Sbornik: *Mathematics* **202**:10 1537–1564

Matematicheskiĭ Sbornik **202**:10 131–160 DOI 10.1070/SM2011v202n10ABEH004198

The continuity of the output entropy of positive maps

M. E. Shirokov

Abstract. Global and local continuity conditions for the output von Neumann entropy for positive maps between Banach spaces of trace-class operators in separable Hilbert spaces are obtained. Special attention is paid to completely positive maps: infinite dimensional quantum channels and operations.

It is shown that as a result of some specific properties of the von Neumann entropy (as a function on the set of density operators) several results on the output entropy of positive maps can be obtained, which cannot be derived from the general properties of entropy type functions. In particular, it is proved that global continuity of the output entropy of a positive map follows from its finiteness. A characterization of positive linear maps preserving continuity of the entropy (in the following sense: continuity of the entropy on an arbitrary subset of input operators implies continuity of the output entropy on this subset) is obtained. A connection between the local continuity properties of two completely positive complementary maps is considered.

Bibliography: 21 titles.

Keywords: von Neumann entropy, positive trace-class operator, quantum operation, quantum channel, PCE-property of positive maps.

§ 1. Introduction

Trace-preserving and trace-nonincreasing positive linear maps between Banach spaces of trace-class operators in separable Hilbert spaces are noncommutative analogs of Markov and sub-Markov maps in the classical probability theory [1]. In the statistical structure of quantum theory the notions of a quantum channel (dynamical map) and of a quantum operation play key roles. They are defined respectively as trace-preserving and trace-nonincreasing linear maps between Banach spaces of trace-class operators possessing the complete positivity property [2], § 3.1).

The output von Neumann entropy is an important characteristic of a quantum channel. It can be considered as a noncommutative analogue of the output

This work is partially supported by the “Mathematical Control Theory” programme of the Russian Academy of Sciences, the RFBR (grant nos. 09-01-00424a and 10-01-00139a), the Federal Target Programme “Scientific and Pedagogical Staff of Innovative Russia” (grant no. 1.2.1.938) and by the Analytical Departmental Target Programme “Development of Scientific Potential in Higher Education” (grant no. 2.1.1/500). The research was partially carried out within the programme “Mathematics in quantum information theory” in the Fields Institute (Canada, July–August 2009).

AMS 2010 *Mathematics Subject Classification.* Primary 81R15, 94A17.

Shannon entropy of a Markov map. It is this characteristic that is involved (sometimes implicitly) in expressions for different information capacities of a quantum channel, see [3], Chs. 8–9. The notions of a quantum operation and of its output entropy are key to the theory of quantum measurements [2], Ch. 4.

In the finite-dimensional case the output entropy of quantum channels and operations is a concave continuous non-negative function on the cone of input positive operators, but only concavity and non-negativity are preserved when we pass to infinite dimensions while continuity is replaced by lower semicontinuity with the possible inclusion of $+\infty$ in the set of output values. This is a consequence of the 'pathological' behaviour of the von Neumann entropy in the infinite dimensional case, which is considered in detail in [4]. At the same time, the special properties of the von Neumann entropy can be used for proving continuity of the output entropy on particular subsets of input operators. For example, in [3], § 11.5 it is shown that the output entropy of Gaussian quantum channels is continuous on the set of quantum states (density operators) of the system of quantum oscillators with bounded mean energy. Moreover, there exist nontrivial infinite dimensional quantum channels, whose output entropy is continuous on the whole cone of input operators (see § 3).

Singular analytical properties of the output entropy of quantum channels and operations are real obstacles to the analysis of their statistical and information characteristics, in particular, the capacities of quantum channels. This means we need to study the output entropy with the aim of obtaining global and local continuity conditions. The following questions arise naturally in applications:

1) Under what conditions is the output entropy of a positive map (in particular, a quantum channel or operation) continuous on the whole cone of input operators?

2) Under what conditions is the output entropy of a positive map (in particular, a quantum channel or operation) continuous on any subset of the cone of input operators on which the entropy is continuous?

3) How are continuity properties of the output entropy of completely positive complementary maps connected?

The complementary relation, mentioned in the last question, is defined via the Stinespring representation of a completely positive map (see § 2). It plays an important role in analysing the information properties of quantum channels [3], Ch. 6.

The paper is organized as follows. In § 2 we give all the necessary definitions and known results used in the main part of the paper. §§ 3–5 are devoted to the analysis of questions 1)–3) above respectively. They are at the core of the paper. Possible generalizations of the results in §§ 3–5 to analyse the continuity of the output entropy of a positive map as a function of the pair (map, input operator) are considered in § 6. This analysis is necessary to study the physically motivated question about the continuity of information capacities of a quantum channel as a function of the channel and to realize an approximation approach in the study of quantum channels [5], [6].

§ 2. Preliminaries

Let $\mathscr{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathscr{H})$ and $\mathfrak{T}(\mathscr{H})$ the Banach spaces of all linear bounded operators in $\mathscr{H}$ with operator norm $\|\cdot\|$ and of all trace-class operators in $\mathscr{H}$ with the trace norm $\|\cdot\|_1 = \operatorname{Tr}|\cdot|$, respectively, $\mathfrak{B}_+(\mathscr{H})$ and $\mathfrak{T}_+(\mathscr{H})$ the cones of positive operators in these spaces (see [1], [2]). The closed convex subsets

$$\mathfrak{T}_1(\mathscr{H}) = \{A \in \mathfrak{T}_+(\mathscr{H}) \mid \operatorname{Tr} A \leqslant 1\}, \qquad \mathfrak{S}(\mathscr{H}) = \{A \in \mathfrak{T}_+(\mathscr{H}) \mid \operatorname{Tr} A = 1\}$$

of the cone $\mathfrak{T}_+(\mathscr{H})$ are complete separable metric spaces with the metric defined by the trace norm. Following tradition operators in $\mathfrak{S}(\mathscr{H})$ will be denoted by Greek letters ρ, σ, ω and called *states* since each such operator ρ determines a linear normal functional $A \mapsto \operatorname{Tr} A\rho$ with unit norm on the algebra $\mathfrak{B}(\mathscr{H})$, called a *state* in the theory of operator algebras [7]. The *support* $\operatorname{supp} A$ *of a positive operator* A is the orthogonal complement of its kernel, the *rank* of this operator is the dimension of its support: $\operatorname{rank} A = \dim \operatorname{supp} A$. The range of an arbitrary linear operator A will be denoted $\operatorname{Ran} A$.

We will use the Dirac notation $|\varphi\rangle, |\chi\rangle\langle\psi|, \dots$ for vectors and operators of rank 1 in a Hilbert space (in this notation the action of the operator $|\chi\rangle\langle\psi|$ on a vector $|\varphi\rangle$ gives the vector $\langle\psi, \varphi\rangle\,|\chi\rangle$). As usual, orthonormal sets of vectors $\{|\varphi_i\rangle\}_{i\in I}$, where $I = \{1, 2, \dots, n\}$ or $I = \mathbb{N}$, will be denoted by $\{|i\rangle\}_{i\in I}$.

In what follows $\mathscr{A}$ is a subset of the cone of positive trace-class operators.

Let $\operatorname{cl}(\mathscr{A})$, $\operatorname{co}(\mathscr{A})$, $\overline{\operatorname{co}}(\mathscr{A})$ and $\operatorname{extr}(\mathscr{A})$ denote the closure, the convex hull, the convex closure and the set of extreme points of a set $\mathscr{A}$, respectively, see [8], [9].

A finite or countable collection $\{A_i\}$ of operators (states) in a particular subset $\mathscr{A}$ of the cone $\mathfrak{T}_+(\mathscr{H})$ with the corresponding probability distribution $\{\pi_i\}$ will be called an *ensemble* and denoted by $\{\pi_i, A_i\}$. The operator $\sum_i \pi_i A_i$ in $\overline{\operatorname{co}}(\mathscr{A})$ is called the *average operator* (*state*) of such an ensemble. The set of ensembles of operators from $\mathscr{A}$ with a given average operator A will be denoted $\mathscr{P}^{\mathrm{a}}_{\{A\}}(\mathscr{A})$.[1]

We set $I_{\mathscr{H}}$ and $\mathrm{Id}_{\mathscr{H}}$ to be the identity operator in the Hilbert space $\mathscr{H}$ and the identity transformation of the Banach space $\mathfrak{T}(\mathscr{H})$, respectively.

Let $\mathscr{H}$ and $\mathscr{H}'$ be separable Hilbert spaces, called *input* and *output* spaces, respectively, and let $\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$ be a linear map which is positive and trace-nonincreasing ($\Phi(A) \geqslant 0$ and $\operatorname{Tr}\Phi(A) \leqslant \operatorname{Tr} A$ for any $A \geqslant 0$). The *dual map* $\Phi^*\colon \mathfrak{B}(\mathscr{H}') \to \mathfrak{B}(\mathscr{H})$ (defined by the duality relation $\operatorname{Tr}\Phi(A)B = \operatorname{Tr} A\Phi^*(B)$, $A \in \mathfrak{T}(\mathscr{H})$, $B \in \mathfrak{B}(\mathscr{H}')$) is a positive map such that $\Phi^*(I_{\mathscr{H}'}) \leqslant I_{\mathscr{H}}$. The set of all linear positive trace-nonincreasing maps from $\mathfrak{T}(\mathscr{H})$ into $\mathfrak{T}(\mathscr{H}')$ is denoted by $\mathfrak{L}^+_{\leqslant 1}(\mathscr{H}, \mathscr{H}')$.

A linear map $\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$ is called *completely positive* if for any Hilbert space $\mathscr{K}$ the map $\Phi^* \otimes \mathrm{Id}^*_{\mathscr{K}}$ from the C^*-algebra $\mathfrak{B}(\mathscr{H}' \otimes \mathscr{K})$ into the C^*-algebra $\mathfrak{B}(\mathscr{H} \otimes \mathscr{K})$ is positive (equivalent definitions of complete positivity can be found in [3], § 6.2).

---

[1] This notation is used because an arbitrary ensemble of operators from $\mathscr{A}$ can be identified with the atomic probability measure on the set $\mathscr{A}$.

Definition 1. A linear completely positive trace-nonincreasing map

$$\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$$

is called a *quantum operation.*

A trace-preserving quantum operation is called a *quantum channel.*

Denote the sets of all quantum operations and of all quantum channels from $\mathfrak{T}(\mathscr{H})$ into $\mathfrak{T}(\mathscr{H}')$ by $\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ and $\mathfrak{F}_{=1}(\mathscr{H},\mathscr{H}')$, respectively. Thus

$$\mathfrak{F}_{=1}(\mathscr{H},\mathscr{H}') \subset \mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}') \subset \mathfrak{L}^{+}_{\leqslant 1}(\mathscr{H},\mathscr{H}').$$

An arbitrary quantum operation (channel) $\Phi \in \mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ has the Kraus representation

$$\Phi(\,\cdot\,) = \sum_{i=1}^{+\infty} V_i(\,\cdot\,)V_i^*, \tag{1}$$

determined by the set $\{V_i\}_{i=1}^{+\infty}$ of bounded linear operators from $\mathscr{H}$ into $\mathscr{H}'$ such that $\sum_{i=1}^{+\infty} V_i^*V_i \leqslant I_{\mathscr{H}}$ ($\sum_{i=1}^{+\infty} V_i^*V_i = I_{\mathscr{H}}$, respectively); see [3], § 6.2.

If Φ is a quantum operation (channel) from $\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$, then the Stinespring theorem implies the existence of a Hilbert space $\mathscr{H}''$ and a contraction (isometry) $V\colon \mathscr{H} \to \mathscr{H}' \otimes \mathscr{H}''$ such that

$$\Phi(A) = \operatorname{Tr}_{\mathscr{H}''} VAV^*, \qquad A \in \mathfrak{T}(\mathscr{H}). \tag{2}$$

A quantum operation (channel)

$$\mathfrak{T}(\mathscr{H}) \ni A \mapsto \widetilde{\Phi}(A) = \operatorname{Tr}_{\mathscr{H}'} VAV^* \in \mathfrak{T}(\mathscr{H}'') \tag{3}$$

is called *complementary to the operation* (channel) Φ (see [3], § 6.6).[2] That the definition of the complementary operation is unique (up to unitary equivalence) is shown in [11], where the following representation is also proved:

$$\widetilde{\Phi}(A) = \sum_{i,j=1}^{+\infty} \operatorname{Tr}[V_iAV_j^*]|i\rangle\langle j|, \qquad A \in \mathfrak{T}(\mathscr{H}), \tag{4}$$

where $\{V_i\}_{i=1}^{+\infty}$ is the set of operators from the Kraus representation (1) for the operation Φ and $\{|i\rangle\}$ is an orthonormal basis in the space $\mathscr{H}''$ (this representation can easily be obtained by noting that (2) holds for the operator $V\colon \mathscr{H} \ni |\varphi\rangle \mapsto \sum_{i=1}^{+\infty} |V_i\varphi\rangle \otimes |i\rangle \in \mathscr{H}' \otimes \mathscr{H}''$).

The dual map of a quantum operation Φ with representations (1) and (2) has the form

$$\Phi^*(B) = \sum_{i=1}^{+\infty} V_i^*BV_i = V^*(B \otimes I_{\mathscr{H}''})V, \qquad B \in \mathfrak{B}(\mathscr{H}'). \tag{5}$$

We denote the simplex of all probability distributions with $n \leqslant +\infty$ outcomes by $\mathfrak{P}_n$.

