

Reversibility conditions for quantum channels and their applications

M. E. Shirokov

Abstract. Conditions for a quantum channel (noncommutative Markov operator) to be reversible with respect to complete families of quantum states with bounded rank are obtained. A description of all quantum channels reversible with respect to a given (orthogonal or nonorthogonal) complete family of pure states is given. Some applications in quantum information theory are considered.

Bibliography: 20 titles.

Keywords: quantum states, quantum channels, relative entropy, χ -quantity of ensemble of quantum states.

§ 1. Introduction

Quantum channels are linear trace-preserving completely positive maps between Banach spaces of trace class operators (Schatten class of order 1), and they play the role of dynamical maps in quantum theory [1]. They can be considered as noncommutative analogues of Markov operators in classical probability theory.

The notion of reversibility (sufficiency) of a quantum channel with respect to a given family of states arose in the 1980s, in the analysis of various general questions in quantum theory, in particular, the question of preserving certain quantitative characteristics of states under the action of a quantum channel [2]–[4]. The reversibility of a quantum channel Φ with respect to a family of input states \mathfrak{S} means there exists a quantum channel Ψ from the output space of the channel Φ into its input space such that $\Psi(\Phi(\rho)) = \rho$ for any state ρ in \mathfrak{S} .

The notion of reversibility of a quantum channel is related to Petz's theorem, which states that equality holds in the general inequality

$$H(\Phi(\rho) \parallel \Phi(\sigma)) \leq H(\rho \parallel \sigma)$$

(this expresses the fundamental monotonicity property of the quantum relative entropy $H(\rho \parallel \sigma)$ of the states ρ and σ under the action of the quantum channel Φ) if and only if Φ is reversible with respect to the family $\mathfrak{S} = \{\rho, \sigma\}$ ([2], Theorem 5).

The work was partially supported by the research programme of the Russian Academy of Sciences, “Mathematical control theory and dynamical systems”, and by the Russian Foundation for Basic Research (grant nos. 12-01-00319-a and 13-01-00295-a).

AMS 2010 *Mathematics Subject Classification*. Primary 81P45, 81P68; Secondary 46L60, 46N50.

An important corollary to this theorem is related to the notion of the χ -quantity¹ $\chi(\{\pi_i, \rho_i\})$ of an ensemble of quantum states $\{\pi_i, \rho_i\}$ (a collection of states $\{\rho_i\}$ with the corresponding probability distribution $\{\pi_i\}$). It plays a key role in quantum information theory. This corollary states that a necessary and sufficient condition for the χ -quantity of an ensemble $\{\pi_i, \rho_i\}$ to be preserved under the action of a quantum channel Φ , that is, for there to be equality in the general inequality

$$\chi(\{\pi_i, \Phi(\rho_i)\}) \leq \chi(\{\pi_i, \rho_i\}),$$

is that the channel Φ be reversible with respect to the family $\mathfrak{S} = \{\rho_i\}$.

It has recently been shown that criteria for many other important characteristics to be preserved under the action of a quantum channel also reduce to the reversibility condition [5], [6]. It was also observed that necessary and sufficient conditions for the capacities of two different transmission protocols for classical information through a quantum channel Φ to coincide can be formulated in terms of the reversibility of the complementary channel $\widehat{\Phi}$ (for the definition see §2) with respect to certain families of pure states [7].

Starting with Petz’s works, conditions for quantum channels (and more general maps) to be reversible with respect to families of two states and many states have been studied by many authors (see [6], [8] and the references there). A distinguishing feature of this paper is that in the analysis of reversibility we use results related to the notion of a complementary channel. The importance of its role in various questions in quantum information theory was observed recently [9].

The paper is organized as follows. In §§2,3 we give an overview of the necessary concepts and preliminary results. In §4 we prove necessary conditions for a quantum channel to be reversible with respect to families of bounded rank states expressed in terms of the complementary channel (Theorem 3) and give a description of the class of quantum channels reversible with respect to a given family of pure states (orthogonal and nonorthogonal) possessing the completeness property (Proposition 1 and Theorem 4). In §5 we look at some applications of these results in quantum information theory. A condition for the χ -quantity of an arbitrary (discrete or continuous) ensemble of bounded rank states to be preserved under the action of a quantum channel is obtained (Theorem 5) and some its corollaries are considered.

We present a possible generalization of Petz’s theorem to the case of arbitrary quantum states in the Appendix.

§2. Preliminaries

Let \mathcal{H} be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ the Banach spaces of all bounded operators in \mathcal{H} with the operator norm $\|\cdot\|$ and all trace-class operators in \mathcal{H} with the trace norm $\|\cdot\|_1 = \text{Tr}|\cdot|$ [1], [10]. The closed convex subset

$$\mathfrak{S}(\mathcal{H}) = \{A \in \mathfrak{T}(\mathcal{H}) \mid A \geq 0, \text{Tr} A = 1\}$$

of $\mathfrak{T}(\mathcal{H})$ is a complete separable metric space with the metric defined by the trace norm. By convention we denote operators in $\mathfrak{S}(\mathcal{H})$ by the Greek letters ρ, σ, ω and

¹This characteristic is often called the Holevo quantity; it plays the basic role in quantum information theory, see [4], Ch. 5.

call them *states*, since any such operator ρ determines a normal linear functional $A \mapsto \text{Tr } A\rho$ with unit norm on the algebra $\mathfrak{B}(\mathcal{H})$ [11]. *Pure states* are rank one projectors, they are extreme points of the set $\mathfrak{S}(\mathcal{H})$. The support $\text{supp } \rho$ of a state ρ is the orthogonal complement of its kernel $\ker \rho$; the dimension of the support is called the rank of a state: $\text{rank } \rho = \dim \text{supp } \rho$. A state ρ for which $\ker \rho = \{0\}$, is called nondegenerate.

