M(\overline{\operatorname{extr}\mathscr{A}})\ni\mu\ \mapsto\mathbf{b}(\mu)\in\mathscr{A}$ *is open*;[5]

(iv) *the convex hull of an arbitrary function in* $C(\mathscr{A})$ *is continuous* (*the strong* CE-*property* [9]);

(v) *the convex hull of an arbitrary function in* $Q(\mathscr{A})$ *is continuous* (*the* CE-*property* [11]).

The equivalent properties (i)–(v) *imply the closedness of the set* $\operatorname{extr}\mathscr{A}$.

Remark 5. Property (i) in Theorem 1 is equivalent to the openness of the map $\mathscr{A}\times\mathscr{A}\times[0,1]\ni(x,y,\lambda)\mapsto\lambda x+(1-\lambda)y\in\mathscr{A}$ [14]. Properties (iv) and (v) in Theorem 1 can be formulated as the continuity of the convex closure and its coincidence with the convex hull for any function in $C(\mathscr{A})$ and in $Q(\mathscr{A})$, respectively.

Remark 6. If properties (i)–(v) hold for a convex μ-compact set $\mathscr{A}$, then the family $F(\mathscr{A})$ in Proposition 2 can be chosen consisting of lower semicontinuous functions. Indeed, using property (ii) it is easy to show that the functions f_φ constructed in the proof of Proposition 2 are lower semicontinuous.

Proof of Theorem 1. Note first that by Proposition 8, (v) implies the closedness of the set $\operatorname{extr}\mathscr{A}$ since (v) guarantees the closedness of the set

$$\mathscr{B}_f=\{x\in\mathscr{A}\mid f(x)=\overline{\operatorname{co}}\,f(x)\}$$

for any function $f\in Q(\mathscr{A})$.

(v) $\Longrightarrow$ (iii). The proof of this part of the theorem (as well as the proof of the analogous part of Theorem 3.2 in [10]) can be carried out by means of Lemma 2.1 in [10]. That lemma can be proved without the compactness assumption in view of the following observation: if X is a compact space and Y is an arbitrary topological space then the image of any closed subset of $X\times Y$ under the canonical projection $X\times Y\ni(x,y)\mapsto y\in Y$ is a closed subset of Y.

By representation (3) the convex closure of any function f in $Q(\mathscr{A})$ is determined by the expression

$$\overline{\operatorname{co}}\,f(x)=\inf_{\mu\in M_x(\overline{\operatorname{extr}\mathscr{A}})}\mu(f)\quad\forall\,x\in\mathscr{A},\qquad\text{where }\mu(f)=\int_{\overline{\operatorname{extr}\mathscr{A}}}f(x)\,\mu(dx).$$

Hence (v) yields the continuity and boundedness of the function

$$\mathscr{A}\ni x\mapsto\sup\{\mu(f)\mid\mu\in M(\overline{\operatorname{extr}\mathscr{A}}),\ b(\mu)=x\}$$

[5] This map is surjective by Proposition 5.

for any f in $P(\mathscr{A})$. The above-mentioned generalization of Lemma 2.1 in [10] with $K = M(\mathscr{A})$, $M = M(\operatorname{extr}\mathscr{A})$ and $K' = \mathscr{A}$ implies the openness of the map

$$M(\operatorname{extr}\mathscr{A}) \ni \mu \mapsto \mathbf{b}(\mu) \in \mathscr{A}, \tag{6}$$

provided that the set $M(\operatorname{extr}\mathscr{A})$ is endowed with the topology that has a subbasis consisting of the sets $\{\mu \in M(\operatorname{extr}\mathscr{A}) \mid \mu(f) > 0\}$, $f \in P(\mathscr{A})$. Following the terminology in [10], this will be called the *p-topology*. This is the weakest topology providing the lower semicontinuity of the functionals $\mu \mapsto \mu(f)$ for any function $f \in P(\mathscr{A})$.

Using Lemma 6 (see § 6) we shall show that the openness of the map (6) in the p-topology on $M(\operatorname{extr}\mathscr{A})$ and the closedness of the set $\operatorname{extr}\mathscr{A}$ proved above imply the openness of the map (6) in the weak topology on $M(\operatorname{extr}\mathscr{A})$.[6] For this it suffices to show that for an arbitrary converging sequence $\{x_n\} \subset \mathscr{A}$ and an arbitrary net $\{\mu_\lambda\}_{\lambda\in\Lambda} \subset M(\operatorname{extr}\mathscr{A})$ such that

$$b(\{\mu_\lambda\}_{\lambda\in\Lambda}) \subseteq \{x_n\}, \qquad \exists\, p\text{-}\lim_{\lambda} \mu_\lambda = \mu_0,$$

where μ_0 is a measure in $M(\operatorname{extr}\mathscr{A})$ such that $\mathbf{b}(\mu_0) = \lim_{n\to+\infty} x_n$, there exists a subnet of the net $\{\mu_\lambda\}_{\lambda\in\Lambda}$ that converges weakly to the measure μ_0.