[2]Sometimes the quantum operation $\widetilde{\Phi}$ is also called *conjugate to the operation* Φ [10].

The von Neumann entropy

$$H(\rho) = \mathrm{Tr}\,\eta(\rho)$$

of a state $\rho \in \mathfrak{S}(\mathscr{H})$, where $\eta(x) = -x\log x$, has the following natural extension to the cone $\mathfrak{T}_+(\mathscr{H})$:

$$H(A) = \mathrm{Tr}\,AH\left(\frac{A}{\mathrm{Tr}\,A}\right) = \mathrm{Tr}\,\eta(A) - \eta(\mathrm{Tr}\,A), \qquad A \in \mathfrak{T}_+(\mathscr{H})$$

(see [12]).[3] In what follows the function $A \mapsto H(A)$ on the cone $\mathfrak{T}_+(\mathscr{H})$ will be called the *quantum entropy* while the function

$$\{x_i\} \mapsto H(\{x_i\}) = \sum_i \eta(x_i) - \eta\left(\sum_i x_i\right)$$

on the positive cone of the Banach space ℓ_1, coinciding with the Shannon entropy on the set $\mathfrak{P}_{+\infty}$ of all probability distributions, will be called the *classical entropy*.

The non-negativity, concavity and lower semicontinuity of the quantum entropy on the cone $\mathfrak{T}_+(\mathscr{H})$ follow from the corresponding properties of the von Neumann entropy on the set $\mathfrak{S}(\mathscr{H})$ ([4], [12], [13]). By definition

$$H(\lambda A) = \lambda H(A), \qquad A \in \mathfrak{T}_+(\mathscr{H}), \quad \lambda \geqslant 0. \tag{6}$$

Taken with (6) the concavity of the von Neumann entropy implies that the quantum entropy is monotonic:

$$A \leqslant B \quad \Longrightarrow \quad H(A) \leqslant H(B), \qquad A, B \in \mathfrak{T}_+(\mathscr{H}). \tag{7}$$

By simple approximation it is easy to derive from Theorem 11.10 in [14] that

$$\sum_{i=1}^{n} \lambda_i H(A_i) \leqslant H\left(\sum_{i=1}^{n} \lambda_i A_i\right) \leqslant \sum_{i=1}^{n} \lambda_i H(A_i) + H(\{\lambda_i\}_{i=1}^{n}), \tag{8}$$

which holds for any collection $\{A_i\}_{i=1}^{n} \subset \mathfrak{T}_1(\mathscr{H})$ and any probability distribution $\{\lambda_i\}_{i=1}^{n}$, where $n \leqslant +\infty$. This inequality implies the following:

$$\sum_{i=1}^{n} H(A_i) \leqslant H\left(\sum_{i=1}^{n} A_i\right) \leqslant \sum_{i=1}^{n} H(A_i) + H(\{\mathrm{Tr}\,A_i\}_{i=1}^{n}), \tag{9}$$

which holds for any collection $\{A_i\}_{i=1}^{n} \subset \mathfrak{T}_+(\mathscr{H})$ such that

$$\sum_{i=1}^{n} \mathrm{Tr}\,A_i < +\infty.$$

Equality holds in the second inequality in (9) if $\mathrm{supp}\,A_i \perp \mathrm{supp}\,A_j$ for any $i \neq j$.

[3] Here and in what follows log denotes the natural logarithm.

If V is an arbitrary linear contraction from $\mathscr{H}$ into $\mathscr{H}'$ and A is an arbitrary operator in $\mathfrak{T}_+(\mathscr{H})$ the following inequality holds:

$$H(VAV^*) \leqslant H(A), \tag{10}$$

which can easily be proved by noting that $V \oplus \sqrt{I_{\mathscr{H}} - V^*V}$ is an isometry from $\mathscr{H}$ into $\mathscr{H}' \oplus \mathscr{H}$.

A positive unbounded linear operator in a separable Hilbert space with a discrete spectrum of finite multiplicity will be called an $\mathfrak{H}$-*operator*. If $\{|i\rangle\}_{i=1}^{+\infty}$ is an orthonormal set of eigenvectors and $\{h_i\}_{i=1}^{+\infty}$ the corresponding sequence of eigenvalues of the $\mathfrak{H}$-operator H, then this operator has the 'spectral' representation $H = \sum_{i=1}^{+\infty} h_i|i\rangle\langle i|$ on the domain

$$\mathscr{D}(H) = \biggl\{\varphi \in \overline{\operatorname{lin}}(\{|i\rangle\}_{i=1}^{+\infty}) \biggm| \sum_{i=1}^{+\infty} h_i^2|\langle i,\varphi\rangle|^2 < +\infty\biggr\}.$$

Let

$$H = \sum_{i=1}^{+\infty} h_i|i\rangle\langle i|$$

be an $\mathfrak{H}$-operator in the space $\mathscr{H}$ and let $A \in \mathfrak{T}_+(\mathscr{H})$. We will say that

$$\operatorname{Tr} AH = \sum_{i=1}^{+\infty} h_i\langle i|A|i\rangle \leqslant +\infty$$

if $\operatorname{supp} A \subseteq \overline{\operatorname{lin}}(\{|i\rangle\}_{i=1}^{+\infty})$ and $\operatorname{Tr} AH = +\infty$ otherwise.

An important example of an $\mathfrak{H}$-operator is the operator $-\log A$ for any operator $A = \sum_{i=1}^{+\infty} \lambda_i|i\rangle\langle i|$ in $\mathfrak{T}_1(\mathscr{H})$ with infinite rank ($\lambda_i > 0$, $i = 1, 2, \dots$), which has the representation $-\log A = \sum_{i=1}^{+\infty}(-\log\lambda_i)|i\rangle\langle i|$ on the set

$$\mathscr{D}(-\log A) = \biggl\{\varphi \in \operatorname{supp} A \biggm| \sum_{i=1}^{+\infty}(\log\lambda_i)^2|\langle i,\varphi\rangle|^2 < +\infty\biggr\}.$$

Note that for any operators $A \in \mathfrak{T}_1(\mathscr{H})$ and $B \in \mathfrak{T}_1(\mathscr{K})$ the following identity holds

$$-\log(A \otimes B) = (-\log A) \otimes I_{\mathscr{K}} + I_{\mathscr{H}} \otimes (-\log B), \tag{11}$$

where '=' means the operators coincide on

$$\mathscr{D}(-\log(A \otimes B)) \subseteq \operatorname{supp} A \otimes \operatorname{supp} B.$$

For an $\mathfrak{H}$-operator H we introduce the parameter

$$\mathrm{g}(H) = \inf\{\lambda > 0 \mid \operatorname{Tr} e^{-\lambda H} < +\infty\},$$

where we assume that $\mathrm{g}(H) = +\infty$ if $\operatorname{Tr} e^{-\lambda H} = +\infty$ for all $\lambda > 0$ [15]. It is clear that $\mathrm{g}(-\log A) \leqslant 1$ for any operator A in $\mathfrak{T}_1(\mathscr{H})$.

A given $\mathfrak{H}$-operator H in a Hilbert space $\mathscr{H}$ and a positive number h determine the closed convex set

$$\mathscr{K}_{H,h} = \{A \in \mathfrak{T}_1(\mathscr{H}) \mid \operatorname{Tr} AH \leqslant h\}.$$

We will use the following generalized versions of Proposition 1 in [15], Part I and Proposition 6.6 in [13], which can easily be derived using the construction from the proof of Lemma 3 below.

Proposition 1. *Let H be an $\mathfrak{H}$-operator in a Hilbert space $\mathscr{H}$ and let $h > 0$.*

A) *The quantum entropy is bounded on the set $\mathscr{K}_{H,h}$ if and only if $\mathrm{g}(H) < +\infty$;*

B) *the quantum entropy is continuous on the set $\mathscr{K}_{H,h}$ if and only if $\mathrm{g}(H) = 0$.*

The following result can be derived directly from Corollaries 3 and 4 in [16].

Lemma 1. *Let $\{A_n\}$ and $\{B_n\}$ be sequences of operators in $\mathfrak{T}_+(\mathscr{H})$ converging to operators A_0 and B_0, respectively. Then*

$$\Big\{H(A_n+B_n) \underset{n}{\to} H(A_0+B_0)\Big\} \iff \Big\{H(A_n) \underset{n}{\to} H(A_0)\Big\} \wedge \Big\{H(B_n) \underset{n}{\to} H(B_0)\Big\}.$$

The quantum entropy of an arbitrary operator $A \in \mathfrak{T}_+(\mathscr{H})$ and the classical entropy of the sequence of diagonal elements of its matrix in any orthonormal basis $\{|i\rangle\}_{i=1}^{+\infty}$ of the space $\mathscr{H}$ are related by the inequality

$$H(A) \leqslant H(\{\langle i|A|i\rangle\}_{i=1}^{+\infty}), \tag{12}$$

which follows since the relative entropy is non-negative (see equality (21) in [15], Part I).

Using relations (6) and (12) it is easy to obtain the following continuity condition for quantum entropy from Proposition 5 in [15], Part I.

Proposition 2. *Let $\{|i\rangle\}_{i=1}^{+\infty}$ be an arbitrary orthonormal basis in a Hilbert space $\mathscr{H}$. The quantum entropy on a set $\mathscr{A} \subset \mathfrak{T}_+(\mathscr{H})$ is continuous provided that the classical entropy on the set $\big\{\{\langle i|A|i\rangle\}_{i=1}^{+\infty} \mid A \in \mathscr{A}\big\} \subset (\ell_1)_+$ is continuous.*

For an arbitrary operator C in $\mathfrak{T}_+(\mathscr{H} \otimes \mathscr{K})$ the following triangle inequality holds (see [14]):

$$H(C) \geqslant \big|H(\operatorname{Tr}_{\mathscr{K}} C) - H(\operatorname{Tr}_{\mathscr{H}} C)\big|. \tag{13}$$

For arbitrary map $\Phi \in \mathfrak{L}^+_{\leqslant 1}(\mathscr{H}, \mathscr{H}')$ and operator $A \in \mathfrak{T}_+(\mathscr{H})$ the following estimate holds:

$$H(\Phi(A)) \leqslant \Big[\sup_{\rho \in \operatorname{extr} \mathfrak{S}(\mathscr{H})} H(\Phi(\rho))\Big] \operatorname{Tr} A + H(A), \tag{14}$$

which is easily proved using the spectral decomposition of the operator A and inequality (8).

We will use the following simple result repeatedly (see [3], § 3.1.3).

Lemma 2. *Let A be a rank one operator in $\mathfrak{T}_+(\mathscr{H} \otimes \mathscr{K})$. The operators*

$$\operatorname{Tr}_{\mathscr{K}} A \in \mathfrak{T}_+(\mathscr{H}) \quad \text{and} \quad \operatorname{Tr}_{\mathscr{H}} A \in \mathfrak{T}_+(\mathscr{K})$$

are isomorphic and hence have the same entropy.

The relative entropy of operators A and B in $\mathfrak{T}_+(\mathscr{H})$ is defined as follows:

$$H(A\,\|B)=\sum_{i=1}^{+\infty}\langle i|\,A\log A-A\log B+B-A\,|i\rangle,$$

where $\{|i\rangle\}_{i=1}^{+\infty}$ is the orthonormal basis of eigenvectors for the operator A (or B), and it is assumed that $H(A\,\|B)=+\infty$ if $\operatorname{supp}A\nsubseteq\operatorname{supp}B$ (for details see [12]).

An important property of a quantum operation $\Phi\in\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ consists in the following monotonicity relation for the relative entropy (see [12]):

$$H(\Phi(A)\|\Phi(B))\leqslant H(A\,\|B),\qquad A,B\in\mathfrak{T}_+(\mathscr{H}). \tag{15}$$

For a given natural number k denote by $\mathfrak{T}_+^k(\mathscr{H})$ (by $\mathfrak{S}_k(\mathscr{H})$) the set of operators in $\mathfrak{T}_+(\mathscr{H})$ (states in $\mathfrak{S}(\mathscr{H})$, respectively) having rank $\leqslant k$.

We shall look briefly at the method for approximating concave lower semicontinuous non-negative functions on the set $\mathfrak{S}(\mathscr{H})$, put forward in [16], § 4.

For a given natural number k and a non-negative function f on the set $\mathfrak{S}_k(\mathscr{H})$ we consider the concave function

$$\mathfrak{S}(\mathscr{H})\ni\rho\mapsto\widehat{f}_k^{\sigma}(\rho)=\sup_{\{\pi_i,\rho_i\}\in\mathscr{P}^{\mathrm{a}}_{\{\rho\}}(\mathfrak{S}_k(\mathscr{H}))}\sum_i\pi_i f(\rho_i)\in[0,+\infty] \tag{16}$$

(the supremum is taken over all ensembles of states $\mathfrak{S}_k(\mathscr{H})$ with average state ρ, that is, over all decompositions of the state ρ into a convex combinations of states with rank $\leqslant k$).

The strong stability property of the set $\mathfrak{S}(\mathscr{H})$ (see [16], § 3) means we can show that for any lower semicontinuous non-negative (continuous and bounded) function f on the set $\mathfrak{S}_k(\mathscr{H})$ the function $\widehat{f}_k^{\sigma}$ is lower semicontinuous (continuous) on the set $\mathfrak{S}(\mathscr{H})$.