For vectors and rank one operators in a Hilbert space we will use the Dirac notations $|\varphi\rangle, |\chi\rangle\langle\psi|, \dots$ (in which the action of the operator $|\chi\rangle\langle\psi|$ on the vector $|\varphi\rangle$ gives the vector $\langle\psi, \varphi\rangle|\chi\rangle$). For brevity, orthonormal sets of vectors $\{|\varphi_i\rangle\}_{i \in I}$, where $I = \{1, 2, \dots, n\}$ or $I = \mathbb{N}$, will be denoted by $\{|i\rangle\}_{i \in I}$.

The identity operator in a Hilbert space \mathcal{H} and the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$ will be denoted by $I_{\mathcal{H}}$ and $\text{Id}_{\mathcal{H}}$, respectively.

A set of vectors $\{|\psi_i\rangle\}$ in a Hilbert space \mathcal{H} is called an *overcomplete system* if

$$\sum_i |\psi_i\rangle\langle\psi_i| = I_{\mathcal{H}}.$$

An orthonormal basis in \mathcal{H} is an example of an overcomplete system.

A Hilbert space of finite dimension d (it can be identified with \mathbb{C}^d) will be denoted by \mathcal{H}_d .

Let \mathcal{H}_A and \mathcal{H}_B be Hilbert spaces called input and output, respectively. Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a linear map which is positive and trace-preserving ($\Phi(A) \geq 0$ and $\text{Tr } \Phi(A) = \text{Tr } A$ for any $A \geq 0$). The *dual map* $\Phi^*: \mathfrak{B}(\mathcal{H}_B) \rightarrow \mathfrak{B}(\mathcal{H}_A)$ (defined by the relation $\text{Tr } \Phi(A)B = \text{Tr } A\Phi^*(B)$, $A \in \mathfrak{T}(\mathcal{H}_A)$, $B \in \mathfrak{B}(\mathcal{H}_B)$) is the positive map such that $\Phi^*(I_{\mathcal{H}_B}) = I_{\mathcal{H}_A}$.

A linear map $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called *completely positive* if the map $\Phi \otimes \text{Id}_{\mathcal{H}_d}$ from $\mathfrak{T}(\mathcal{H}_A \otimes \mathcal{H}_d)$ into $\mathfrak{T}(\mathcal{H}_B \otimes \mathcal{H}_d)$ is positive for each natural number d (equivalent definitions of complete positivity can be found in [4], § 6.2).

Definition 1. A linear completely positive trace-preserving map $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called a *quantum channel*.

This definition of a quantum channel corresponds to the Schrodinger picture in which the dynamics of a quantum system is described via the evolution of states. A quantum channel in the Heisenberg picture is the dual map $\Phi^*: \mathfrak{B}(\mathcal{H}_B) \rightarrow \mathfrak{B}(\mathcal{H}_A)$ describing the evolution of quantum observables [1], Ch. 3.

An important example of a quantum channel is the operation of partial trace

$$\mathfrak{T}(\mathcal{H} \otimes \mathcal{K}) \ni C \mapsto \text{Tr}_{\mathcal{K}} C \in \mathfrak{T}(\mathcal{H}),$$

which transforms the operator $A \otimes B$ into the operator $A \text{Tr } B$ and is extended to operators of general form by linearity and continuity (see the strict definition in [1], [4]). This operation is a noncommutative analogue of the transition from a joint distribution of random variables to their partial distributions in classical probability theory.

Using Stinespring’s theorem on representations of completely positive maps on C^* -algebras and some properties of the algebra $\mathfrak{B}(\mathcal{H})$ one can obtain the following representation of an arbitrary quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$: there exist a Hilbert space \mathcal{H}_E and an isometry $V: \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(A) = \text{Tr}_{\mathcal{H}_E} VAV^* \quad \forall A \in \mathfrak{T}(\mathcal{H}_A) \tag{2.1}$$

(see [1], [4]). This representation will be called the Stinespring representation of the channel Φ , while the operator V is the Stinespring isometry.

The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni A \mapsto \widehat{\Phi}(A) = \text{Tr}_{\mathcal{H}_B} VAV^* \in \mathfrak{T}(\mathcal{H}_E) \tag{2.2}$$

is said to be *complementary* to the channel Φ (see [4], § 6.6, [9])². The complementary channel is uniquely defined: if $\widehat{\Phi}': \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_{E'})$ is the channel defined by (2.2) via the Stinespring isometry $V': \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_{E'}$ then the channels $\widehat{\Phi}$ and $\widehat{\Phi}'$ are isometrically equivalent in the sense of the following definition (see the Appendix in [9]).