Let $\{x_n\}$ and $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be the above sequence and net, respectively. Since the sequence is relatively compact, the μ-compactness of the set $\mathscr{A}$ and the inclusions $\mathbf{b}(\{\mu_\lambda\}_{\lambda\in\Lambda}) \subseteq \{x_n\}$ imply the relative compactness of the net $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in the weak topology and hence the existence of a subnet $\{\mu_{\lambda_\pi}\}_{\pi\in\Pi}$ weakly converging to some measure $\nu \in M(\operatorname{extr}\mathscr{A})$. By the definitions of the weak topology and p-topology

$$\nu(f) = \lim_{\pi} \mu_{\lambda_\pi}(f) \geqslant p\text{-}\liminf_{\lambda} \mu_\lambda(f) \geqslant \mu_0(f) \qquad \forall\, f \in P(\mathscr{A}).$$

This means that $\nu \succ \mu_0$ (in the Choquet ordering). The closedness of the set $\operatorname{extr}\mathscr{A}$ implies the maximality in $M(\mathscr{A})$ of any measure in $M(\operatorname{extr}\mathscr{A})$. This can be proved using Theorem 2.2 in [5] and the arguments from the proof of Theorem 1.1 in [7], but it can also be immediately shown by using property (v) and the coincidence of any function in $Q(\mathscr{A})$ with its convex hull on the set $\operatorname{extr}\mathscr{A}$. Thus μ_0 is the maximal measure in $M(\mathscr{A})$ and hence $\nu = \mu_0$.

(iii) $\Longrightarrow$ (i). This implication follows from Proposition 5 and Proposition 17 below (with $X = \overline{\operatorname{extr}\mathscr{A}}$).

By Remark 5 the equivalence of properties (i), (ii) and (iv) for convex μ-compact subsets of a Banach space is proved in [9], Theorem 1. This proof is easily extended to the class of sets considered in this paper.

The implication (iv) $\Longrightarrow$ (v) is obvious.

In the proof of Theorem 1 we have involved the following result of measure theory (see [24], Theorem 2.4).

Proposition 17. *Let X be a complete separable metric space. Then the map $M(X) \times M(X) \ni (\mu,\nu) \mapsto \frac{1}{2}(\mu+\nu) \in M(X)$ is open.*

[6] In the case of a compact set $\mathscr{A}$ the coincidence of these topologies on $M(\operatorname{extr}\mathscr{A})$ is proved in [10], Lemma 3.4. In the case of a μ-compact set $\mathscr{A}$ we cannot prove such a coincidence.

Since the set $M(X)$ is μ-compact (Corollary 4), we arrive at the following observation.

Corollary 5. *Properties (i)–(v) in Theorem 1 hold for the set of Borel probability measures on a complete separable metric space endowed with the weak-convergence topology.*

The μ-compactness condition is essential in the proof of Theorem 1; it cannot be removed without changing the whole structure of the proof. This motivates the conjecture that the class of μ-compact convex sets is the maximal class of convex metrizable sets for which the Vesterstrom-O'Brien theory can be generalized. This conjecture can be justified by the following example, showing that even pointwise μ-compactness is not sufficient for the proof of Theorem 1.

Proposition 18. *For any $p > 1$ the pointwise μ-compact simplex*

$$\Delta_p = \left\{ x \in l_p \mid x \geqslant 0,\ \sum_{i=1}^{\infty} x^i \leqslant 1 \right\}$$

in l_p is stable, that is, it possesses property (i) in Theorem 1, which implies (ii), but it does not possess properties (iii)–(v).

Note that for $p = 1$ the μ-compact simplex $\Delta_1 = \mathscr{A}_1$ already has properties (i)–(v) in Theorem 1.

Proof. Example 1 shows that the simplex Δ_p does not possess properties (iv) and (v). Let us show that it does not possess (iii) either. Note that $\operatorname{extr}\Delta_p = \{0, e_i,\ i \in \mathbb{N}\}$, and the set Δ_p is a simplex: for any point x in it there exists a unique measure on $\operatorname{extr}\Delta_p$ with barycentre x. The sequence of points $x_n = (1/n, \dots, 1/n, 0, \dots) \in \Delta_p$ (the first n coordinates are equal to $1/n$ and all other are zero) converges to zero in l_p as $n \to +\infty$, but it is easy to see that the corresponding sequence of measures on $\operatorname{extr}\Delta_p$ does not converge to the single-atom measure supported at the point 0.