If f is a concave lower semicontinuous non-negative function on the set $\mathfrak{S}(\mathscr{H})$, then the nondecreasing sequence $\{\widehat{f}_k^{\sigma}\}_k$ converges pointwise to the function f. Thus the above remark implies the following continuity condition: if the function f has a continuous restriction to the set $\mathfrak{S}_k(\mathscr{H})$ for each $k\in\mathbb{N}$, then a sufficient condition for the function f to be continuous on a set $\mathscr{A}\subset\mathfrak{S}(\mathscr{H})$ is that the sequence $\{\widehat{f}_k^{\sigma}\}_k$ converges uniformly on this set. If $\mathscr{A}$ is compact, then this condition is also necessary by Dini's lemma.

In [16], using this method it is shown that a sufficient condition for the quantum entropy to be continuous on a set $A\subset\mathfrak{T}_+(\mathscr{H})$ is that the uniform approximation property (briefly, the UA-property) holds for this set. It can be expressed as follows:

$$\lim_{k\to+\infty}\sup_{A\in\mathscr{A}}\Delta_k(A)=0,$$

where

$$\Delta_k(A)=\inf_{\{\pi_i,A_i\}\in\mathscr{P}^{\mathrm{a}}_{\{A\}}(\mathfrak{T}_+^k(\mathscr{H}))}\sum_i\pi_i H(A_i\|A),\qquad k\in\mathbb{N}$$

(the infimum is taken over all decompositions of the operator A into a convex combination of operators with rank $\leqslant k$). If the set $\mathscr{A}$ is compact, then the UA-property is also a necessary condition for the quantum entropy to be continuous on this set.

In § 4 the above approximation method will be used to analyse the local continuity of the output entropy of a positive map.

Remark 1. In what follows when we say a function f is continuous on a subset $\mathscr{A}$ of the cone $\mathfrak{T}_+(\mathscr{H})$ we mean the restriction of this function to this subset is continuous. We assume that if a function is continuous, this implies it is finite (by contrast with lower or upper semicontinuity).

§ 3. The continuity of the output entropy on the cone of positive operators

3.1. General case. Let $\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$ be a positive linear map. The output entropy $H_\Phi \doteq H \circ \Phi$ of this map is a concave lower semicontinuous non-negative function on the cone $\mathfrak{T}_+(\mathscr{H})$. The following theorem shows that this function cannot be both finite and discontinuous, that is, it is either continuous or takes infinite values on some input operators.

Theorem 1. *Let Φ be a map in $\mathfrak{L}^+_{\leqslant 1}(\mathscr{H}, \mathscr{H}')$. The following statements are equivalent:*

(i) *the function $A \mapsto H_\Phi(A)$ is finite on the set $\mathfrak{S}(\mathscr{H}) \subset \mathfrak{T}_+(\mathscr{H})$;*
(ii) *the function $A \mapsto H_\Phi(A)$ is continuous on the cone $\mathfrak{T}_+(\mathscr{H})$;*
(iii) *there exists an orthonormal basis $\{|i\rangle\}_{i=1}^{+\infty}$ of the space $\mathscr{H}'$ such that the function $A \mapsto H(\{\langle i|\Phi(A)|i\rangle\}_{i=1}^{+\infty})$ is continuous on the cone $\mathfrak{T}_+(\mathscr{H})$;*[4]
(iv) *there exists an orthonormal basis $\{|i\rangle\}_{i=1}^{+\infty}$ of the space $\mathscr{H}'$ and a sequence $\{h_i\}_{i=1}^{+\infty}$ of non-negative numbers such that*

$$\biggl\|\sum_{i=1}^{+\infty} h_i \Phi^*(|i\rangle\langle i|)\biggr\| < +\infty, \qquad \sum_{i=1}^{+\infty} e^{-h_i} < +\infty,$$

where $\Phi^*\colon \mathfrak{B}(\mathscr{H}') \to \mathfrak{B}(\mathscr{H})$ is the map dual to the map Φ.

The set $\mathfrak{S}(\mathscr{H})$ in (i) *can be replaced by an arbitrary convex closed bounded subset $\mathscr{A} \subset \mathfrak{T}_+(\mathscr{H})$ such that*

$$\sup_n \sup_{A\in\mathscr{A}} \operatorname{Tr} AB_n < +\infty \quad \Longrightarrow \quad \sup_n \|B_n\| < +\infty$$

for any increasing sequence $\{B_n\}$ of operators in $\Phi^*(\mathfrak{B}_+(\mathscr{H}'))$.

Restrictions on the choice of the basis $\{|i\rangle\}_{i=1}^{+\infty}$ in statements (iii) and (iv) of this theorem are considered in Remark 4 below.

Proof. (i) $\Longrightarrow$ (ii). Let $\mathscr{A}$ be a closed convex bounded subset of the cone $\mathfrak{T}_+(\mathscr{H})$ satisfying the condition in the final assertion of the theorem. We can assume that $\mathscr{A} \subseteq \mathfrak{T}_1(\mathscr{H})$. If the function $A \mapsto H_\Phi(A)$ is finite on $\mathscr{A}$ then it is bounded on this set. Indeed, if for any natural number n there exists an operator $A_n \in \mathscr{A}$ such that $H_\Phi(A_n) \geqslant 2^n$ then

$$\sum_{n=1}^{+\infty} 2^{-n} A_n \in \mathscr{A} \quad \text{and} \quad H_\Phi\biggl(\sum_{n=1}^{+\infty} 2^{-n} A_n\biggr) \geqslant \sum_{n=1}^{+\infty} 2^{-n} H_\Phi(A_n) = +\infty$$

[4]By Proposition 2 this statement is formally stronger than the previous one.

by the discrete Jensen inequality (it is easy to verify that it holds for the concave non-negative function $A \mapsto H_\Phi(A)$ on the set $\mathscr{A}$).

Thus Lemma 3 below implies that an $\mathfrak{H}$-operator $H = -\log T$ exists in the space $\mathscr{H}'$ such that $\mathrm{g}(H) \leqslant 1$ and $\operatorname{Tr} H\Phi(A) \leqslant h$ for all $A \in \mathscr{A}$ and for some $h > 0$. Let $H = \sum_{i=1}^{+\infty} h_i |i\rangle\langle i|$. We can assume that $\{|i\rangle\}_{i=1}^{+\infty}$ is a basis in $\mathscr{H}'$. Since the function

$$A \mapsto \operatorname{Tr} H\Phi(A) = \sum_{i=1}^{+\infty} h_i \langle i|\Phi(A)|i\rangle = \operatorname{Tr}\Biggl[\sum_{i=1}^{+\infty} h_i \Phi^*(|i\rangle\langle i|)\Biggr] A \tag{17}$$

is bounded on the set $\mathscr{A}$, the linear operator in the square brackets is bounded, that is, using the assumption on the set $\mathscr{A}$ it lies in $\mathfrak{B}(\mathscr{H})$. Thus function (17) is continuous on the cone $\mathfrak{T}_+(\mathscr{H})$. For an arbitrary compact set $\mathscr{C} \subset \mathfrak{T}_+(\mathscr{H})$ Dini's lemma implies that the series $\sum_{i=1}^{+\infty} h_i \langle i|\Phi(A)|i\rangle$ converges uniformly on this compact set and hence a nondecreasing sequence $\{y_i^{\mathscr{C}}\}_{i=1}^{+\infty}$ of positive numbers exists, which converges to $+\infty$ and is such that $\sup_{A\in\mathscr{C}} \sum_{i=1}^{+\infty} y_i^{\mathscr{C}} h_i \langle i|\Phi(A)|i\rangle < +\infty$. Note that $H^{\mathscr{C}} = \sum_{i=1}^{+\infty} y_i^{\mathscr{C}} h_i |i\rangle\langle i|$ is an $\mathfrak{H}$-operator with $\mathrm{g}(H^{\mathscr{C}}) = 0$. We have

$$\sup_{A\in\mathscr{C}} \operatorname{Tr} H^{\mathscr{C}}\Phi(A) = \sup_{A\in\mathscr{C}} \sum_{i=1}^{+\infty} y_i^{\mathscr{C}} h_i \langle i|\Phi(A)|i\rangle < +\infty. \tag{18}$$

By Proposition 1, B) the function $A \mapsto H(\Phi(A))$ is continuous on the set $\mathscr{C}$ and hence on the whole cone $\mathfrak{T}_+(\mathscr{H})$ (since $\mathscr{C}$ is an arbitrary compact subset of $\mathfrak{T}_+(\mathscr{H})$).

(i) $\Longrightarrow$ (iv). In the proof of (i) $\Longrightarrow$ (ii) the existence of a basis $\{|i\rangle\}_{i=1}^{+\infty}$ and a sequence $\{h_i\}_{i=1}^{+\infty}$ with the required properties was shown.

(iv) $\Longrightarrow$ (iii). This follows from the proof of (i) $\Longrightarrow$ (ii) since (18) implies the function $A \mapsto H(\{\langle i|\Phi(A)|i\rangle\}_{i=1}^{+\infty})$ is continuous on the set $\mathscr{C}$ by the classical analogue of Proposition 1, B).

(iii) $\Longrightarrow$ (i). This follows from relation (12).

Lemma 3. *Let $\mathscr{A} \subset \mathfrak{T}_1(\mathscr{H})$ be an arbitrary convex set on which the quantum entropy is bounded. Then there exists an operator $T \in \mathfrak{T}_1(\mathscr{H})$ such that*

$$\sup_{A\in\mathscr{A}} \operatorname{Tr} A(-\log T) < +\infty \quad \textit{and} \quad UT = TU$$

for any unitary operator U in $\mathfrak{B}(\mathscr{H})$ such that $UAU^* \in \mathscr{A}$ for all $A \in \mathscr{A}$.

Proof. Let $\mathscr{K}$ be the one dimensional space generated by the vector $|0\rangle$. Consider the convex set

$$\mathscr{A}^{\mathrm{e}} = \bigl\{\rho_A = A \oplus (1 - \operatorname{Tr} A)|0\rangle\langle 0| \mid A \in \mathscr{A}\bigr\}$$

of states in $\mathfrak{S}(\mathscr{H} \oplus \mathscr{K})$. For an arbitrary operator $A \in \mathscr{A}$ we have

$$H(\rho_A) = \operatorname{Tr}\eta(A) + \eta(1 - \operatorname{Tr} A) = H(A) + \eta(\operatorname{Tr} A) + \eta(1 - \operatorname{Tr} A) \leqslant H(A) + 1.$$

Since the von Neumann entropy is bounded on the convex set $\mathscr{A}^{\mathrm{e}}$, the χ-capacity $\overline{C}(\mathscr{A}^{\mathrm{e}})$ of this set is finite [15]. Theorem 1 in [15], Part I implies a unique state $\Omega(\mathscr{A}^{\mathrm{e}})$ in $\mathrm{cl}(\mathscr{A}^{\mathrm{e}})$ exists (called the optimal average state of the set $\mathscr{A}^{\mathrm{e}}$) such that

$$H(\rho\|\Omega(\mathscr{A}^{\mathrm{e}})) \leqslant \overline{C}(\mathscr{A}^{\mathrm{e}}), \qquad \rho \in \mathscr{A}^{\mathrm{e}}.$$

The state $\Omega(\mathscr{A}^{\mathrm{e}})$ has the form $T \oplus \lambda|0\rangle\langle 0|$, where $T \in \mathfrak{T}_1(\mathscr{H})$ and $\lambda \geqslant 0$. Note that

$$\begin{aligned}\operatorname{Tr} A(-\log T) &\leqslant \operatorname{Tr} \rho_A(-\log \Omega(\mathscr{A}^{\mathrm{e}})) \\ &= H(\rho_A \| \Omega(\mathscr{A}^{\mathrm{e}})) + H(\rho_A) \leqslant \overline{C}(\mathscr{A}^{\mathrm{e}}) + H(A) + 1, \qquad A \in \mathscr{A}.\end{aligned}$$

For an arbitrary unitary operator U in $\mathfrak{B}(\mathscr{H})$ such that $U\mathscr{A}U^* = \mathscr{A}$ we have $(U \oplus I_{\mathscr{K}})\Omega(\mathscr{A}^{\mathrm{e}}) = \Omega(\mathscr{A}^{\mathrm{e}})(U \oplus I_{\mathscr{K}})$ by Corollary 4 in [15], Part II. Hence $UT = TU$.

Remark 2. Theorem 1 does not assert that if the quantum entropy is finite on the set $\Phi(\mathfrak{S}(\mathscr{H}))$ it is continuous on this set since the continuity of the function $A \mapsto H_\Phi(A) \doteq H(\Phi(A))$ on the noncompact set $\mathfrak{S}(\mathscr{H})$ does not imply that the function $A \mapsto H(A)$ is continuous on the set $\Phi(\mathfrak{S}(\mathscr{H}))$. This is confirmed by the following example.