Definition 2. Two channels $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ and $\Phi': \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}'_B)$ are *isometrically equivalent* if there is a partial isometry $W: \mathcal{H}_B \rightarrow \mathcal{H}'_B$ such that

$$\Phi'(A) = W\Phi(A)W^*, \quad \Phi(A) = W^*\Phi'(A)W, \quad A \in \mathfrak{T}(\mathcal{H}_A). \tag{2.3}$$

The notion of isometrical equivalence is very close to the notion of unitary equivalence. Indeed, if the channels Φ, Φ' are isometrically equivalent, they are unitary equivalent, provided that the output spaces $\mathcal{H}_B, \mathcal{H}'_B$ are replaced by the subspaces

$$\mathcal{H}_B^\Phi = \bigvee_{\rho \in \mathfrak{S}(\mathcal{H}_A)} \text{supp } \Phi(\rho), \quad \mathcal{H}'_B^{\Phi'} = \bigvee_{\rho \in \mathfrak{S}(\mathcal{H}_A)} \text{supp } \Phi'(\rho).$$

The concept of isometrical equivalence is convenient, since when we deal with a particular representation of a quantum channel Φ it is not always easy to determine the corresponding subspace \mathcal{H}_B^Φ .

A Stinespring representation (2.1) is called *minimal* if the set

$$\mathcal{M} = \{(X \otimes I_{\mathcal{H}_E})V|\varphi \mid \varphi \in \mathcal{H}_A, X \in \mathfrak{B}(\mathcal{H}_B)\}$$

is dense in $\mathcal{H}_B \otimes \mathcal{H}_E$. The complementary channel $\widehat{\Phi}$ defined by (2.2) via a minimal Stinespring representation has the following property:

$$\ker \rho = \{0\} \implies \ker \widehat{\Phi}(\rho) = \{0\}. \tag{2.4}$$

By using the Stinespring representation (2.1) it is easy to obtain the Kraus representation

$$\Phi(A) = \sum_k V_k AV_k^*, \quad A \in \mathfrak{T}(\mathcal{H}_A), \tag{2.5}$$

where $\{V_k\}$ is a collection of linear bounded operators from \mathcal{H}_A into \mathcal{H}_B such that $\sum_k V_k^* V_k = I_{\mathcal{H}_A}$. These operators are defined by the relation

$$\langle \varphi | V_k \psi \rangle = \langle \varphi \otimes k | V \psi \rangle, \quad \varphi \in \mathcal{H}_B, \quad \psi \in \mathcal{H}_A,$$

where $\{|k\rangle\}$ is an orthonormal basis in the space \mathcal{H}_E . It is easy to see that the complementary channel (2.2) has the representation

$$\widehat{\Phi}(A) = \sum_{k,l} \text{Tr}[V_k AV_l^*] |k\rangle\langle l|, \quad A \in \mathfrak{T}(\mathcal{H}_A). \tag{2.6}$$

²The quantum channel $\widehat{\Phi}$ is also called *conjugate* to the channel Φ [12].

The following class of quantum channels plays an essential role in this paper [4], [13].

Definition 3. A quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called *classical-quantum* (briefly, a *c-q channel*) if it has the representation

$$\Phi(A) = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k|Ak \rangle \sigma_k, \quad A \in \mathfrak{T}(\mathcal{H}_A),$$

where $\{|k\rangle\}$ is an orthonormal basis in \mathcal{H}_A and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$.

Following [6], [8], we now introduce the basic notion of the paper.

Definition 4. A quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called *reversible* with respect to a family $\mathfrak{S} \subseteq \mathfrak{S}(\mathcal{H}_A)$ if there exists a quantum channel $\Psi: \mathfrak{T}(\mathcal{H}_B) \rightarrow \mathfrak{T}(\mathcal{H}_A)$ such that $\rho = \Psi \circ \Phi(\rho)$ for all $\rho \in \mathfrak{S}$.³

The channel Ψ in this definition will be called a *reversing channel* for the channel Φ with respect to the family \mathfrak{S} .

Note that reversibility is a property common to isometrically equivalent channels.

Lemma 1. Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ and $\Phi': \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_{B'})$ be isometrically equivalent quantum channels. The channel Φ being reversible with respect to a family $\mathfrak{S} \subseteq \mathfrak{S}(\mathcal{H}_A)$ is equivalent to Φ' being reversible with respect to this family.

Proof. Let Ψ be a reversing channel for the channel Φ with respect to the family \mathfrak{S} . Let $\Theta(\cdot) = W^*(\cdot)W + \sigma \text{Tr}(I_{\mathcal{H}_{B'}} - WW^*)(\cdot)$ be a channel from $\mathfrak{T}(\mathcal{H}_{B'})$ into $\mathfrak{T}(\mathcal{H}_B)$, where W is a partial isometry from (2.3) and σ is a fixed state in $\mathfrak{S}(\mathcal{H}_B)$. Then $\Psi \circ \Theta$ is a reversing channel for the channel Φ' with respect to the family \mathfrak{S} .

The von Neumann entropy of a state ρ in $\mathfrak{S}(\mathcal{H})$ is defined as follows:⁴

$$H(\rho) = -\text{Tr} \rho \log \rho = -\sum_{i=1}^{+\infty} \lambda_i \log \lambda_i,$$

where $\{\lambda_i\}$ is the set of eigenvalues of the state ρ (see [3], [4], [15]).