Let us now show that for an arbitrary $p > 1$ the set Δ_p is stable, that is, has property (i) which implies (ii) (see the proof of Theorem 1 in [9]). It suffices to prove that for arbitrary points $a, b \subset \Delta_p$, $c = \frac{1}{2}(a+b)$ and for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ with the following property: for any $z \in \Delta_p$ such that $\|z - c\|_p < \delta$ there exists a closed interval $[x, y] \subset \Delta_p$ with centre at z for which $\|x - a\|_p < \varepsilon$ and $\|y - b\|_p < \varepsilon$. By taking sufficiently small ε we can assume that $\|a\|_p < 1 - \varepsilon$ and that $\|b\|_p < 1 - \varepsilon$. Otherwise the points a and b can be replaced by sufficiently close points belonging to the interior of $[a, b]$. Since the l_p-norm is strictly convex, the norms of a and of b will be less then 1. Then we choose a large N so that for each of a and b the norm of the 'tail' starting from the $(N+1)$th coordinate is less than $\frac{1}{6}\,\varepsilon$, that is, $\left(\sum_{k=N+1}^{\infty} (a^k)^p\right)^{1/p} < \frac{1}{6}\,\varepsilon$ and $\left(\sum_{k=N+1}^{\infty} (b^k)^p\right)^{1/p} < \frac{1}{6}\,\varepsilon$. Consider the space $\mathbb{R}^N$ generated by the first N coordinates. We denote by $\widetilde{\Delta}_p$ and $\widetilde{s}$ the restrictions of the set Δ_p and an arbitrary element $s \in l_p$ to this space. Since $\widetilde{\Delta}_p$ is a simplex in $\mathbb{R}^N$, it is stable (see [13]) and one can take $\delta > 0$ such that there always exist points $\widetilde{x}, \widetilde{y} \in \widetilde{\Delta}_p$ for which $\frac{1}{2}(\widetilde{x} + \widetilde{y}) = \widetilde{z}$, $\|\widetilde{x} - \widetilde{a}\|_p < \frac{1}{3}\,\varepsilon$ and $\|\widetilde{y} - \widetilde{b}\|_p < \frac{1}{3}\,\varepsilon$

once $\|\widetilde{z}-\widetilde{c}\|_p < \delta$. Now for fixed $z \in \Delta_p$ and arbitrary $t \in [-1,1]$ we define points $x(t), y(t) \in l_p$ as follows:

$$x^k(t) = \begin{cases} \widetilde{x}^k, & k \leqslant N, \\ (1+t)z^k, & k > N, \end{cases} \qquad y^k(t) = \begin{cases} \widetilde{y}^k, & k \leqslant N, \\ (1-t)z^k, & k > N. \end{cases}$$

By construction $\frac{1}{2}(x(t)+y(t)) = z$ for any t, while the norms of the elements $x(t)$ and $y(t)$ do not exceed $1-\frac{1}{3}\,\varepsilon+2\delta$. Indeed,

$$\|\widetilde{x}\|_p \leqslant \|\widetilde{a}\|_p + \|\widetilde{x}-\widetilde{a}\|_p \leqslant 1-\varepsilon+\frac{1}{3}\,\varepsilon = 1-\frac{2}{3}\,\varepsilon,$$

while the norm of the 'tail' of $(1+t)z$ does not exceed the sum of the norms of the 'tails' of $2c$ and $2(z-c)$, that is, it does not exceed $2\bigl(\frac{1}{6}\,\varepsilon+\delta\bigr)$. Taking the sum we obtain $\|x(t)\|_p \leqslant 1-\frac{1}{3}\,\varepsilon+2\delta$, and the same holds for $y(t)$. For $\delta \leqslant \frac{1}{6}\,\varepsilon$ we obtain $\|x(t)\|_p \leqslant 1$ and $\|y(t)\|_p \leqslant 1$. We shall show that there exists $\tau \in [-1,1]$ for which $\|x(\tau)\|_1 \leqslant 1$ and $\|y(\tau)\|_1 \leqslant 1$, so that $x(\tau), y(\tau) \in \Delta_p$. It is clear that $\|\widetilde{x}\|_1 \leqslant 1$ and $\|\widetilde{y}\|_1 \leqslant 1$. For definiteness suppose $\|\widetilde{x}\|_1 \geqslant \|\widetilde{y}\|_1$. If $\|y(-1)\|_1 \leqslant 1$, then one can take $\tau = -1$ since $\|x(-1)\|_1 = \|\widetilde{x}\|_1 \leqslant 1$. If $\|y(-1)\|_1 > 1$, then $\|y(-1)\|_1 > \|x(-1)\|_1$, and since $\|y(1)\|_1 \leqslant \|x(1)\|_1$, using the continuity argument we conclude that there exists $\tau \in [-1,1]$ such that $\|y(\tau)\|_1 = \|\,x(\tau)\|_1$. Since $\frac{1}{2}(x(\tau)+y(\tau)) = z$, it follows that $\|x(\tau)\|_1 = \|y(\tau)\|_1 = \|z\|_1 \leqslant 1$.