Let $\mathscr{A}$ be a closed convex subset of the set $\mathfrak{S}(\mathscr{H}')$ on which the von Neumann entropy is discontinuous but bounded (see examples in [15]). Let $\{\sigma_n\}_{n=1}^{+\infty}$ be a sequence of states in $\mathscr{A}$ converging to the state σ_0 such that $\lim_{n\to+\infty} H(\sigma_n) \neq H(\sigma_0)$. Consider the map $\Phi\colon A \mapsto \sum_{n\geqslant 0}\langle n|A|n\rangle\sigma_n$, where $\{|n\rangle\}_{n\geqslant 0}$ is a particular orthonormal basis in $\mathscr{H}$. By Theorem 1 the function $A \mapsto H_\Phi(A)$ is continuous on the set $\mathfrak{S}(\mathscr{H})$, but the function $A \mapsto H(A)$ is not continuous on the set $\Phi(\mathfrak{S}(\mathscr{H}))$ containing the sequence $\{\sigma_n\}_{n=1}^{+\infty}$ and the state σ_0.

The continuity of the function $A \mapsto H_\Phi(A)$ on the set $\mathfrak{S}(\mathscr{H})$ means that the function $A \mapsto H(A)$ is continuous on any set of the form $\Phi(\mathscr{C})$, where $\mathscr{C}$ is a compact subset of the set $\mathfrak{S}(\mathscr{H})$.

Remark 3. The main assertion of Theorem 1 (the implication (i) $\Longrightarrow$ (ii)) is based on the specific properties of the von Neumann entropy, it cannot be proved using only general properties of entropy type functions such as concavity, lower semicontinuity, etc. The simplest example confirming this assertion is the function

$$A \mapsto R_0(\Phi(A)) \doteq \|\Phi(A)\|_1 \log \operatorname{rank}(\Phi(A)),$$

the output 0-order Renyi entropy of the map Φ.

Lemma 3 plays a key role in the proof of Theorem 1; it is based on results related to the notion of the χ-capacity of subsets of quantum states [15].

Remark 4. Statement (iv) in Theorem 1 can be considered as a continuity criterion for the output entropy of the map Φ in terms of the dual map Φ^*. Using this criterion we will prove Proposition 3 in the next subsection.

There are some restrictions on the choice of the basis $\{|i\rangle\}$ in statements (iii) and (iv) of Theorem 1, which follow from the proof of this theorem and Lemma 3. Namely, $\{|i\rangle\}$ is the basis of eigenvectors of some operator T in $\operatorname{cl}(\Phi(\mathfrak{S}(\mathscr{H})))$ which commutes with any unitary operator U such that $U\Phi(\mathfrak{S}(\mathscr{H}))U^* \subseteq \Phi(\mathfrak{S}(\mathscr{H}))$. In particular, if the set $\Phi(\mathfrak{S}(\mathscr{H}))$ consists of commuting operators then $\{|i\rangle\}$ is the basis in which these operators have a diagonal matrix. The last remark can be used to 'reformulate' Theorem 1 for the case of a positive map Φ from $\mathfrak{T}(\mathscr{H})$ into ℓ_1 since such a map is naturally identified with a map of the form mentioned in this remark.

Using Theorem 1 we can obtain a continuity condition for the output entropy of quantum channels of the following class.

Example 1. Let G be a compact group, $\{V_g\}_{g\in G}$ a unitary representation of G in a Hilbert space $\mathscr{H}'$, M a positive operator-valued measure (POVM) on G taking values in $\mathfrak{B}(\mathscr{H})$. For a given arbitrary state σ in $\mathfrak{S}(\mathscr{H}')$ consider the quantum channel

$$\Phi_\sigma(A) = \int_G V_g \sigma V_g^* \operatorname{Tr} AM\,(dg), \qquad A \in \mathfrak{T}(\mathscr{H}).$$

For an appropriate choice of the parameters (G, V_g, M, σ) this channel possesses specific properties of infinite-dimensional quantum channels; in particular, it is an entanglement-breaking channel having no Kraus representation with rank one operators [17].[5]

By Theorem 1 and the concavity of the von Neumann entropy the output entropy of the channel Φ_σ is continuous if the state

$$\omega(G, V_g, \sigma) = \int_G V_g \sigma V_g^* \mu_H\,(dg),$$

where μ_H is the Haar measure on the group G, has finite entropy. It is easy to show that this condition is also necessary if the set of probability measures $\{\operatorname{Tr} \rho M(\,\cdot\,)\}_{\rho\in\mathfrak{S}(\mathscr{H})}$ is weakly dense in the set of all probability measures on G.

By inequality (9) Theorem 1 implies the following assertion.

Corollary 1. *Let $\{\Phi_i\}_{i\in I}$ be a finite or countable collection of maps from $\mathfrak{L}^+_{\leqslant 1}(\mathscr{H}, \mathscr{H}')$ such that*

$$\sup_{\rho\in\mathfrak{S}(\mathscr{H})} \sum_{i\in I} \operatorname{Tr} \Phi_i(\rho) < +\infty.$$

The output entropy of the map $\sum_{i\in I} \Phi_i$ is continuous if

$$\sum_{i\in I} H(\Phi_i(\rho)) < +\infty, \quad H(\{\operatorname{Tr} \Phi_i(\rho)\}_{i\in I}) < +\infty, \qquad \rho \in \mathfrak{S}(\mathscr{H}).$$

This is a necessary condition for the output entropy of the map $\sum_{i\in I} \Phi_i$ to be continuous if either the set I is finite or for each $\rho \in \mathfrak{S}(\mathscr{H})$ there is n such that

$$\operatorname{supp} \Phi_i(\rho) \perp \operatorname{supp} \Phi_j(\rho)$$

for all $i, j \geqslant n$, $i \neq j$.

We complete this subsection by considering a commutative variant of Theorem 1, which can be used to analyse the output Shannon entropy of Markov and sub-Markov operators.

[5] An arbitrary finite-dimensional entanglement-breaking channel has Kraus representation with rank one operators [18].

Corollary 2. *Let $\|\phi_{ij}\|$ be a matrix of a positive bounded linear transformation in the Banach space ℓ_1. The following statements are equivalent*:

(i) *the function $\{x_i\} \mapsto H(\{\sum_j \phi_{ij}x_j\}_{i=1}^{+\infty})$ is finite on the set $\mathfrak{P}_{+\infty} \subset (\ell_1)_+$;*

(ii) *the function $\{x_i\} \mapsto H(\{\sum_j \phi_{ij}x_j\}_{i=1}^{+\infty})$ is continuous on the cone $(\ell_1)_+$;*

(iii) *there exists a sequence $\{h_i\}_{i=1}^{+\infty}$ of non-negative numbers such that*

$$\sup_j \sum_{i=1}^{+\infty} h_i\phi_{ij} < +\infty \quad \textit{and} \quad \sum_{i=1}^{+\infty} e^{-h_i} < +\infty.$$

3.2. The case of completely positive maps. The simplest completely positive linear map from $\mathfrak{T}_+(\mathscr{H})$ into $\mathfrak{T}_+(\mathscr{H}')$ has the form $A \mapsto VAV^*$, where V is a linear bounded operator from $\mathscr{H}$ into $\mathscr{H}'$. Using Theorem 1 it is easy to obtain a necessary and sufficient condition for the output entropy of this map to be continuous (its special role is motivated by representations (1) and (2)).

Proposition 3. *Let V be a bounded linear operator from $\mathscr{H}$ into $\mathscr{H}'$. The function $A \mapsto H(VAV^*)$ is continuous on the cone $\mathfrak{T}_+(\mathscr{H})$ if and only if the operator V is compact and has a sequence of singular values $\{\nu_i\}$ (eigenvalues of the operator $\sqrt{V^*V}$) such that $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} < +\infty$ for some $\lambda > 0$ ($e^{-\lambda/0} \doteq 0$). If this condition holds, then*

$$\sup_{\rho\in\mathfrak{S}(\mathscr{H})} H(V\rho V^*) = \lambda^*(V), \tag{19}$$

where $\lambda^*(V)$ is the unique solution of the equation $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} = 1$ if it exists and $\lambda^*(V) = \mathrm{g}(\{\nu_i^{-2}\}) = \inf\{\lambda > 0 \mid \sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} < +\infty\}$ otherwise.[6]

In what follows we shall use the parameter $\lambda^*(V)$ for an arbitrary operator $V \in \mathfrak{B}(\mathscr{H})$, under the assumption that $\lambda^*(V) = +\infty$ if either the operator V is not compact or it has a sequence of singular values $\{\nu_i\}$ such that $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} = +\infty$ for all $\lambda > 0$. By Theorem 1 relation (19) holds in this case as well.

Proof. We can assume that $\mathscr{H} = \mathscr{H}'$, $V = \sqrt{V^*V}$, $\|V\| \leqslant 1$ and $\operatorname{Ker} V = \{0\}$.

Let $V = \sum_{i=1}^{+\infty} \nu_i|i\rangle\langle i|$. If $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} < +\infty$ for some $\lambda > 0$ then statement (iv) in Theorem 1 holds for the basis $\{|i\rangle\}_{i=1}^{+\infty}$ and the sequence $\{h_i = \lambda\nu_i^{-2}\}_{i=1}^{+\infty}$ (since in this case $\Phi^*(\,\cdot\,) = V(\,\cdot\,)V$ and hence $\Phi^*(|i\rangle\langle i|) = \nu_i^2|i\rangle\langle i|$).

The assertion concerning the supremum of the function $A \mapsto H(VAV)$ on the set $\mathfrak{S}(\mathscr{H})$ is easily proved using Lemma 6 and inequality (12).

If the function $A \mapsto H(VAV)$ is continuous on the cone $\mathfrak{T}_+(\mathscr{H})$, then the entropy is bounded on the convex set $\{V\rho V \mid \rho \in \mathfrak{S}(\mathscr{H})\}$ and hence this set is relatively compact by Corollary 7 in [15], Part I (used with the construction from the proof of Lemma 3). Thus the operator V is compact (since otherwise there exists a sequence of unit vectors $\{|\varphi_n\rangle\}$ such that the sequence $\{V|\varphi_n\rangle\}$ is not relatively compact). Lemma 3 implies an operator $T \in \mathfrak{T}_1(\mathscr{H})$ exists such that $\sup_{\rho\in\mathfrak{S}(\mathscr{H})} \operatorname{Tr} V\rho V(-\log T) < +\infty$ and $UT = TU$ for any unitary operator U commuting with the operator V. This last property of the operator T shows

[6] If $\mathrm{g}(\{\nu_i^{-2}\}) < +\infty$, the equation $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} = 1$ has no solutions if $\sum_{i=1}^{+\infty} e^{-\mathrm{g}(\{\nu_i^{-2}\})/\nu_i^2} < 1$.

that this operator is diagonalizable in the basis $\{|i\rangle\}_{i=1}^{+\infty}$, that is, $T=\sum_{i=1}^{+\infty}\tau_i|i\rangle\langle i|$, where $\{\tau_i\}_{i=1}^{+\infty}$ is a sequence of non-negative numbers such that $\sum_{i=1}^{+\infty}\tau_i\leqslant 1$. Thus

$$\sup_{\rho\in\mathfrak{S}(\mathscr{H})}\operatorname{Tr}V\rho V(-\log T)=\sup_{\rho\in\mathfrak{S}(\mathscr{H})}\sum_{i=1}^{+\infty}\langle i|\rho|i\rangle\nu_i^2(-\log\tau_i)=\lambda<+\infty$$

and hence $\nu_i^2(-\log\tau_i)\leqslant\lambda$ for all i. This shows that $\lambda^*(V)<+\infty$.

In Remark 2 we show that the continuity of the quantum entropy on the set $\Phi(\mathfrak{S}(\mathscr{H}))$ is not a necessary condition for the output entropy of the map Φ to be continuous. By Proposition 3 the map $A\mapsto VAV^*$, where

$$V=\sum_{i>1}(\log(i))^{-1/2}|i\rangle\langle i|,$$

gives another example confirming this observation, since it is easy to see that the classical entropy is discontinuous on the set $\{\{(\log(i))^{-1}x_i\}_{i>1}\mid\{x_i\}_{i>1}\in\mathfrak{P}_{+\infty}\}$.

An arbitrary quantum operation $\Phi\colon\mathfrak{T}(\mathscr{H})\to\mathfrak{T}(\mathscr{H}')$ has Kraus representation (1). Proposition 3 and Corollary 1 imply a necessary and sufficient condition for the output entropy of a quantum operation having a Kraus representation with a finite number of nonzero summands to be continuous.

Corollary 3. *The output entropy of the map* $\Phi(\,\cdot\,)=\sum_{i=1}^{m}V_i(\,\cdot\,)V_i^*$, *where* $\{V_i\}_{i=1}^m$ *is a finite set of bounded linear operators from* $\mathscr{H}$ *into* $\mathscr{H}'$, *is continuous if and only if* $\lambda^*(V_i)<+\infty$ *for all* $i=1,\dots,m$.

This corollary shows, in particular, that any quantum channel with continuous output entropy has no Kraus representation with a finite number of nonzero summands, since the condition $\sum_{i=1}^{m}V_i^*V_i=I_{\mathscr{H}}$ is inconsistent with the condition that the operators V_i, $i=1,\dots,m$, be compact.

For a quantum operation having Kraus representation (1) with a countable number of nonzero summands the condition '$\lambda^*(V_i)<+\infty$ for all i' is only a necessary condition for the output entropy to be continuous. Using Theorem 1, Proposition 3, Corollary 1 and some other results we can obtain several sufficient conditions for the output entropy of such quantum operation to be continuous in terms of its Kraus operators. These conditions, as well as examples where they are applied, are considered in [19], § 3.2.