The quantum relative entropy of states ρ and σ in $\mathfrak{S}(\mathcal{H})$ is defined as follows

$$H(\rho||\sigma) = \sum_{i=1}^{+\infty} \langle i|[\rho \log \rho - \rho \log \sigma]i \rangle,$$

where $\{|i\rangle\}_{i=1}^{+\infty}$ is an orthonormal basis of eigenvectors of the state ρ (or σ) and it is assumed that $H(\rho||\sigma) = +\infty$ if $\text{supp} \rho \not\subseteq \text{supp} \sigma$ (see [3], [4], [15]).

We will use the following concept concerning the structure of the set of states in a Hilbert space $\mathcal{H} \otimes \mathcal{K}$ describing composite quantum systems.

³In most early papers this property was called the sufficiency of the channel Φ with respect to the family \mathfrak{S} (see [2], [14]).

⁴Here and in what follows \log denotes the natural logarithm.

The Schmidt rank of a pure state $|\psi\rangle\langle\psi|$ in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ is defined as the number of nonzero summands in the Schmidt decomposition

$$|\psi\rangle = \sum_i \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle,$$

where $\{|\alpha_i\rangle\}$ and $\{|\beta_i\rangle\}$ are orthonormal bases in \mathcal{H} and \mathcal{K} ; it coincides with the rank of partial states $\text{Tr}_{\mathcal{K}} |\psi\rangle\langle\psi|$ and $\text{Tr}_{\mathcal{H}} |\psi\rangle\langle\psi|$ (see [16]).

The Schmidt class \mathfrak{S}_r of order $r \in \mathbb{N}$ is the minimal closed convex subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ containing all pure states with Schmidt rank not exceeding r , that is, \mathfrak{S}_r is the convex closure of these pure states [16], [17].⁵ In this notation \mathfrak{S}_1 is the set of all separable (nonentangled) states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ (see [4], [15]).

A quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called *entanglement-breaking* if for any Hilbert space \mathcal{K} the state $\Phi \otimes \text{Id}_{\mathcal{K}}(\omega)$ is separable in $\mathfrak{S}(\mathcal{H}_B \otimes \mathcal{K})$ for each state $\omega \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{K})$ (see [13]). This notion has the following natural generalization given in [16].

Definition 5. A quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called *partially entanglement-breaking of order r* (briefly, an r -PEB channel) if for any Hilbert space \mathcal{K} the state $\Phi \otimes \text{Id}_{\mathcal{K}}(\omega)$ belongs to the Schmidt class $\mathfrak{S}_r \subset \mathfrak{S}(\mathcal{H}_B \otimes \mathcal{K})$ for each state $\omega \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{K})$.

In this notation entanglement-breaking quantum channels are 1-PEB channels. Properties of finite-dimensional r -PEB channels are studied in [16], where it is proved, in particular, that the class of r -PEB channels coincides with the class of channels having Kraus representation (2.5) such that $\text{rank } V_k \leq r$ for all k . But in infinite dimensions the first class is essentially wider than the second; moreover, for each r there exist r -PEB channels such that all the operators in any of their Kraus representations have infinite rank [17].

§ 3. Petz’s theorem and reversibility criteria

A fundamental property of quantum relative entropy is its monotonicity (non-increasing) under the action of a quantum channel; this is expressed by the inequality

$$H(\Phi(\rho) \parallel \Phi(\sigma)) \leq H(\rho \parallel \sigma), \tag{3.1}$$

which is valid for an arbitrary channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ and any states ρ and σ in $\mathfrak{S}(\mathcal{H}_A)$.

We will consider states ρ and σ such that $H(\rho \parallel \sigma) < +\infty$. This implies $\text{supp } \rho \subseteq \text{supp } \sigma$. Thus, we will assume that σ and $\Phi(\sigma)$ are nondegenerate states in $\mathfrak{S}(\mathcal{H}_A)$ and in $\mathfrak{S}(\mathcal{H}_B)$, respectively (the general case is reduced to this one by replacing \mathcal{H}_A by $\text{supp } \sigma$ and \mathcal{H}_B by $\text{supp } \Phi(\sigma)$).

Petz’s theorem, which follows, characterizes the case of equality in (3.1).

⁵In finite dimensions the convex closure coincides with the convex hull by Carathéodory’s theorem, but in infinite dimensions even the set of all *countable* convex mixtures of pure states with Schmidt rank $\leq r$ is a proper subset of \mathfrak{S}_r for each r [17].

Theorem 1. *Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel, and ρ and σ states in $\mathfrak{S}(\mathcal{H}_A)$ such that $H(\rho\|\sigma) < +\infty$. Let $\Theta_\sigma: \mathfrak{T}(\mathcal{H}_B) \rightarrow \mathfrak{T}(\mathcal{H}_A)$ be the quantum channel predual to the map*

$$\mathfrak{B}(\mathcal{H}_A) \ni A \mapsto \Theta_\sigma^*(A) = C\Phi(BAB)C \in \mathfrak{B}(\mathcal{H}_B), \quad B = [\sigma]^{1/2}, \quad C = [\Phi(\sigma)]^{-1/2}.$$

Then the following statements are equivalent:

- (i) $H(\Phi(\rho)\|\Phi(\sigma)) = H(\rho\|\sigma)$;
- (ii) $\rho = \Theta_\sigma(\Phi(\rho))$;
- (iii) *the channel Φ is reversible with respect to the states ρ and σ .*

Note that the implication (iii) \implies (i) in this theorem follows immediately from the monotonicity of relative entropy, while the implication (ii) \implies (iii) is obvious, since it is easy to verify that $\sigma = \Theta_\sigma(\Phi(\sigma))$.