Finally, the norm of the difference $\|x(\tau)-a\|_p$ as concerns the first N coordinates, does not exceed $\frac{1}{3}\,\varepsilon$, and as concerns the other coordinates it does not exceed the maximal norm of the two 'tails': of the element a and of $2z$. Hence

$$\|x(\tau)-a\|_p \leqslant \frac{1}{3}\,\varepsilon + \max\biggl\{\frac{1}{6}\,\varepsilon, 2\biggl(\frac{1}{6}\,\varepsilon+\delta\biggr)\biggr\} = \frac{2}{3}\,\varepsilon+2\delta.$$

For $\delta < \frac{1}{6}\varepsilon$ we obtain $\|x(\tau)-a\|_p < \varepsilon$, and similarly $\|y(\tau)-b\|_p < \varepsilon$. Setting $x = x(\tau)$ and $y = y(\tau)$ we complete the proof.

§ 4. Applications to quantum information theory

An important example of convex μ-compact sets for which the equivalent properties in Theorem 1 hold is the set $\mathfrak{S}(\mathscr{H})$ of quantum states. Quantum states are density operators (positive operators with trace equal to 1) in a separable Hilbert space $\mathscr{H}$.[7] The set of extreme points of the set $\mathfrak{S}(\mathscr{H})$ consists of one-dimensional projectors called pure states [25]. The μ-compactness and the stability property (i) in Theorem 1 of the set $\mathfrak{S}(\mathscr{H})$ are established in [22], Proposition 2 and in [26], Lemma 3, respectively. These properties are used essentially in the study of characteristics of quantum states and of quantum channels. For instance, the μ-compactness of the set $\mathfrak{S}(\mathscr{H})$ makes it possible to prove that any nonentangled state (see below) of a composite quantum system can be represented as an average (barycentre) state of some generalized ensemble of pure product states (probability measure on the set of product pure states) [27]. The stability property of the set

[7] The set $\mathfrak{S}(\mathscr{H})$ is compact if and only if $\dim\mathscr{H} < +\infty$.

$\mathfrak{S}(\mathscr{H})$ plays the crucial role in the proof of the lower semicontinuity property of the χ-function for an arbitrary quantum channel, which is an important characteristic related to the classical capacity of this channel [26].

In this section we consider a result following directly from the generalized Vesterstrom-O'Brien theorem (Theorem 1) applied to the stable set $\mathfrak{S}(\mathscr{H})$.

According to the quantum mechanical formalism, states of a composite quantum system arising as a result of joining two quantum systems represented by two Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ correspond to the density operators in the tensor product $\mathscr{H}\otimes\mathscr{K}$ of these spaces. A specific property of the quantum mechanical statistical model (in comparison with the classical one) is the existence of so-called *entangled* states of the composite system, which cannot be represented as convex combinations of product states describing independent subsystems. Entanglement can be regarded as a special purely quantum correlation, which is the base for the construction of different quantum algorithms, quantum cryptographical protocols and systems of information transmissions, which has attracted a lot of attention of scientists in the last two decades (see [25], Chapter 3). This is why the study of entanglement and, in particular, of its quantitative characteristics, is one of the main problems of quantum information theory.

Let $\mathscr{H}$ and $\mathscr{K}$ be separable Hilbert spaces. A state $\omega\in\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ is called *nonentangled* if it belongs to the convex closure of the set of product states, that is, of states of the form $\rho\otimes\sigma$, where $\rho\in\mathfrak{S}(\mathscr{H})$ and $\sigma\in\mathfrak{S}(\mathscr{K})$; otherwise it is called *entangled*.

Entanglement monotone is an arbitrary function E on the set $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ that possesses the following properties (see [28], [29]):

E1) $\{E(\omega)=0\}\iff\{\text{the state }\omega\text{ is nonentangled}\}$;

E2) Monotonicity under Local Operations and Classical Communications (LOCC), which means that

$$E(\omega)\geqslant\sum_i\pi_iE(\omega_i) \tag{7}$$

for an arbitrary state $\omega\in\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ and an arbitrary LOCC-operation transforming the state ω into a set $\{\omega_i\}$ of states with probability distribution $\{\pi_i\}$ (see details in [29]);

E3) the convexity of the function E on the set $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$, which means that

$$E\biggl(\sum_i\pi_i\omega_i\biggr)\leqslant\sum_i\pi_iE(\omega_i)$$

for an arbitrary finite set $\{\omega_i\}$ of states in $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ and probability distribution $\{\pi_i\}$.

The standard method of 'generation' of entanglement monotones (EM) in the case of finite-dimensional spaces $\mathscr{H}$ and $\mathscr{K}$ is the *convex roof construction* (see [29], [30]). In accordance with this method, for an arbitrary concave continuous nonnegative function f on the set $\mathfrak{S}(\mathscr{H})$ such that

$$f^{-1}(0)=\operatorname{extr}\mathfrak{S}(\mathscr{H})\quad\text{and}\quad f(\rho)=f(U\rho U^*) \tag{8}$$

for an arbitrary state ρ in $\mathfrak{S}(\mathscr{H})$ and an arbitrary unitary operator U in the space $\mathscr{H}$ the corresponding EM E^f is defined by

$$E^f(\omega) = \inf_{\{\pi_i,\omega_i\}\in M_\omega(\operatorname{extr}\mathfrak{S}(\mathscr{H}\otimes\mathscr{K}))} \sum_i \pi_i f\circ\Theta(\omega_i), \qquad \omega\in\mathfrak{S}(\mathscr{H}\otimes\mathscr{K}), \tag{9}$$

where $\Theta\colon \omega\mapsto \operatorname{Tr}_{\mathscr{K}}\omega$ is the partial trace [27] (by the spectral theorem the right-hand side of (9) is well defined). If the von Neumann entropy $H(\rho) = -\operatorname{Tr}\rho\log\rho$ (see [29]) is used as a function f, then this method provides the construction of the Entanglement of Formation (EoF), which is one of the most useful entanglement measures.[8]

In what follows we consider properties of the function E^f defined by (9) in the case of infinite-dimensional spaces $\mathscr{H}$ and $\mathscr{K}$.