Applications in quantum information theory and the results of the next section (see the remark after Corollary 6 and Corollary 8) require conditions which ensure the continuity of the output entropy of the quantum operation complementary to the initial one. Theorem 1 yields the following conditions.

Proposition 4. *Let*

$$\Phi(\,\cdot\,)=\sum_{i=1}^{+\infty}V_i(\,\cdot\,)V_i^*$$

be a quantum operation in $\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$. *The complementary operation* $\widetilde{\Phi}$ *has continuous output entropy if one of the following conditions holds* (*they are related by the implications* c $\Longrightarrow$ b $\Longleftrightarrow$ a):

a) $H(\{\operatorname{Tr} V_i\rho V_i^*\}_{i=1}^{+\infty}) < +\infty$ *for all* $\rho \in \mathfrak{S}(\mathscr{H})$;
b) *there exists a sequence* $\{h_i\}_{i=1}^{+\infty}$ *of non-negative numbers such that*

$$\left\|\sum_{i=1}^{+\infty} h_i V_i^* V_i\right\| < +\infty, \qquad \sum_{i=1}^{+\infty} e^{-h_i} < +\infty;$$

c)

$$H(\{\|V_i\|^2\}_{i=1}^{+\infty}) < +\infty.$$

If $\operatorname{Ran} V_i \perp \operatorname{Ran} V_j$ *for all sufficiently large* $i \neq j$, *then* a $\Longleftrightarrow$ b *is a necessary condition for the output entropy of the quantum operation* $\widetilde{\Phi}$ *to be continuous.*

Proof. By Theorem 1 condition a) is equivalent to the continuity of the output entropy of the map

$$\mathfrak{T}(\mathscr{H}) \ni A \mapsto \Psi(A) = \sum_{i=1}^{+\infty} \operatorname{Tr} V_i A V_i^* |i\rangle\langle i| \in \mathfrak{T}(\mathscr{H}''),$$

where $\{|i\rangle\}$ is the orthonormal basis from representation (4) of the quantum operation $\widetilde{\Phi}$. Thus the continuity of the output entropy of the quantum operation $\widetilde{\Phi}$ follows from condition a) by Proposition 2.

The equivalence of conditions a) and b) follows because (ii) and (iv) in Theorem 1 are equivalent by Remark 4, since the map dual to the map Ψ has the form

$$\Psi^*(B) = \sum_{i=1}^{+\infty} \langle i|B|i\rangle V_i^* V_i, \qquad B \in \mathfrak{B}(\mathscr{H}'').$$

That c) $\Longrightarrow$ b) is obvious, because if condition c) holds then the sequence $\{h_i = -\log\|V_i\|^2\}$ has the desired properties.

Let $\operatorname{Ran} V_i \perp \operatorname{Ran} V_j$ for all $i, j \geqslant n$, $i \neq j$. Then

$$P\widetilde{\Phi}(\,\cdot\,)P = \sum_{i=n}^{+\infty} \operatorname{Tr} V_i(\,\cdot\,)V_i^* |i\rangle\langle i|,$$

where $P = \sum_{i=n}^{+\infty} |i\rangle\langle i|$ and $\{|i\rangle\}_{i=1}^{+\infty}$ is the basis from representation (4) of the quantum operation $\widetilde{\Phi}$. Using (10), it follows from $H(\widetilde{\Phi}(\rho)) < +\infty$ that

$$H(P\widetilde{\Phi}(\rho)P) = H(\{\operatorname{Tr} V_i\rho V_i^*\}_{i=n}^{+\infty}) < +\infty$$

for any $\rho \in \mathfrak{S}(\mathscr{H})$, which is equivalent to condition a).

Condition c) shows that the output entropy of the complementary quantum operation will be continuous if the rate at which the sequence of norms of the Kraus operators of the initial operation decreases is sufficiently fast. But this condition is rather rough, since it does not take into account the 'geometry' of this sequence, that is, the mutual relations between the Kraus operators. This is illustrated by the following example.

Example 2. Let $\{V_i\}_{i=1}^{+\infty}$ be a sequence of operators from $\mathfrak{B}(\mathscr{H})$ such that

$$\sum_{i=1}^{+\infty} V_i^*V_i \leqslant I_{\mathscr{H}}, \qquad \operatorname{Ran} V_i^* \perp \operatorname{Ran} V_j^*$$

for all sufficiently large $i \neq j$ and $\|V_i\|^2 \leqslant C\log^{-\alpha}(i)$ for all i, where $\alpha \geqslant 0$ and $C > 0$. Since $V_i^*V_i \leqslant C\log^{-\alpha}(i)P_i$, where P_i is the projector on the subspace $\operatorname{Ran} V_i^*$, condition b) of Proposition 4 is valid for the quantum operation $\Phi_\alpha(\,\cdot\,) = \sum_{i=1}^{+\infty} V_i(\,\cdot\,)V_i^*$ if $\alpha \geqslant 1$ (one can take the sequence $\{h_i = \log(i)\}$). Hence the output entropy of the complementary operation $\widetilde{\Phi}_\alpha$ is continuous if $\alpha \geqslant 1$.

The last assertion of Proposition 4 shows that the output entropy of the quantum operation $\widetilde{\Phi}_\alpha$ is not continuous if $\alpha < 1$ and $V_i = \sqrt{C\log^{-\alpha}(i)}P_i$.

§ 4. Preserving the continuity of the entropy

The output entropy of a positive linear map being continuous on the whole cone of input operators is a very strong property provided by the special features of this map. In this section we consider a significantly weaker property of positive linear maps, namely the continuity of the output entropy on any subset of the cone of input operators on which the quantum entropy is continuous. To study this property we will use the method for approximating concave lower semicontinuous functions described briefly at the end of § 2.

Let $\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$ be a positive linear map and $H_\Phi \doteq H\circ\Phi$ be its output entropy, a concave lower semicontinuous non-negative function on the cone $\mathfrak{T}_+(\mathscr{H})$. For each natural k consider the concave function

$$H_\Phi^k(A) \doteq \sup_{\{\pi_i, A_i\}\in\mathscr{P}^{\mathrm{a}}_{\{A\}}(\mathfrak{T}_+^k(\mathscr{H}))} \sum_i \pi_i H_\Phi(A_i) \tag{20}$$

on the cone $\mathfrak{T}_+(\mathscr{H})$ (the supremum is taken over all decompositions of the operator A into a countable convex combination of operators with rank $\leqslant k$). Using (6) it is easy to show that the restriction of the function H_Φ^k to the set $\mathfrak{S}(\mathscr{H})$ coincides with the function $(\widehat{H_\Phi})_k^\sigma$ defined by formula (16) with $f = H_\Phi$ and that

$$H_\Phi^k(\lambda A) = \lambda H_\Phi^k(A), \qquad A\in\mathfrak{T}_+(\mathscr{H}), \quad \lambda\geqslant 0.$$

Thus the results presented at the end of § 2 show that the function H_Φ^k is lower semicontinuous on the cone $\mathfrak{T}_+(\mathscr{H})$ for each k and that the increasing sequence $\{H_\Phi^k\}$ converges pointwise to the function H_Φ. We will call the function H_Φ^k the *approximator of the output entropy of the map* Φ *of order* k.

Using spectral decomposition we can show that the sequence $\{H_\Phi^k\}$ converges uniformly to the function H_Φ on those compact subsets of the cone $\mathfrak{T}_+(\mathscr{H})$ on which the quantum entropy is continuous.

Lemma 4. *If the quantum entropy is continuous on a compact subset* $\mathscr{A}$ *of the cone* $\mathfrak{T}_+(\mathscr{H})$ *then*

$$\lim_{k\to+\infty}\ \sup_{A\in\mathscr{A},\, \Phi\in\mathfrak{L}^+_{\leqslant 1}(\mathscr{H},\mathscr{H}')} (H_\Phi(A) - H_\Phi^k(A)) = 0. \tag{21}$$

Proof. We can assume that $\mathscr{A}\subset\mathfrak{T}_1(\mathscr{H})$. Let $\lambda_i^k(A)$ be the sum of the eigenvalues $\lambda_{(i-1)k+1},\dots,\lambda_{ik}$ of the operator A (arranged in nonincreasing order) and let P_i^k be the spectral projection of this operator corresponding to the above collection of eigenvalues. Since the ensemble $\{\pi_i^k,(\pi_i^k)^{-1}P_i^kA\}$, where $\pi_i^k=\|A\|_1^{-1}\lambda_i^k(A)$, belongs to the set $\mathscr{P}^{\mathrm{a}}_{\{A\}}(\mathfrak{T}_+^k(\mathscr{H}))$, using inequality (9) and the monotonicity property (7) we obtain

$$\begin{aligned}H_\Phi(A)-H_\Phi^k(A)&\leqslant H(\Phi(A))-\sum_i\pi_i^kH\big(\Phi((\pi_i^k)^{-1}P_i^kA)\big)\\&=H(\Phi(A))-\sum_i H(\Phi(P_i^kA))\leqslant H(\{\operatorname{Tr}\Phi(P_i^kA)\})\leqslant H(\{\lambda_i^k(A)\})\end{aligned}$$

for any map Φ in $\mathfrak{L}^+_{\leqslant 1}(\mathscr{H},\mathscr{H}')$. Thus the assertion of the lemma follows from Lemma 9 in [16] showing that $\lim_{k\to+\infty}\sup_{A\in\mathscr{A}}H(\{\lambda_i^k(A)\})=0$.

Note that the concavity of the function $\eta(x)=-x\log x$ implies that

$$H_\Phi(A)-H_\Phi^k(A)\leqslant\inf_{\{\pi_i,A_i\}\in\mathscr{P}^{\mathrm{a}}_{\{A\}}(\mathfrak{T}_+^k(\mathscr{H}))}\sum_i\pi_iH(\Phi(A_i)\|\Phi(A)),\qquad A\in\mathfrak{T}_+(\mathscr{H}),$$

which shows that (21) holds for any subset $\mathscr{A}$ of the cone $\mathfrak{T}_+(\mathscr{H})$ possessing the UA-property (it need not be compact) if the set $\mathfrak{L}^+_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ of all positive maps is replaced by the set $\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ of all quantum operations (or by any other set of positive linear maps for which the relative entropy satisfies the monotonicity relation given in (15).

Remark 5. Since the function H_Φ^k is lower semicontinuous on the cone $\mathfrak{T}_+(\mathscr{H})$ for each k, the generalized Dini lemma (in which the condition that the functions in the increasing sequence are continuous is replaced by the condition that they are lower semicontinuous) shows that if H_Φ is continuous on a compact set $\mathscr{A}\subset\mathfrak{T}_+(\mathscr{H})$ then the sequence $\{H_\Phi^k\}$ is uniformly convergent to the function H_Φ on this set. The converse assertion obviously holds if the functions H_Φ^k are continuous on the set $\mathscr{A}$ for all k.

The above observations give the following result, which answers the second question stated in § 1.

Theorem 2. *Let Φ be a map in $\mathfrak{L}^+_{\leqslant 1}(\mathscr{H},\mathscr{H}')$. The following properties are equivalent*:

(i) *the function $A\mapsto H_\Phi(A)$ is continuous on the cone*

$$\mathfrak{T}_+^1(\mathscr{H})=\{A\in\mathfrak{T}_+(\mathscr{H})\mid\operatorname{rank}A\leqslant 1\};$$

(ii) *the function $A\mapsto H_\Phi^k(A)$ is continuous on the cone $\mathfrak{T}_+(\mathscr{H})$ for each k*;

(iii) *the function $A\mapsto H_\Phi(A)$ is continuous on any subset $\mathscr{A}$ of the cone $\mathfrak{T}_+(\mathscr{H})$ on which the quantum entropy is continuous.*

Property (i) *is equivalent the function $A\mapsto H_\Phi(A)$ being bounded and continuous on the set* $\operatorname{extr}\mathfrak{S}(\mathscr{H})$ *and hence it follows from the* UA-*property of the set* $\Phi(\operatorname{extr}\mathfrak{S}(\mathscr{H}))$.

Proof. (i) $\Longrightarrow$ (ii). We first show that (i) implies the function $A \mapsto H_\Phi(A)$ is continuous on the cone $\mathfrak{T}_+^k(\mathscr{H})$ for each k. Suppose there exists a sequence $\{A_n\} \subset \mathfrak{T}_+^k(\mathscr{H})$ converging to an operator A_0 such that

$$\lim_{n\to+\infty} H_\Phi(A_n) > H_\Phi(A_0). \tag{22}$$

For each $n \in \mathbb{N}$ we have $A_n = \sum_{i=1}^k A_i^n$, where $\{A_i^n\}_{i=1}^k$ is a collection of operators from $\mathfrak{T}_+^1(\mathscr{H})$. Since the set $\{A_n\}_{n\geqslant 0}$ is compact, the compactness criterion for subsets of the cone $\mathfrak{T}_+(\mathscr{H})$ shows relative compactness of the sequence $\{A_i^n\}_n$ for each $i = 1, \ldots, k$ (see [16], Lemma 10). Hence we can assume that

$$\lim_{n\to+\infty} A_i^n = A_i^0 \in \mathfrak{T}_+^1(\mathscr{H})$$

exists for each $i = 1, \ldots, k$. It is clear that $\sum_{i=1}^k A_i^0 = A_0$. It follows from (i) that $\lim_{n\to+\infty} H_\Phi(A_i^n) = H_\Phi(A_i^0)$. Hence Lemma 1 gives a contradiction to (22).