Note also that the action of the channel Θ_σ on states ϱ in $\mathfrak{S}(\mathcal{H}_B)$ such that $\lambda\varrho \leq \Phi(\sigma)$ for some $\lambda > 0$ is given by the explicit formula

$$\Theta_\sigma(\varrho) = B\Phi^*(C\varrho C)B, \quad B = [\sigma]^{1/2}, \quad C = [\Phi(\sigma)]^{-1/2}$$

(the condition $\lambda\varrho \leq \Phi(\sigma)$ guarantees the operator $C\varrho C$ is bounded).

Theorem 1 was formulated and proved in [2] in terms of von Neumann algebras and normal states on these algebras. In [2] both ρ and σ were assumed to be faithful (nondegenerate in our notation). So the assertion of Theorem 1 follows directly from the theorem in [2] only for nondegenerate states ρ . A possible generalization to the case of arbitrary states ρ is presented in the Appendix. Note that in finite dimension this generalization follows from the theorem in [5], § 5.1.

Definition 6. A family \mathfrak{S} of states in $\mathfrak{S}(\mathcal{H})$ is called *complete* if for any positive operator A in $\mathfrak{B}(\mathcal{H})$ there exists a state ρ in \mathfrak{S} such that $\text{Tr } A\rho > 0$.

A family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ of pure states in $\mathfrak{S}(\mathcal{H})$ is complete if and only if the linear hull of the family $\{|\varphi_\lambda\rangle\}_{\lambda \in \Lambda}$ is dense in \mathcal{H} . It is easy to show that an arbitrary complete family of any states in $\mathfrak{S}(\mathcal{H})$ contains a countable complete subfamily [14], Lemma 2.

A general criterion for the reversibility of a channel with respect to complete families of states was obtained in [14]. We will use this criterion in the following restricted form (in which $\Theta_{\bar{\rho}}$ is the channel defined in Theorem 1 with $\sigma = \bar{\rho}$).

Theorem 2. *A quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is reversible with respect to a complete countable family $\{\rho_i\}$ of states in $\mathfrak{S}(\mathcal{H}_A)$ if and only if $\rho_i = \Theta_{\bar{\rho}}(\Phi(\rho_i))$ for all i , where $\bar{\rho} = \sum_i \pi_i \rho_i$ and $\{\pi_i\}$ is any nondegenerate probability distribution.*

Note that the assertion of Theorem 2 can be deduced from Theorem 1, since using properties of the quantum relative entropy we can show that $H(\rho_i\|\bar{\rho}) < +\infty$ for all i .

§ 4. Conditions for the reversibility of a quantum channel with respect to complete families of states

4.1. Families of states with bounded rank. Using Theorem 2 we can obtain a necessary condition for a quantum channel to be reversible with respect to complete families of states with bounded rank, expressed in terms of the complementary channel.

Theorem 3. Let $\mathfrak{S} = \{\rho_i\}_{i=1}^n, n \leq +\infty$, be a complete family of states in $\mathfrak{S}(\mathcal{H}_A)$ such that $\text{rank } \rho_i \leq r$ for all i . If a quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is reversible with respect to the family \mathfrak{S} then its complementary channel $\widehat{\Phi}: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_E)$ has a Kraus representation (2.5) with $n \cdot \min\{\dim \ker \Phi^* + r^2, \dim \mathcal{H}_B\}$ summands, such that $\text{rank } V_k \leq r$ for all k , and hence $\widehat{\Phi}$ is a PEB-channel of order r .

If the above condition holds for $r = 1$, that is, $\rho_i = |\varphi_i\rangle\langle\varphi_i|$ for all i , then

$$\widehat{\Phi}(A) = \sum_{i=1}^n \langle\phi_i|A\phi_i\rangle \sum_{k=1}^m |\psi_{ik}\rangle\langle\psi_{ik}|, \quad A \in \mathfrak{T}(\mathcal{H}_A), \tag{4.1}$$

where $m = \min\{\dim \ker \Phi^* + 1, \dim \mathcal{H}_B\}$, $\{|\phi_i\rangle\}_{i=1}^n$ is an overcomplete system of vectors in \mathcal{H}_A defined by means of an arbitrary nondegenerate probability distribution $\{\pi_i\}_{i=1}^n$ as follows:

$$|\phi_i\rangle = \sqrt{\pi_i \bar{\rho}_\pi^{-1}} |\varphi_i\rangle, \quad \bar{\rho}_\pi = \sum_{i=1}^n \pi_i |\varphi_i\rangle\langle\varphi_i|, \tag{4.2}$$

and $\{|\psi_{ik}\rangle\}$ is a collection of vectors in a Hilbert space \mathcal{H}_E such that $\sum_{k=1}^m \|\psi_{ik}\|^2 = 1$ and $\langle\psi_{il}|\psi_{ik}\rangle = 0$ for all $k \neq l$ for each $i = 1, \dots, n$. Hence the channel Φ is isometrically equivalent to the channel

$$\Phi'(A) = \sum_{i,j=1}^n \langle\phi_i|A\phi_j\rangle |i\rangle\langle j| \otimes \sum_{k,l=1}^m \langle\psi_{jl}|\psi_{ik}\rangle |k\rangle\langle l| \tag{4.3}$$

from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_n \otimes \mathcal{H}_m)$, where $\{|i\rangle\}_{i=1}^n$ and $\{|k\rangle\}_{k=1}^m$ are arbitrary orthonormal bases in \mathcal{H}_n and in \mathcal{H}_m , respectively.