An important problem in constructing EM is to analyse the continuity properties, in particular, to prove its continuity on the entire state space $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ (formally, the last property is not included in the definition of EM, but in finite dimensions it is considered a natural requirement). Note that the continuity of the function E^f is not obvious even in the finite-dimensional case and is proved in general using the explicit form of the function f. By Theorem 1 the μ-compactness and stability of the set $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ guarantee the continuity of the function E^f on this set for an arbitrary continuous function f in both the finite- and infinite-dimensional cases.

Theorem 2. *Let f be a concave continuous nonnegative function on the set $\mathfrak{S}(\mathscr{H})$ satisfying conditions (8). Then the function E^f defined by (9) is an entanglement monotone that is continuous on the set $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$.*

Proof. The nonnegativity, concavity, and continuity of the function f imply its boundedness. By Theorem 1 the stability property of the μ-compact set $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ guarantees the continuity of the function $\operatorname{co}(f\circ\Theta)$ and therefore its coincidence with the function $\overline{\operatorname{co}}(f\circ\Theta)$, which by Proposition 6 has the following representation:

$$\overline{\operatorname{co}}(f\circ\Theta)(\omega) = \inf_{\mu\in M_\omega(\mathfrak{S}(\mathscr{H}\otimes\mathscr{K}))} \int_{\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})} (f\circ\Theta)(\varpi)\,\mu(d\varpi), \qquad \omega\in\mathfrak{S}(\mathscr{H}\otimes\mathscr{K}), \tag{10}$$

where the infimum is attained at a particular measure μ_ω in $M_\omega(\mathfrak{S}(\mathscr{H}\otimes\mathscr{K}))$. By the concavity, continuity, and boundedness of the function $f\circ\Theta$ one can assume that μ_ω is a measure in $M_\omega(\operatorname{extr}\mathfrak{S}(\mathscr{H}\otimes\mathscr{K}))$. Hence the definition of the function E^f and the concavity of the function $f\circ\Theta$ imply $E^f = \operatorname{co}(f\circ\Theta) = \overline{\operatorname{co}}(f\circ\Theta)$.

By (8) the nonnegative function $f\circ\Theta$ vanishes on a pure state in $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ if and only if this state is product. As shown in [27], a state ω is nonentangled if and only if there exists a measure μ_ω supported by pure product states such that $\mathbf{b}(\mu_\omega) = \omega$. Thus, the above remark shows that condition E1) is fulfilled for the function $E^f = \overline{\operatorname{co}}(f\circ\Theta)$.

Condition E2) for the function E^f is easily established in the same way as in the finite-dimensional case (see [29]).

Condition E3) for the function E^f follows from its definition.

[8] An entanglement measure is an EM having some particular properties [29].

Example 2. Generalizing the argument in [30] to the infinite-dimensional case we consider the family of continuous concave functions

$$f_\alpha(\rho) = 2(1 - \operatorname{Tr}\rho^\alpha), \qquad \alpha > 1,$$

on the set $\mathfrak{S}(\mathscr{H})$ with $\dim\mathscr{H} \leqslant +\infty$. It is easy to see that all the functions in this family satisfy assumptions (8). By Theorem 2, $\{E^{f_\alpha}\}_{\alpha>1}$ is a family of entanglement monotones, which are continuous and bounded on the set $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ with $\dim\mathscr{H} \leqslant +\infty$ and $\dim\mathscr{K} \leqslant +\infty$. The case $\alpha = 2$ is of particular interest since the entanglement monotone E^{f_2} can be considered as an infinite-dimensional generalization of the notion of I-tangle [31].