If H_Φ is continuous on $\mathfrak{T}_+^k(\mathscr{H})$ then it is bounded on $\mathfrak{S}_k(\mathscr{H})$ (this can easily be verified by assuming the converse and taking the fact that $0 \in \mathfrak{T}_+^k(\mathscr{H})$ into account). By Corollary 1 in [16] the function H_Φ^k is continuous and bounded on the set $\mathfrak{S}(\mathscr{H})$ (where it coincides with $(\widehat{H_\Phi})_k^\sigma$) and hence this function is continuous on $\mathfrak{T}_+(\mathscr{H})$.

The implication (ii) $\Longrightarrow$ (iii) follows directly from Lemma 4 while (iii) $\Longrightarrow$ (i) is obvious.

The last assertion of the theorem follows from Theorem 2, A) in [16] (since the quantum entropy is bounded on any bounded set possessing the UA-property).

Remark 6. As in Theorem 1, the main assertion of Theorem 2 (the implication (i) $\Longrightarrow$ (iii)) is based on the specific properties of the von Neumann entropy, it cannot be proved if we only use general properties of entropy type functions. In this case the simplest example is again given by the output 0-order Renyi entropy of the map Φ,

$$A \mapsto R_0(\Phi(A)) = \|\Phi(A)\|_1 \log \operatorname{rank}(\Phi(A)).$$

Indeed, if $\Phi(A) = \frac{1}{2}(A + UAU^*)$, where U is a unitary operator having no eigenvectors, then $R_0(\Phi(A)) = \|A\|_1 \log 2$ for all $A \in \mathfrak{T}_+^1(\mathscr{H})$, but the function $A \mapsto R_0(\Phi(A))$ is not continuous on the set $\mathfrak{T}_+^2(\mathscr{H}) \setminus \mathfrak{T}_+^1(\mathscr{H})$, on which $R_0(A) = \|A\|_1 \log 2$.

The second inequality in (9) (see the proof of Lemma 4) and the implication '$\Longleftarrow$' in Lemma 1 play essential roles in the proof of Theorem 2.

Using the same terminology as [19] we introduce the following definition.

Definition 2. Property (iii) in Theorem 2 will be called the PCE-*property*. A positive linear map (quantum operation or quantum channel) Φ possessing this property will be called a PCE-*map* (PCE-*operation* or PCE-*channel*, respectively).

The abbreviation 'PCE' is used here because a map Φ possessing property (iii) in Theorem 2 can be called a map 'preserving the continuity of the entropy'.

The simplest examples of PCE-maps are the completely positive linear maps with Kraus representation (1) consisting of a finite number of nonzero summands, for which property (i) in Theorem 2 can be verified directly.

By the last assertion of Theorem 2, to prove a map Φ has the PCE-property it suffices to show that

$$\Phi(\operatorname{extr}\mathfrak{S}(\mathscr{H})) \subseteq \Lambda(\mathscr{A}),$$

where Λ is a finite composition of transformations preserving the UA-property (see Proposition 4 in [16]), and $\mathscr{A}$ is a compact set on which the entropy is continuous. This gives the following sufficient condition for positive maps to have the PCE-property.

Corollary 4. *A map Φ in $\mathfrak{L}^{+}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ has the* PCE*-property if there exists a separable Hilbert space $\mathscr{K}$, a family $\{A_\psi\}_{\psi\in\mathscr{H},\|\psi\|=1}$ of operators belonging to some compact subset $\mathscr{A}$ of the cone $\mathfrak{T}_+(\mathscr{K})$, on which the quantum entropy is continuous, and a family $\{V_\psi\}_{\psi\in\mathscr{H},\|\psi\|=1}$ of linear contractions from $\mathscr{K}$ into $\mathscr{H}'$ such that $\Phi(|\psi\rangle\langle\psi|) = V_\psi A_\psi V_\psi^*$ for each unit vector ψ in $\mathscr{H}$.*

If Φ is a quantum operation with Kraus representation (1) consisting of k nonzero summands then it is easy to verify that the hypothesis in Corollary 4 holds if we take a k-dimensional Hilbert space $\mathscr{K}$. A nontrivial application of Corollary 4 is proving that the following family of quantum channels has the PCE-property.

Example 3. Let $\mathscr{H}_a$ be the Hilbert space $\mathscr{L}_2([-a,+a])$, where $a < +\infty$, and let $\{U_t\}_{t\in\mathbb{R}}$ be the group of unitary operators in $\mathscr{H}_a$ defined by

$$(U_t\varphi)(x) = e^{-\mathrm{i}tx}\varphi(x), \qquad \varphi \in \mathscr{H}_a.$$

For a given probability density function $p(t)$ consider the quantum channel

$$\Phi^a_p\colon \mathfrak{T}(\mathscr{H}_a) \ni A \mapsto \int_{-\infty}^{+\infty} U_t A U_t^* p(t)\,dt \in \mathfrak{T}(\mathscr{H}_a).$$

In Appendix 5.2 in [20] it is shown that the channel Φ^a_p satisfies the hypothesis of Corollary 4 with the Hilbert space $\mathscr{K} = \mathscr{L}_2(\mathbb{R})$ and a particular family of unitary operators $\{V_\psi\}$ from $\mathscr{K}$ into $\mathscr{H}_a$ provided that the differential entropy of the distribution $p(t)$ is finite and the function $p(t)$ is bounded on $\mathbb{R}$ and monotonic on $(-\infty,-b]$ and $[+b,+\infty)$ for sufficiently large b.

If the PCE-property holds for two positive maps then it also holds for their composition. Hence Theorem 2 implies the following result.

Corollary 5. *If property* (i) *in Theorem 2 holds for the positive linear maps*

$$\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}') \quad \textit{and} \quad \Psi\colon \mathfrak{T}(\mathscr{H}') \to \mathfrak{T}(\mathscr{H}''),$$

then it holds for the map $\Psi\circ\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}'')$.

In quantum information theory the notion of the convex closure of the output entropy (CCoOE) of a quantum channel is used. It is defined as the maximal convex closed (that is, lower semicontinuous) function on the set of input states of this channel not exceeding the output entropy, see [20], [21]. By generalizing the proof of Proposition 2 in [20] we can show that property (i) in Theorem 2 is equivalent to the CCoOE of the map Φ being continuous and bounded on the set $\mathfrak{S}(\mathscr{H})$. Thus Corollary 5 shows that if the CCoOE of the positive linear maps $\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$

and $\Psi\colon \mathfrak{T}(\mathscr{H}') \to \mathfrak{T}(\mathscr{H}'')$ are continuous and bounded then so is the CCoOE of the map $\Psi\circ\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}'')$.

If Φ is a quantum operation (a completely positive trace-nonincreasing linear map) having representation (2) then the complementary operation $\widetilde{\Phi}$ has representation (3). Since by Lemma 2 the output entropies of the quantum operations Φ and $\widetilde{\Phi}$ coincide on the set of rank one operators, Theorem 2 implies the following assertion.

Corollary 6. *A quantum operation Φ has the* PCE-*property if and only if the complementary operation $\widetilde{\Phi}$ has the* PCE-*property.*

According to this corollary to prove a quantum operation Φ has the PCE-property it suffices to show that the output entropy of the complementary operation $\widetilde{\Phi}$ is continuous, which can be done using the sufficient conditions in Proposition 4.

§ 5. The output entropy of complementary completely positive maps

The output entropies of two complementary quantum operations (completely positive trace-nonincreasing linear maps related via representations (2) and (3)) coincide on the set of input operators of rank 1 (by Lemma 2), but in general these are different functions on the cone of input operators, whose analytical properties may be essentially different (one can confirm this remark by considering the identity map, since its complementary map is the completely depolarizing map [3], Example 6.4.1). Nevertheless the following relationship between local continuity properties of these functions holds.

Theorem 3. *Let*

$$\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}') \quad \textit{and} \quad \widetilde{\Phi}\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}'')$$

be complementary quantum operations and let $\mathscr{A}$ be a subset of the cone $\mathfrak{T}_+(\mathscr{H})$ on which $\min\{H_\Phi(A), H_{\widetilde{\Phi}}(A)\} < +\infty$. If the quantum entropy is continuous on the set $\mathscr{A}$ then $A \mapsto (H_\Phi(A) - H_{\widetilde{\Phi}}(A))$ is continuous on the set $\mathscr{A}$.

Remark 7. It follows from (10), where V is the contraction from representations (2) and (3), and (13) that

$$|H_\Phi(A) - H_{\widetilde{\Phi}}(A)| \leqslant H(A), \qquad A \in \mathfrak{T}_+(\mathscr{H}). \tag{23}$$

Theorem 3 shows that if the right-hand side of this inequality is continuous on a particular subset of the cone $\mathfrak{T}_+(\mathscr{H})$ then the expression within the modulus sign in the left-hand side is continuous on this subset. The condition

$$\min\{H_\Phi(A), H_{\widetilde{\Phi}}(A)\} < +\infty$$

in Theorem 3 can be removed if we include operators A such that

$$H_\Phi(A) = H_{\widetilde{\Phi}}(A) = +\infty \quad \text{but} \quad H(A) < +\infty$$

in the domain of the function $A \mapsto (H_\Phi(A) - H_{\widetilde{\Phi}}(A))$.

Remark 8. If Φ is a quantum channel then $H_\Phi(\rho) - H_{\widetilde{\Phi}}(\rho)$ is the coherent information $I_c(\rho, \Phi)$ of this channel at the state ρ; see [3], [14].

Proof. Let $\{\rho_n\}$ be a sequence of states from $\mathfrak{S}(\mathscr{H})$ converging to a state ρ_0 such that $H(\rho_n) < +\infty$ and $\min\{H_\Phi(\rho_n), H_{\widetilde{\Phi}}(\rho_n)\} < +\infty$ for all $n \geqslant 0$, and $\lim_{n\to+\infty} H(\rho_n) = H(\rho_0)$. Inequality (23) implies the values of $H_\Phi(\rho_n)$ and $H_{\widetilde{\Phi}}(\rho_n)$ are finite for all $n \geqslant 0$.

Let $a_n = H_\Phi(\rho_n) - H_{\widetilde{\Phi}}(\rho_n)$ for each $n \geqslant 0$. By symmetry, to prove $\lim_{n\to+\infty} a_n = a_0$ it suffices to show that

$$\liminf_{n\to+\infty} a_n \geqslant a_0. \tag{24}$$

Let $\mathscr{K}$ be a separable Hilbert space and $\{|\varphi_n\rangle\}$ a sequence of unit vectors from $\mathscr{H} \otimes \mathscr{K}$ converging to the vector $|\varphi_0\rangle$ such that $\operatorname{Tr}_{\mathscr{K}} |\varphi_n\rangle\langle\varphi_n| = \rho_n$ for all $n \geqslant 0$ (see Lemma 3 in [16]). Lemma 2 implies that

$$H(\Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)) = H(\widetilde{\Phi}(\rho_n)) \tag{25}$$

for each n. Indeed, by representation (2) the operator $\Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)$ coincides with the partial trace of the operator $V \otimes I_{\mathscr{K}} |\varphi_n\rangle\langle\varphi_n| V^* \otimes I_{\mathscr{K}}$ in $\mathfrak{T}_+^1(\mathscr{H}' \otimes \mathscr{K} \otimes \mathscr{H}'')$ over the space $\mathscr{H}''$, while by representation (3) the operator $\widetilde{\Phi}(\rho_n)$ coincides with the partial trace of the same operator over the space $\mathscr{H}' \otimes \mathscr{K}$.

As the values $H_\Phi(\rho_n)$ and $H_{\widetilde{\Phi}}(\rho_n)$ are finite, by (11) and (25) we obtain

$$\begin{aligned} b_n &\doteq H\bigl(\Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|) \| \Phi(\rho_n) \otimes \rho_n\bigr) \\ &= \operatorname{Tr} \Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)\bigl(-\log(\Phi(\rho_n) \otimes \rho_n)\bigr) \\ &\qquad - \operatorname{Tr} \Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)\bigl(-\log(\Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|))\bigr) \\ &= H(\Phi(\rho_n)) + c_n - H(\Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)) = a_n + c_n, \end{aligned}$$

where $c_n = \operatorname{Tr} \Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)(I_{\mathscr{H}'} \otimes (-\log \rho_n))$.

The relative entropy in both arguments is lower semicontinuous and so

$$\liminf_{n\to+\infty} b_n \geqslant b_0.$$

Thus one can prove (24) by showing that

$$\limsup_{n\to+\infty} c_n \leqslant c_0. \tag{26}$$

Consider the quantum channel $\Psi = \Phi + \Delta$, where

$$\Delta(\,\cdot\,) = \sigma \operatorname{Tr}((I_{\mathscr{H}} - \Phi^*(I_{\mathscr{H}'}))(\,\cdot\,))$$

is a quantum operation in $\mathfrak{F}_{\leqslant 1}(\mathscr{H}, \mathscr{H}')$ defined by means of a particular state σ in $\mathfrak{S}(\mathscr{H}')$. Since $\lim_{n\to+\infty} H(\rho_n) = H(\rho_0) < +\infty$ and

$$H(\rho_n) = \operatorname{Tr} \Psi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)(I_{\mathscr{H}'} \otimes (-\log \rho_n)) = c_n + d_n, \qquad n = 0, 1, 2, \dots,$$

where $d_n = \operatorname{Tr} \Delta \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)(I_{\mathscr{H}'} \otimes (-\log \rho_n))$, to prove (26) it suffices to show that

$$\liminf_{n\to+\infty} d_n \geqslant d_0. \tag{27}$$

We have $d_n = \operatorname{Tr} B_n(-\log \rho_n)$, where $B_n = \operatorname{Tr}_{\mathscr{H}'} \Delta \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|)$ is an operator in $\mathfrak{T}_+(\mathscr{K})$. Since $B_n \leqslant B_n + \operatorname{Tr}_{\mathscr{H}'} \Phi \otimes \mathrm{Id}_{\mathscr{K}}(|\varphi_n\rangle\langle\varphi_n|) = \rho_n$, the value $H(B_n)$ is finite and hence $d_n = H(B_n) + H(B_n \| \rho_n) + \eta(\operatorname{Tr} B_n) + \operatorname{Tr} B_n - 1$. Since the quantum entropy and the relative entropy are lower semicontinuous, this gives (27).