The first assertion in Theorem 3 means that the channel $\widehat{\Phi}$ has the following property: for an arbitrary Hilbert space \mathcal{X} and any state ω in $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{X})$ the state $\widehat{\Phi} \otimes \text{Id}_{\mathcal{X}}(\omega)$ is a countably decomposable state in the Schmidt class $\mathfrak{S}_r \subset \mathfrak{S}(\mathcal{H}_E \otimes \mathcal{X})$, that is, it can be represented as a countable convex mixture of pure states with Schmidt rank $\leq r$ (there exist states in \mathfrak{S}_r which are not countably decomposable [17]).

Proof. Let $\widehat{\Phi}(\rho) = \sum_{k=1}^d V_k \rho V_k^*$, $d \leq +\infty$, be the Kraus representation of the channel $\widehat{\Phi}: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_E)$, obtained via its minimal Stinespring representation with the isometry $V: \mathcal{H}_A \rightarrow \mathcal{H}_E \otimes \mathcal{H}_C$ (see §2). The complementary channel $\Psi = \widehat{\widehat{\Phi}}$ to the channel $\widehat{\Phi}$ determined by means of this representation has the form

$$\mathfrak{T}(\mathcal{H}_A) \ni A \mapsto \Psi(A) = \sum_{k,l=1}^d [\text{Tr } V_k A V_l^*] |k\rangle\langle l| \in \mathfrak{T}(\mathcal{H}_C),$$

where $\{|k\rangle\}_{k=1}^d$ is an orthonormal basis in d -dimensional Hilbert space \mathcal{H}_C .

Since $\Psi = \widehat{\widehat{\Phi}}$, the channels Φ and Ψ are isometrically equivalent (see §2). By Lemma 1 the channel Ψ is reversible with respect to the family $\{\rho_i\}$.

Let $\{\pi_i\}_{i=1}^n$ be an arbitrary nondegenerate probability distribution and $\bar{\rho}_\pi = \sum_{i=1}^n \pi_i \rho_i$. It follows from (2.4) that $\Psi(\bar{\rho}_\pi)$ is a nondegenerate state in $\mathfrak{S}(\mathcal{H}_C)$. By Theorem 2 the reversibility of Ψ implies $A_i = \Psi^*(B_i)$ for all i , where $A_i = \pi_i [\bar{\rho}_\pi]^{-1/2} \rho_i [\bar{\rho}_\pi]^{-1/2}$ and $B_i = \pi_i [\Psi(\bar{\rho}_\pi)]^{-1/2} \Psi(\rho_i) [\Psi(\bar{\rho}_\pi)]^{-1/2}$ are positive operators in $\mathfrak{B}(\mathcal{H}_A)$ and in $\mathfrak{B}(\mathcal{H}_C)$, respectively.

Note that

$$\Psi^*(C) = \sum_{k,l=1}^d \langle l|Ck\rangle V_l^* V_k, \quad C \in \mathfrak{B}(\mathcal{H}_C).$$

Since $A_i = \Psi^*(B_i)$ is an operator of rank $\leq r$, Lemma 2 below implies that $B_i = \sum_{j=1}^m |\psi_{ij}\rangle\langle\psi_{ij}|$, where $m = \min\{\dim \ker \Psi^* + r^2, \dim \mathcal{H}_C\}$ and $\{|\psi_{ij}\rangle\}_{j=1}^m$ is a set of vectors in \mathcal{H}_C , for each $i = 1, \dots, n$. Since the state $\Psi(\bar{\rho}_\pi) \in \mathfrak{S}(\mathcal{H}_C)$ is nondegenerate it follows that

$$\sum_{i=1}^n \sum_{j=1}^m |\psi_{ij}\rangle\langle\psi_{ij}| = \sum_{i=1}^n B_i = I_{\mathcal{H}_C}.$$

By Lemma 3 below

$$\widehat{\Phi}(\cdot) = \sum_{i=1}^n \sum_{j=1}^m W_{ij}(\cdot) W_{ij}^*, \tag{4.4}$$

where $W_{ij} = \sum_{k=1}^d \langle\psi_{ij}|k\rangle V_k$. Since $A_i = \Psi^*(\sum_{j=1}^m |\psi_{ij}\rangle\langle\psi_{ij}|)$ is an operator of rank $\leq r$ for each i and

$$\Psi^*(|\psi_{ij}\rangle\langle\psi_{ij}|) = \sum_{k,l=1}^d \langle l|\psi_{ij}\rangle\langle\psi_{ij}|k\rangle V_l^* V_k = W_{ij}^* W_{ij}, \tag{4.5}$$

the collection $\{W_{ij}\}$ consists of operators of rank $\leq r$. To complete the proof of the first part of the theorem it suffices to note that the partial isometry in the relation similar to (2.3), which expresses the isometrical equivalence of the channels Φ and Ψ , is an isometrical embedding of \mathcal{H}_C into \mathcal{H}_B (since $\Psi(\bar{\rho}_\pi)$ is a nondegenerate state in $\mathfrak{S}(\mathcal{H}_C)$). Hence $\dim \mathcal{H}_C \leq \dim \mathcal{H}_B$ and $\dim \ker \Psi^* \leq \dim \ker \Phi^*$.