§ 5. Possible generalizations and open questions

Proposition 1 actually gives an equivalent definition of the μ-compactness property for convex sets of the class considered in this paper. A convex set $\mathscr{A}$ is *μ-compact* if and only if for an arbitrary compact set $\mathscr{K} \subseteq \mathscr{A}$ and arbitrary $\varepsilon > 0$ there exists a compact set $\mathscr{K}_\varepsilon \subseteq \mathscr{A}$ such that for any expansion of a point $x \in \mathscr{K}$ in a convex combination of points in $\mathscr{A}$ the total weight of points belonging to the set $\mathscr{K}_\varepsilon$ is at least $1-\varepsilon$. This property of (arbitrary!) convex sets can be called *generalized μ-compactness*, or briefly, *$\widetilde{\mu}$-compactness*. By Proposition 1 the $\widetilde{\mu}$-compactness property means μ-compactness for convex bounded subsets of locally convex spaces that are complete separable metric spaces. The above definition of $\widetilde{\mu}$-compactness is translated without any change to any convex closed subsets of linear topological spaces, not necessarily bounded. By contrast, the definition of μ-compactness is not generalized to unbounded sets, since for unbounded sets the barycentre map may not be well defined: the integral

$$\mathbf{b}(\mu) = \int_{\mathscr{A}} x\,\mu(dx)$$

does not necessarily exist for some measures $\mu \in M(\mathscr{A})$. Thus, the notion of $\widetilde{\mu}$-compactness generalizes the notion of μ-compactness to a wider class of convex sets. Similarly to the case of μ-compact sets, the intersection and the Cartesian product of a finite or countable family of $\widetilde{\mu}$-compact sets is $\widetilde{\mu}$-compact, a convex closed subset of a $\widetilde{\mu}$-compact set is $\widetilde{\mu}$-compact. The proof of these assertions is literally the same as the proof of Proposition 4. A complete analogue of Proposition 3 for continuous transformations of μ-compact sets also holds. Nontrivial examples of $\widetilde{\mu}$-compact sets appear even in the finite dimensional case.

Lemma 3. *An arbitrary convex closed pointed* (*not containing lines*) *cone in* $\mathbb{R}^d$ *is $\widetilde{\mu}$-compact.*

Proof. Let $\mathscr{C} \subset \mathbb{R}^d$ be a convex pointed cone. Then there exists a vector $a \in \mathbb{R}^d$ such that $\inf_{x\in\mathscr{C},\,\|x\|=1}(x,a) > 0$ (see [32], p. 53). For each $r > 0$ the truncated cone $\mathscr{C}_r = \{x \in \mathscr{C},\,(x,a) \leqslant r\}$ is compact. An arbitrary compact subset $\mathscr{K}$ of $\mathscr{C}$ can be put in some truncated cone $\mathscr{C}_r$. Then for each $\varepsilon > 0$ the compact set $\mathscr{K}_\varepsilon = \mathscr{C}_{r/\varepsilon}$ has the required property.

The following result of convex geometry is well known, so we omit the proof.

Lemma 4. *The following properties of a convex closed set $\mathscr{A} \subset \mathbb{R}^d$ are equivalent*:

(i) *$\mathscr{A}$ lies in a convex pointed cone*;
(ii) *$\mathscr{A}$ has at least one extreme point*;
(iii) *$\mathscr{A}$ contains no straight line*;
(iv) *the polar of the set $\mathscr{A}$ has a nonempty interior.*

Applying Lemma 3 we conclude that property (i) implies the $\widetilde{\mu}$-compactness of the set $\mathscr{A}$. On the other hand, a line is not $\widetilde{\mu}$-compact, therefore $\widetilde{\mu}$-compactness implies property (iii). Thus, we obtain the following result.

Proposition 19. *In the space $\mathbb{R}^d$ $\widetilde{\mu}$-compactness is equivalent to each of properties* (i)–(iv) *in Lemma 4.*

Positive cones in the spaces l_p and $L_p(X)$ are $\widetilde{\mu}$-compact in the weak topology. The proof is analogous to the proof of Lemma 3 and is based on the weak compactness of bounded sets in these spaces. The positive cone in l_1 (Proposition 9), the cone of finite Borel measures on a complete separable metric space (Proposition 10) and the cone of positive operators in the Shatten class of order $p = 1$ (Proposition 11) are $\widetilde{\mu}$-compact and, by contrast with these propositions, one need not take bounded parts of these cones. Thus, $\widetilde{\mu}$-compactness substantially extends the notion of μ-compactness. This motivates our first question.

Question 1. To what extent are the results of this paper generalized to $\widetilde{\mu}$-compact sets?

The following questions concern μ-compact sets.

Question 2. Do there exist μ-compact noncompact sets in the spaces L_p and l_p for $p > 1$?

Question 3. Consider a Banach lattice. Under what conditions is the bounded part of the positive cone in it μ-compact?

Question 4. Under what conditions on a convex set $\mathscr{A}$, is the convex hull of an arbitrary continuous bounded function $\mathscr{A}$ continuous?

The last property holds for stable μ-compact sets (Theorem 1), but it does not hold for stable pointwise μ-compact sets that are not μ-compact (Proposition 7). The unit ball in the space l_2 possesses this property (Remark 3), but the positive part of this ball does not (Example 1).

§ 6. Appendix

6.1. The compactness criterion for subsets of the cone $\mathfrak{L}_+(\mathscr{H}, \mathscr{H}')$. Let $\mathfrak{L}_+(\mathscr{H}, \mathscr{H}')$ be the cone of linear continuous positive maps from the Banach space $\mathfrak{T}(\mathscr{H})$ of trace-class operators in a separable Hilbert space $\mathscr{H}$ into the similar Banach space $\mathfrak{T}(\mathscr{H}')$. The compactness criterion for subsets of this cone in the strong operator topology is presented in the following lemma.