Thus the assertion of the theorem is proved in the case $\mathscr{A} \subset \mathfrak{S}(\mathscr{H})$. The general assertion is easily reduced to this case using property (6), since for an arbitrary sequence $\{A_n\}$ converging to zero inequality (23) implies that

$$\lim_{n\to+\infty} H(A_n) = 0 \quad \Longrightarrow \quad \lim_{n\to+\infty} (H_\Phi(A_n) - H_{\widetilde{\Phi}}(A_n)) = 0.$$

Corollary 7. *Let*

$$\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$$

be a quantum channel and

$$\widetilde{\Phi}\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}'')$$

the complementary channel. If any two functions from the triple $\{H, H_\Phi, H_{\widetilde{\Phi}}\}$ *are continuous on a set* $\mathscr{A} \subset \mathfrak{T}_+(\mathscr{H})$ *then the third function is also continuous on this set.*

This assertion holds for any quantum operation $\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$ *such that*

$$\lambda^*\bigl(\sqrt{I_{\mathscr{H}} - \Phi^*(I_{\mathscr{H}'})}\,\bigr) < +\infty.^7 \tag{28}$$

Since $\Phi^*(I_{\mathscr{H}'}) \leqslant I_{\mathscr{H}}$ for any quantum operation Φ and the equality in this inequality holds if and only if Φ is a quantum channel, (28) can be treated as a condition on the 'closeness' of the quantum operation to a quantum channel, which provides the corresponding behaviour of its output entropy. This condition is symmetric with respect to $(\Phi, \widetilde{\Phi})$ since $\Phi^*(I_{\mathscr{H}'}) = \widetilde{\Phi}^*(I_{\mathscr{H}''})$ by representations (2), (3) and (5).

Proof. By representations (2) and (3) the first assertion in the corollary can be derived directly from Theorem 3 and Proposition 5.

The second assertion in the corollary is derived from the first by means of Lemma 5 below since by representations (2), (3) and (5) we have $\Phi = \Theta \circ \Lambda$, $\widetilde{\Phi} = \widetilde{\Theta} \circ \Lambda$ and $\Phi^*(I_{\mathscr{H}'}) = V^*V$, where $\Theta(\,\cdot\,) = \operatorname{Tr}_{\mathscr{H}''}(\,\cdot\,)$ is a quantum channel from $\mathfrak{T}(\mathscr{H}' \otimes \mathscr{H}'')$ into $\mathfrak{T}(\mathscr{H}')$ and $\Lambda(\,\cdot\,) = V(\,\cdot\,)V^*$ is a quantum operation from $\mathfrak{T}(\mathscr{H})$ into $\mathfrak{T}(\mathscr{H}' \otimes \mathscr{H}'')$.

Lemma 5. *If V is a linear contraction from $\mathscr{H}$ into $\mathscr{H}'$ such that*

$$\lambda^*\bigl(\sqrt{I_{\mathscr{H}} - V^*V}\,\bigr) < +\infty,$$

then the quantum entropy is continuous on the set $\mathscr{A} \subset \mathfrak{T}_+(\mathscr{H})$ *provided that the function* $A \mapsto H(VAV^*)$ *is continuous on this set.*

Note that the converse assertion holds for an arbitrary linear contraction V by Theorem 2.

[7] The parameter $\lambda^*(\,\cdot\,)$ is defined in Proposition 3.

Proof. Consider the quantum channel

$$\mathfrak{T}(\mathscr{H}) \ni A \mapsto \Psi(A) = VAV^* \oplus \sqrt{I_{\mathscr{H}} - V^*V}A\sqrt{I_{\mathscr{H}} - V^*V} \in \mathfrak{T}(\mathscr{H}' \oplus \mathscr{H}).$$

By Proposition 3 the function $A \mapsto H\bigl(\sqrt{I_{\mathscr{H}} - V^*V}A\sqrt{I_{\mathscr{H}} - V^*V}\,\bigr)$ is continuous on the set $\mathfrak{T}_+(\mathscr{H})$. Hence the function $A \mapsto H_\Psi(A)$ is continuous on the set $\mathscr{A}$. Since the complementary channel $\widetilde{\Psi}$ has a two-dimensional output space (by representation (4)), the assertion of the lemma follows from the first assertion of Corollary 7.

Corollary 8. *Let $\Phi\colon \mathfrak{T}(\mathscr{H}) \to \mathfrak{T}(\mathscr{H}')$ be a quantum channel (or a quantum operation satisfying condition (28)) such that the complementary channel (operation) $\widetilde{\Phi}$ has continuous output entropy. The function $A \mapsto H_\Phi(A)$ is continuous on the set $\mathscr{A} \subset \mathfrak{T}_+(\mathscr{H})$ if and only if the quantum entropy is continuous on this set.*

The condition in Corollary 8 holds for quantum channels having Kraus representation (1) with a finite number of nonzero summands, since their complementary channels have a finite-dimensional output space (by representation (4)).

Sufficient conditions, expressed in terms of the Kraus operators of the quantum operation Φ, for the output entropy of the quantum operation $\widetilde{\Phi}$ to be continuous (which is equivalent to it being finite by Theorem 1) are represented in Proposition 4.

The following assertion can be considered as a generalization of Proposition 2 (since if we apply it to the channel $\Phi(A) = \sum_{i=1}^{+\infty} \langle i|A|i\rangle |i\rangle\langle i|$ it gives the assertion of this proposition).

Corollary 9. *Let $\Phi(\,\cdot\,) = \sum_{i=1}^{+\infty} V_i(\,\cdot\,)V_i^*$ be a quantum channel (or a quantum operation satisfying condition (28)) in $\mathfrak{F}_{\leqslant 1}(\mathscr{H}, \mathscr{H}')$ such that $\operatorname{Ran} V_i \perp \operatorname{Ran} V_j$ for all sufficiently large $i \neq j$. If $A \mapsto H_\Phi(A)$ is continuous on a set $\mathscr{A} \subset \mathfrak{T}_+(\mathscr{H})$ then the quantum entropy is continuous on the set $\mathscr{A}$.*

Proof. Suppose that $\operatorname{Ran} V_i \perp \operatorname{Ran} V_j$ for all $i, j \geqslant n$, $i \neq j$. Consider the quantum operations $\Phi_1(\,\cdot\,) = \sum_{i=1}^{n-1} V_i(\,\cdot\,)V_i^*$ and $\Phi_2(\,\cdot\,) = \sum_{i=n}^{+\infty} V_i(\,\cdot\,)V_i^*$. By Lemma 1 if $\mathscr{A} \ni A \mapsto H(\Phi(A)) = H(\Phi_1(A) + \Phi_2(A))$ is continuous, then so is the function $\mathscr{A} \ni A \mapsto H(\Phi_2(A))$. By hypothesis

$$H(\Phi_2(A)) = \sum_{i=n}^{+\infty} H(V_iAV_i^*) + H(\{\operatorname{Tr} V_iAV_i^*\}_{i=n}^{+\infty}) = \sum_{i=n}^{+\infty} H(V_iAV_i^*) + H(\widetilde{\Phi}_2(A)),$$

where $\widetilde{\Phi}_2(A) = \sum_{i=n}^{+\infty} \operatorname{Tr}[V_iAV_i^*]|i\rangle\langle i|$, and $\{|i\rangle\}$ is the basis from the representation (4) of the operation $\widetilde{\Phi}$. Since both the terms on the right-hand side of this expression are lower semicontinuous functions of the operator A, if $\mathscr{A} \ni A \mapsto H(\Phi_2(A))$ is continuous, so is $\mathscr{A} \ni A \mapsto H(\widetilde{\Phi}_2(A))$.

Consider the quantum channel $\Pi(\,\cdot\,) = P(\,\cdot\,)P + (I_{\mathscr{H}''} - P)(\,\cdot\,)(I_{\mathscr{H}''} - P)$ in $\mathfrak{F}_{=1}(\mathscr{H}'', \mathscr{H}'')$, where $P = \sum_{i=1}^{n-1} |i\rangle\langle i|$. Since

$$\Pi(\widetilde{\Phi}(A)) = \sum_{i,j=1}^{n-1} \operatorname{Tr}[V_iAV_j^*]|i\rangle\langle j| + \widetilde{\Phi}_2(A),$$

if $\mathscr{A}\ni A\mapsto H(\widetilde{\Phi}_2(A))$ is continuous then, by Lemma 1, so is $\mathscr{A}\ni A\mapsto H(\Pi(\widetilde{\Phi}(A)))$, and by Corollary 8 this is equivalent to the function $\mathscr{A}\ni A\mapsto H(\widetilde{\Phi}(A))$ being continuous. Hence Corollary 7 implies that $\mathscr{A}\ni A\mapsto H(A)$ is too.

Remark 9. The assertions of Corollaries 7, 8 and 9 are not valid for a quantum operation Φ unless it satisfies (28). This is confirmed by the following assertion.

Corollary 10. *Let Φ be a quantum operation in $\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ and let*

$$\mathfrak{T}_{\Phi}=\bigl\{A\in\mathfrak{T}_{+}(\mathscr{H})\mid \min\{H_{\Phi}(A),H_{\widetilde{\Phi}}(A)\}<+\infty\bigr\}.$$

If $\lambda^{*}(\sqrt{\Phi^{*}(I_{\mathscr{H}'})})<+\infty$, then the function $A\mapsto(H_{\Phi}(A)-H_{\widetilde{\Phi}}(A))$ is continuous on the cone $\mathfrak{T}_{\Phi}$ and its absolute value does not exceed $\lambda^{*}(\sqrt{\Phi^{*}(I_{\mathscr{H}'})})\|A\|_{1}$.

If the functions $A\mapsto H_{\Phi}(A)$ and $A\mapsto H_{\widetilde{\Phi}}(A)$ are continuous on the cone $\mathfrak{T}_{+}(\mathscr{H})$ then the operator $\Phi^{*}(I_{\mathscr{H}'})$ satisfies the above condition.

Proof. Representation (5) shows that $\Phi^{*}(I_{\mathscr{H}'})=V^{*}V$, where V is the contraction from representation (2) of the quantum operation Φ. Thus using representations (2) and (3) one can derive the first assertion of the corollary from Proposition 3 and Theorem 3, while the second comes from Proposition 3 and Proposition 5.

§ 6. The output entropy as a function of the pair (map, input operator)

When we analyse the physically motivated question about the continuity of the information characteristics of a quantum channel as a function of the channel (that is continuity with respect to 'perturbations' of the channel) we need to consider the output entropy as a function of the pair (channel, input state) and to explore the continuity of this function in the topology of the Cartesian product on the set of such pairs, under the assumption that the set of quantum channels is endowed with an appropriate (sufficiently weak) topology (see [5], [6]). The same problem arises when we study quantum channels by means of an approximation to them by quantum operations with 'good' analytical properties [5].

We will assume that the set $\mathfrak{L}^{+}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ is endowed with the topology of strong convergence generated by the strong operator topology on the set of all bounded linear maps between the Banach spaces $\mathfrak{T}(\mathscr{H})$ and $\mathfrak{T}(\mathscr{H}')$ (the properties of this topology on the set $\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ of quantum operations are investigated in [5]). A sequence $\{\Phi_n\}\in\mathfrak{L}^{+}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ converges to a map $\Phi_0\in\mathfrak{L}^{+}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ in the topology of strong convergence if and only if

$$\lim_{n\to+\infty}\Phi_n(A)=\Phi_0(A),\qquad A\in\mathfrak{T}(\mathscr{H}).$$

Several of our earlier results concerning the properties of the function $A\mapsto H_{\Phi}(A)$ can be generalized to give continuity conditions for the function $(\Phi,A)\mapsto H_{\Phi}(A)$.

The following assertion is a generalization of the main result of Theorem 2.