If $\rho_i = |\varphi_i\rangle\langle\varphi_i|$ then $A_i = |\phi_i\rangle\langle\phi_i|$ for each i , where $|\phi_i\rangle$ is the vector defined in (4.2). So, it follows from (4.5) that

$$|\phi_i\rangle\langle\phi_i| = \sum_{j=1}^m \Psi^*(|\psi_{ij}\rangle\langle\psi_{ij}|) = \sum_{j=1}^m W_{ij}^* W_{ij},$$

and hence $W_{ij} = |\eta_{ij}\rangle\langle\phi_i|$ for all $j = 1, \dots, m$, where $\{|\eta_{ij}\rangle\}$ is a set of vectors in \mathcal{H}_B such that $\sum_{j=1}^m \|\eta_{ij}\|^2 = 1$ for each i .

It follows from (4.4) that

$$\widehat{\Phi}(A) = \sum_{i=1}^n \langle\phi_i|A\phi_i\rangle \sum_{j=1}^m |\eta_{ij}\rangle\langle\eta_{ij}|, \quad A \in \mathfrak{T}(\mathcal{H}_A).$$

Using the spectral decomposition of the states $\sum_{j=1}^m |\eta_{ij}\rangle\langle\eta_{ij}|$, $i = 1, \dots, n$, we obtain the representation (4.1).

The representation (4.3) is derived from (4.1) by means of the representation (2.6).

Lemma 2. Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel. If B is a positive operator in $\mathfrak{B}(\mathcal{H}_B)$ such that $\text{rank } \Phi^*(B) = r < +\infty$ then $B = \sum_{j=1}^m |\psi_j\rangle\langle\psi_j|$, where $m = \min\{\dim \ker \Phi^* + r^2, \dim \mathcal{H}_B\}$ and $\{|\psi_j\rangle\}_{j=1}^m$ is a set of vectors in \mathcal{H}_B .

Proof. Note that $B = \sum_{j=1}^{\dim \mathcal{H}_B} |\psi_j\rangle\langle\psi_j|$, where $|\psi_j\rangle = B^{1/2}|j\rangle$, for any orthonormal basis $\{|j\rangle\}$ in \mathcal{H}_B . So, it suffices to consider the case $m = \dim \ker \Phi^* + r^2 < \dim \mathcal{H}_B$.

We may assume that the first $n = \text{rank } B$ vectors in $\{|\psi_j\rangle\}$ are linearly independent, and hence the operators $|\psi_j\rangle\langle\psi_j|, j = 1, \dots, n$, generate an n -dimensional subspace $\mathfrak{B}_n \subset \mathfrak{B}(\mathcal{H}_A)$. Now, $B \geq \sum_{j=1}^n |\psi_j\rangle\langle\psi_j|$ and the support of the operator $\Phi^*(B)$ is contained in some r -dimensional subspace $\mathcal{H}_r \subset \mathcal{H}_A$, and so $\Phi^*(|\psi_j\rangle\langle\psi_j|) \in \mathfrak{B}(\mathcal{H}_r), j = 1, \dots, n$. Thus, $\Phi^*(\mathfrak{B}_n) \subseteq \mathfrak{B}(\mathcal{H}_r)$ and hence

$$\text{rank } B = n = \dim \mathfrak{B}_n \leq \dim \ker \Phi^* + \dim \mathfrak{B}(\mathcal{H}_r) = \dim \ker \Phi^* + r^2 = m.$$

It follows from the finite-dimensional spectral theorem that $B = \sum_{j=1}^m |\psi'_j\rangle\langle\psi'_j|$, where $\{|\psi'_j\rangle\}$ is an orthogonal set of eigenvectors for the operator B .

Lemma 3. Let $\Phi(A) = \sum_{k=1}^d V_k A V_k^*$ be a quantum channel and $\{|k\rangle\}_{k=1}^d$ an orthonormal basis in the Hilbert space $\mathcal{H}_d, d \leq +\infty$. An arbitrary overcomplete system $\{|\psi_i\rangle\}$ of vectors in \mathcal{H}_d generates the Kraus representation $\Phi(A) = \sum_i W_i A W_i^*$ of the channel Φ in which $W_i = \sum_{k=1}^d \langle\psi_i|k\rangle V_k$.

Proof. Since $\sum_i |\psi_i\rangle\langle\psi_i| = I_{\mathcal{H}_d}$, we have

$$\begin{aligned} \sum_i W_i A W_i^* &= \sum_{k,l=1}^d V_k A V_l^* \sum_i \langle\psi_i|k\rangle\langle l|\psi_i\rangle \\ &= \sum_{k,l=1}^d V_k A V_l^* \sum_i \text{Tr } |k\rangle\langle l| |\psi_i\rangle\langle\psi_i| = \sum_{k=1}^d V_k A V_k^*. \end{aligned}$$

4.2. Orthogonal families of pure states. From Theorem 3 we can obtain the following description of the class of all quantum channels reversible with respect to a given complete family of orthogonal pure states.