Lemma 5. 1) *A closed bounded subset* $\mathfrak{L}_0 \subseteq \mathfrak{L}_+(\mathscr{H}, \mathscr{H}')$ *is compact in the strong operator topology if in* $\mathfrak{S}(\mathscr{H})$ *there exists a full-rank state* σ *such that* $\{\Phi(\sigma)\}_{\Phi\in\mathfrak{L}_0}$ *is a compact subset of* $\mathfrak{T}(\mathscr{H}')$.

2) *If a subset* $\mathfrak{L}_0 \subseteq \mathfrak{L}_+(\mathscr{H}, \mathscr{H}')$ *is compact in the strong operator topology, then* $\{\Phi(\sigma)\}_{\Phi\in\mathfrak{L}_0}$ *is a compact subset of* $\mathfrak{T}(\mathscr{H}')$ *for any state* σ *in* $\mathfrak{S}(\mathscr{H})$.

Proof. 1) Let $\{|i\rangle\}$ be the basis of eigenvectors of the state σ arranged in non-increasing order and $\mathscr{H}_m$ be the eigensubspace generated by the first m vectors of this basis. Let $\{\Phi_n\}$ be an arbitrary sequence of maps in $\mathfrak{L}_0$.

We claim that for each m for an arbitrary operator A in $\mathfrak{T}(\mathscr{H}_m)$ there exists a subsequence $\{\Phi_{n_k}\}$ such that the sequence $\{\Phi_{n_k}(A)\}_k$ converges in $\mathfrak{T}(\mathscr{H}')$. Suppose first that $A \geqslant 0$. Since $A \in \mathfrak{T}(\mathscr{H}_m)$, there is $\lambda_A > 0$ such that $\lambda_A A \leqslant \sigma$. By the compactness criterion for subsets of $\mathfrak{T}(\mathscr{H}')$ (see the appendix in [21]) for arbitrary $\varepsilon > 0$ there exists $P_\varepsilon \in \mathfrak{B}(\mathscr{H}')$ such that $\mathrm{Tr}(I_{\mathscr{H}'} - P_\varepsilon)\Phi(\sigma) < \varepsilon$, and hence $\mathrm{Tr}(I_{\mathscr{H}'} - P_\varepsilon)\Phi(A) < \lambda_A^{-1}\varepsilon$ for all $\Phi \in \mathfrak{L}_0$. By the same compactness criterion the set $\{\Phi(A)\}_{\Phi\in\mathfrak{L}_0}$ is compact. This implies the existence of the desired subsequence for a positive operator A. The existence of such a subsequence for an arbitrary operator $A \in \mathfrak{T}(\mathscr{H}_m)$ follows from the representation of this operator as a linear combination of positive operators in $\mathfrak{T}(\mathscr{H}_m)$.

Thus for each m an arbitrary sequence $\{\Phi_n\} \subset \mathfrak{L}_0$ contains a subsequence $\{\Phi_{n_k}\}$ such that the limits

$$\lim_{k\to+\infty} \Phi_{n_k}(|i\rangle\langle j|) = C^m_{ij} \tag{11}$$

exist for all $i, j = 1, \dots, m$, where $\{C^m_{ij}\}$ are some operators in $\mathfrak{T}(\mathscr{H}')$.

For arbitrary $m' > m$, by applying the above observation to the sequence $\{\Phi_{n_k}\}_k$ we obtain a subsequence of the sequence $\{\Phi_n\}$ such that (11) holds for all $i, j = 1, \dots, m'$ with a set of operators $\{C^{m'}_{ij}\}$ such that $C^{m'}_{ij} = C^m_{ij}$ for all $i, j = 1, \dots, m$.

By using this construction one can show the existence of the set $\{C_{ij}\}_{i,j=1}^{+\infty}$ of operators having the following property: for each m there exists a subsequence $\{\Phi_{n_k}\}$ of the sequence $\{\Phi_n\}$ such that (11) holds with $C^m_{ij} = C_{ij}$ for all $i, j = 1, \dots, m$.

On the set $\bigcup_{m\in\mathbb{N}} \mathfrak{T}(\mathscr{H}_m)$ consider the map

$$\Phi_*\colon \sum_{i,j} a_{ij}|i\rangle\langle j| \mapsto \sum_{i,j} a_{ij}\, C_{ij} \in \mathfrak{T}(\mathscr{H}').$$

This map is linear by construction. It is easy to prove its positivity and boundedness. Indeed, by the property of the set $\{C_{ij}\}$ for an arbitrary operator $A \in \bigcup_m \mathfrak{T}(\mathscr{H}_m)$ there exists a subsequence $\{\Phi_{n_k}\}$ of the sequence $\{\Phi_n\}$ such that $\Phi_*(A) = \lim_{k\to+\infty} \Phi_{n_k}(A)$. Thus, the positivity and boundedness of the map Φ_* follows from the positivity of the maps in the sequence $\{\Phi_n\}$ and the uniform boundedness of these maps. Since the set $\bigcup_m \mathfrak{T}(\mathscr{H}_m)$ is dense in $\mathfrak{T}(\mathscr{H})$, the map Φ_* can be extended to a linear positive bounded map from $\mathfrak{T}(\mathscr{H})$ into $\mathfrak{T}(\mathscr{H}')$ (denoted by the same symbol Φ_*).