Theorem 4. *Let $\{\Phi_n\} \subset \mathfrak{L}^{+}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ be a sequence of maps converging to a map Φ_0. The following properties are equivalent:*

(i) $\lim_{n\to+\infty} H_{\Phi_n}(A_n) = H_{\Phi_0}(A_0) < +\infty$ *for any sequence $\{A_n\} \subset \mathfrak{T}^1_+(\mathscr{H})$ converging to the operator A_0;*[8]

(ii) $\lim_{n\to+\infty} H_{\Phi_n}(A_n) = H_{\Phi_0}(A_0) < +\infty$ *for any sequence $\{A_n\} \subset \mathfrak{T}_+(\mathscr{H})$ converging to an operator A_0 such that*

$$\lim_{n\to+\infty} H(A_n) = H(A_0) < +\infty.$$

Theorem 4 implies, in particular, that the function $(\Phi, A) \mapsto H_\Phi(A)$ is continuous on the set $\mathfrak{F}^n_{\leqslant 1}(\mathscr{H},\mathscr{H}') \times \mathscr{A}$ for each $n \in \mathbb{N}$, where $\mathfrak{F}^n_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ is the set of all quantum operations having Kraus representation (1) with the number of nonzero summands $\leqslant n$ and $\mathscr{A}$ is a subset of the cone $\mathfrak{T}_+(\mathscr{H})$ on which the quantum entropy is continuous.

The following assertion is a generalization of Theorem 3.

Theorem 5. *Let $\{\Phi_n\}$ and $\{\widetilde{\Phi}_n\}$ be sequences of quantum operations from $\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}')$ and from $\mathfrak{F}_{\leqslant 1}(\mathscr{H},\mathscr{H}'')$, respectively, converging to operations Φ_0 and $\widetilde{\Phi}_0$ such that $(\Phi_n, \widetilde{\Phi}_n)$ is a complementary pair for each $n = 0, 1, 2, \ldots$ and let $\{A_n\}$ be a sequence of operators from $\mathfrak{T}_+(\mathscr{H})$ converging to an operator A_0 such that*

$$\lim_{n\to+\infty} H(A_n) = H(A_0) < +\infty \quad \textit{and} \quad \min\{H_{\Phi_n}(A_n), H_{\widetilde{\Phi}_n}(A_n)\} < +\infty, \quad n \geqslant 0.$$

Then

$$\lim_{n\to+\infty} \big(H_{\Phi_n}(A_n) - H_{\widetilde{\Phi}_n}(A_n)\big) = H_{\Phi_0}(A_0) - H_{\widetilde{\Phi}_0}(A_0) < +\infty.$$

The hypotheses of Theorem 5 hold in the case

$$\Phi_n(\,\cdot\,) = \sum_{i=1}^{+\infty} V_i^n(\,\cdot\,)(V_i^n)^* \quad \text{and} \quad \widetilde{\Phi}_n(\,\cdot\,) = \sum_{i,j=1}^{+\infty} \operatorname{Tr} V_i^n(\,\cdot\,)(V_j^n)^* |i\rangle\langle j|, \qquad n \geqslant 0,$$

where $\{V_i^n\}_n$ is a sequence of operators from $\mathscr{H}$ into $\mathscr{H}'$ converging strongly to an operator V_i^0 for each i such that

$$\lim_{n\to+\infty} \sum_{i=1}^{+\infty} (V_i^n)^* V_i^n = \sum_{i=1}^{+\infty} (V_i^0)^* V_i^0$$

in the weak operator topology[9] and $\{|i\rangle\}_{i=1}^{+\infty}$ is an orthonormal basis in a separable Hilbert space $\mathscr{H}''$. Using the above generalization of Theorem 3 and Proposition 2 it is easy to obtain the following continuity condition.

[8] $\mathfrak{T}^1_+(\mathscr{H})$ is the cone of positive trace-class operators in $\mathscr{H}$ with rank $\leqslant 1$.

[9] This condition guarantees the sequences $\{\Phi_n\}$ and $\{\widetilde{\Phi}_n\}$ converge to the quantum operations Φ_0 and $\widetilde{\Phi}_0$. It always holds if these sequences consist of quantum channels.

Corollary 11. *Let* $\{A_n\}$ *be a sequence of operators from* $\mathfrak{T}_+(\mathscr{H})$ *converging to an operator* A_0 *such that* $\lim_{n\to+\infty} H(A_n) = H(A_0) < +\infty$*. To prove that*

$$\lim_{n\to+\infty} H_{\Phi_n}(A_n) = H_{\Phi_0}(A_0)$$

it suffices to show that

$$\lim_{n\to+\infty} H\big(\{\operatorname{Tr} V_i^n A_n (V_i^n)^*\}_{i=1}^{+\infty}\big) = H\big(\{\operatorname{Tr} V_i^0 A_0 (V_i^0)^*\}_{i=1}^{+\infty}\big) < +\infty.$$

This condition has a clear physical interpretation in terms of the theory of quantum measurements (see [19], Example 7).

The proofs of all the results presented in this section can be found in [19], § 6.

§ 7. Appendix

Proposition 5. *Let* $\mathscr{C}$ *be a subset of the cone* $\mathfrak{T}_+(\mathscr{H}\otimes\mathscr{K})$*. If the quantum entropy is continuous on the sets* $\operatorname{Tr}_{\mathscr{K}}\mathscr{C}\subset\mathfrak{T}_+(\mathscr{H})$ *and* $\operatorname{Tr}_{\mathscr{H}}\mathscr{C}\subset\mathfrak{T}_+(\mathscr{K})$ *then the quantum entropy is continuous on the set* $\mathscr{C}$*.*

Proof. Let $\{C_n\}\subseteq\mathscr{C}$ be a subsequence converging to an operator $C_0\in\mathscr{C}$. If $C_0\neq 0$, then by hypothesis we have

$$\lim_{n\to+\infty} H\biggl(\frac{C_n^{\mathscr{H}}}{\operatorname{Tr} C_n}\biggr) = H\biggl(\frac{C_0^{\mathscr{H}}}{\operatorname{Tr} C_0}\biggr) \quad\text{and}\quad \lim_{n\to+\infty} H\biggl(\frac{C_n^{\mathscr{K}}}{\operatorname{Tr} C_n}\biggr) = H\biggl(\frac{C_0^{\mathscr{K}}}{\operatorname{Tr} C_0}\biggr),$$

where $C_n^{\mathscr{H}} = \operatorname{Tr}_{\mathscr{K}} C_n$ and $C_n^{\mathscr{K}} = \operatorname{Tr}_{\mathscr{H}} C_n$ for all n. Proposition 8 from [15], Part II shows that

$$\lim_{n\to+\infty} H\biggl(\frac{C_n}{\operatorname{Tr} C_n}\biggr) = H\biggl(\frac{C_0}{\operatorname{Tr} C_0}\biggr)$$

and hence

$$\lim_{n\to+\infty} H(C_n) = H(C_0).$$

If $C_0 = 0$ then the sequence $\{H(C_n)\}$ converges to zero if the sequences $\{H(C_n^{\mathscr{H}})\}$ and $\{H(C_n^{\mathscr{K}})\}$ converge to zero, by the subadditivity of the quantum entropy, that is, using the inequality $H(C_n)\leqslant H(C_n^{\mathscr{H}}) + H(C_n^{\mathscr{K}})$, $n\in\mathbb{N}$.

In the proof of Proposition 3 we used the following lemma.

Lemma 6. *Let* $\{\pi_i\}_{i=1}^{+\infty}$ *be a sequence of positive numbers. Then*

$$\sup_{\{x_i\}\in\mathfrak{P}_{+\infty}} H(\{\pi_i x_i\}_{i=1}^{+\infty}) = \lambda^*,$$

where λ^* *is the unique finite solution of the equation* $\sum_{i=1}^{+\infty} e^{-\lambda/\pi_i} = 1$ *if it exists and* $\lambda^* = \mathrm{g}(\{\pi_i^{-1}\}) = \inf\{\lambda>0 \mid \sum_{i=1}^{+\infty} e^{-\lambda/\pi_i} < +\infty\}$ *otherwise.*[10]

[10] We assume that $\inf\emptyset = +\infty$. The equation $\sum_{i=1}^{+\infty} e^{-\lambda/\pi_i} = 1$ has no solutions if either $\mathrm{g}(\{\pi_i^{-1}\}) = +\infty$ or $\sum_{i=1}^{+\infty} e^{-\mathrm{g}(\{\pi_i^{-1}\})/\pi_i} < 1$.

Proof. Using the Lagrange method it is easy to show that the function

$$\mathfrak{P}_n \ni \{x_i\}_{i=1}^n \mapsto H(\{\pi_i x_i\}_{i=1}^n)$$

attains its maximum at the vector

$$\{x_i^* = c\pi_i^{-1} e^{-\lambda_n^*/\pi_i}\},$$

where λ_n^* is a solution of the equation $\sum_{i=1}^n e^{-\lambda/\pi_i} = 1$ and

$$c = \left[\sum_{i=1}^n \pi_i^{-1} e^{-\lambda_n^*/\pi_i}\right]^{-1}.$$

Hence

$$\max_{\{x_i\}\in\mathfrak{P}_n} H(\{\pi_i x_i\}_{i=1}^n) = \lambda_n^*. \tag{29}$$

Since the classical entropy is lower semicontinuous the assertion of the lemma follows from (29) and the remark that sequence $\{\lambda_n^*\}$ converges to λ^* as $n \to +\infty$.

Applications of the continuity conditions we have obtained for the output entropy of positive maps in quantum information theory are investigated in [19], §§ 7 and 8.

I am grateful to M. B. Ruskai and to the other organizers of the workshop *Thematic Program on Mathematics in Quantum Information* at the Fields Institute (Canada, July–August 2009), where work on this paper began. I am also grateful to A. S. Holevo and to the participants his seminar "Quantum probability, statistics, information" (the Steklov Mathematical Institute) for their interest in this work and the useful discussions we had.

Bibliography

[1] I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators in a Hilbert space*, Nauka, Moscow 1965; English transl., I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators*, Transl. Math. Monogr., vol. 18, Amer. Math. Soc., Providence, RI 1969.

[2] A. S. Holevo, *Statistical structure of quantum theory*, Space Research Institute, Moscow–Izhevsk 2003; English transl., Lect. Notes in Physics. Monographs, vol. 67, Springer-Verlag, Berlin 2001.

[3] A. S. Holevo, *Quantum systems, channels, information*, Moscow Center for Continuous Mathematical Education, Moscow 2010. (Russian)

[4] A. Wehrl, "General properties of entropy", *Rev. Modern Phys.* **50**:2 (1978), 221–260.

[5] M. E. Shirokov and A. S. Holevo, "On approximation of infinite-dimensional quantum channels", *Problemy Peredachi Informatsii* **44**:2 (2008), 3–22; English transl. in *Probl. Inf. Transm.* **44**:2 (2008), 73–90.

[6] D. Leung and G. Smith, "Continuity of quantum channel capacities", *Comm. Math. Phys.* **292**:1 (2009), 201–215.

[7] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*, vol. 1: *C^*- and W^*-algebras, symmetry groups, decomposition of states*, Texts and Monographs in Physics, Springer-Verlag, New York–Heidelberg 1979.

[8] A. D. Ioffe and V. M. Tihomirov, *Theory of extremal problems*, Nauka, Moscow 1974; English transl., Stud. Math. Appl., vol. 6, North-Holland, Amsterdam–New York 1979.

[9] E. S. Polovinkin and M. V. Balashov, *Elements of convex and strongly convex analysis*, 1st ed., Fizmatlit, Moscow 2004 (Russian); 2nd corrected ed., Fizmatlit, Moscow 2007.

[10] C. King, K. Matsumoto, M. Nathanson and M. B. Ruskai, "Properties of conjugate channels with applications to additivity and multiplicativity", *Markov Process. Related Fields* **13**:2 (2007), 391–423.

[11] A. S. Holevo, "Complementary channels and the additivity problem", *Teor. Veroyatnost. i Primenen.* **51**:1 (2006), 133–143; English transl. in *Theory Probab. Appl.* **51**:1 (2007), 92–100.

[12] G. Lindblad, "Expectations and entropy inequalities for finite quantum systems", *Comm. Math. Phys.* **39**:2 (1974), 111–119.

[13] M. Ohya and D. Petz, *Quantum entropy and its use*, Texts Monogr. Phys., Springer-Verlag, Berlin 1993.

[14] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*, Cambridge Univ. Press, Cambridge 2000.

[15] M. E. Shirokov, "Entropy characteristics of subsets of states. I", *Izv. Ross. Akad. Nauk. Ser. Mat.* **70**:6 (2006), 193–222; "II" **71**:1 (2007), 187–224; English transl. in *Izv. Math.* **70**:6 (2006), 1265–1292; "II" **71**:1 (2007), 181–218.

[16] M. E. Shirokov, "Continuity of the von Neumann entropy", *Comm. Math. Phys.* **296**:3 (2010), 625–654.

[17] A. S. Holevo, "Entanglement-breaking channels in infinite dimensions", *Problemy Peredachi Informatsii* **44**:3 (2008), 3–18; English transl. in *Probl. Inf. Transm.* **44**:3 (2008), 171–184.

[18] M. Horodecki, P. W. Shor and M. B. Ruskai, "Entanglement breaking channels", *Rev. Math. Phys.* **15**:6 (2003), 629–641.

[19] M. E. Shirokov, *The output entropy of quantum channels and operations*, arXiv: abs/1002.0230.

[20] M. E. Shirokov, *The convex closure of the output entropy of infinite dimensional channels and the additivity problem*, arXiv: quant-ph/0608090.

[21] K. M. R. Audenaert and S. L. Braunstein, "On strong superadditivity of the entanglement of formation", *Comm. Math. Phys.* **246**:3 (2004), 443–452.

M. E. Shirokov
Steklov Mathematical Institute,
the Russian Academy of Sciences
E-mail: msh@mi.ras.ru

Received 5/APR/10
Translated by M. SHIROKOV