We show that the map Φ_* is a limit point of the sequence $\{\Phi_n\}$ in the strong operator topology. This topology on bounded subsets of $\mathfrak{L}_+(\mathscr{H}, \mathscr{H}')$ can be determined by a countable family of seminorms $\Phi \mapsto \|\Phi(\rho_i)\|_1$, where $\{\rho_i\}$ is an arbitrary countable dense subset of the set $\mathfrak{S}(\mathscr{H})$.[9] We choose the set of states in $\bigcup_m \mathfrak{T}(\mathscr{H}_m)$ as this subset. An arbitrary neighbourhood of the map Φ_* contains a neighbourhood of the form

$$\{\Phi \in \mathfrak{L}(\mathscr{H}, \mathscr{H}') \mid \|(\Phi - \Phi_*)(\rho_{i_t})\|_1 < \varepsilon,\ t = 1, \dots, p\},$$

where $\{\rho_{i_t}\}_{t=1}^p$ is a finite subset of the above set of states and $\varepsilon > 0$. Since $\{\rho_{i_t}\}_{t=1}^p \subset \mathfrak{T}(\mathscr{H}_m)$ for a particular m, the construction of the map Φ_* implies the existence of a subsequence $\{\Phi_{n_k}\}$ of the sequence $\{\Phi_n\}$ such that $\Phi_*(\rho_{i_t}) = \lim_{k\to+\infty} \Phi_{n_k}(\rho_{i_t})$ for all $t = 1, \dots, p$. Hence at least one element of the sequence $\{\Phi_n\}$ is contained in the above neighbourhood.

Thus, the map Φ_* is a limit point of the sequence $\{\Phi_n\}$ in the strong operator topology. By metrizability of the strong operator topology on bounded subsets of the cone $\mathfrak{L}_+(\mathscr{H}, \mathscr{H}')$ this implies the existence of a subsequence of the sequence $\{\Phi_n\}$ converging to the map Φ_*. This proves the compactness of the set $\mathfrak{L}_0$.

2) This assertion immediately follows from the definition of the strong operator topology.

6.2. The openness criterion.

Lemma 6. *Let φ be a map from a topological space X into a metric space Y. Then the following assertions are equivalent:*

(i) *the map φ is open;*

(ii) *for arbitrary $x_0 \in X$ and an arbitrary sequence $\{y_n\} \subset Y$ converging to $y_0 = \varphi(x_0)$ there exists a subnet $\{y_{n_\lambda}\}_{\lambda\in\Lambda}$ of the sequence $\{y_n\}$ and a net $\{x_\lambda\}_{\lambda\in\Lambda}$ converging to x_0 such that $\varphi(x_\lambda) = y_{n_\lambda}$.*

Proof. (i) $\Longrightarrow$ (ii). Let $\mathfrak{U}$ be the set of all neighbourhoods of the point x_0. Then the set Λ of all pairs $\lambda = (U, k)$, where $U \in \mathfrak{U}$ and $k \in \mathbb{N}$, with the partial order

$$\{\lambda_1 = (U_1, k_1) \succ \lambda_2 = (U_2, k_2)\} \quad \Longleftrightarrow \quad \{k_1 \geqslant k_2 \text{ and } U_1 \subseteq U_2\}$$

is directed. For each $\lambda = (U, k)$ the set $W_\lambda = \varphi(U) \cap V_k$, where V_k is the open ball in Y with centre at y_0 and radius $1/k$, is a neighbourhood of the point y_0. Hence there exists a minimal positive integer n_λ such that $y_{n_\lambda} \in W_\lambda$. It is easy to see that $\{y_{n_\lambda}\}_{\lambda\in\Lambda}$ is a subnet of the sequence $\{y_n\}$. For each $\lambda = (U, k)$ there exists $x_\lambda \in U$ such that $\varphi(x_\lambda) = y_{n_\lambda}$. It is clear that the net $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to x_0.

(ii) $\Longrightarrow$ (i). If there exists an open set $U \subseteq X$ such that the set $\varphi(U)$ is not open then there exist $y_0 = \varphi(x_0) \in \varphi(U)$ and sequence $\{y_n\} \subset Y \setminus \varphi(U)$ converging to y_0. Using (ii) it is easy to obtain a contradiction.

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[9] Here we use the possibility to express an arbitrary operator in $\mathfrak{T}(\mathscr{H})$ as a linear combination of four states in $\mathfrak{S}(\mathscr{H})$.

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