Proposition 1. Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel,

$$m = \min\{\dim \ker \Phi^* + 1, \dim \mathcal{H}_B\},$$

and $\mathfrak{S} = \{|\varphi_i\rangle\langle\varphi_i|\}$ a complete family of orthogonal pure states in $\mathfrak{S}(\mathcal{H}_A)$. The following statements are equivalent:

- (i) the channel Φ is reversible with respect to the family \mathfrak{S} ;
- (ii) $\widehat{\Phi}$ is a c - q channel with the representation $\widehat{\Phi}(A) = \sum_{i=1}^{\dim \mathcal{H}_A} \langle\varphi_i|A\varphi_i\rangle\sigma_i$, where $\{\sigma_i\}$ is a set of states in $\mathfrak{S}(\mathcal{H}_B)$ such that $\text{rank } \sigma_i \leq m$ for all i ;
- (iii) the channel Φ is isometrically equivalent to the channel

$$\Phi'(A) = \sum_{i,j=1}^{\dim \mathcal{H}_A} \langle\varphi_i|A\varphi_j\rangle |\varphi_i\rangle\langle\varphi_j| \otimes \sum_{k,l=1}^m \langle\psi_{jl}|\psi_{ik}\rangle |k\rangle\langle l|$$

from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_A \otimes \mathcal{H}_m)$, where $\{|\psi_{ik}\rangle\}$ is a set of vectors in a separable Hilbert space such that $\sum_{k=1}^m \|\psi_{ik}\|^2 = 1$ and $\langle\psi_{il}|\psi_{ik}\rangle = 0$ for all $k \neq l$ for each i and $\{|k\rangle\}$ is an orthonormal basis in \mathcal{H}_m .

Proof. The implication (i) \implies (ii) follows from Theorem 3, since in this case $\phi_i = \varphi_i$ for all i .

(ii) \implies (iii). If $\sigma_i = \sum_{k=1}^m |\psi_{ik}\rangle\langle\psi_{ik}|$ then $\widehat{\Phi}(\rho) = \sum_{i,k} W_{ik}\rho W_{ik}^*$, where $W_{ik} = |\psi_{ik}\rangle\langle\varphi_i|$, and then (2.6) shows that $\widehat{\Phi} = \Phi'$.

The implication (iii) \implies (i) follows from Lemma 1, since $\Psi(\cdot) = \text{Tr}_{\mathcal{H}_m}(\cdot)$ is a reversing channel for the channel Φ' with respect to the family \mathfrak{S} .

Proposition 1 makes it possible to obtain the following criterion for reversibility in terms of the dual map to a quantum channel.

Corollary 1. *A quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is reversible with respect to a complete family of orthogonal pure states $\{|\varphi_i\rangle\langle\varphi_i|\}$ if and only if there exists a partial isometry $W: \mathcal{H}_A \otimes \mathcal{H}_m \rightarrow \mathcal{H}_B$ such that*

$$|\varphi_i\rangle\langle\varphi_i| = \Phi^*(W[|\varphi_i\rangle\langle\varphi_i| \otimes I_{\mathcal{H}_m}]W^*) \quad \forall i, \tag{4.6}$$

where $m = \min\{\dim \ker \Phi^* + 1, \dim \mathcal{H}_B\}$ and $\Phi^*: \mathfrak{B}(\mathcal{H}_B) \rightarrow \mathfrak{B}(\mathcal{H}_A)$ is a dual map to the channel Φ .

Note that condition (4.6) implies that $\Phi^*(WW^*) = I_{\mathcal{H}_A}$, and hence WW^* is the projector on a subspace containing the supports of all states $\Phi(\rho)$, $\rho \in \mathfrak{S}(\mathcal{H}_A)$.

Proof. The necessity of condition (4.6) directly follows from Proposition 1.

To prove its sufficiency consider the channel $\Phi'(A) = W^*\Phi(A)W$ from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_A \otimes \mathcal{H}_m)$. By the above remark

$$W\Phi'(A)W^* = WW^*\Phi(A)WW^* = \Phi(A), \quad A \in \mathfrak{T}(\mathcal{H}_A),$$

and hence the channels Φ and Φ' are isometrically equivalent. By Lemma 1 it suffices to show that Φ' is reversible with respect to the family $\{|\varphi_i\rangle\langle\varphi_i|\}$.

Condition (4.6) implies

$$\text{Tr}[|\varphi_i\rangle\langle\varphi_i| \otimes I_{\mathcal{H}_m}]\Phi'(|\varphi_j\rangle\langle\varphi_j|) = \text{Tr}\Phi^*(W[|\varphi_i\rangle\langle\varphi_i| \otimes I_{\mathcal{H}_m}]W^*)|\varphi_j\rangle\langle\varphi_j| = \delta_{ij}.$$

So, the support of $\Phi'(|\varphi_i\rangle\langle\varphi_i|)$ is contained in the subspace $\{\lambda|\varphi_i\rangle\} \otimes \mathcal{H}_m$, and hence $\text{Tr}_{\mathcal{H}_m} \Phi'(|\varphi_i\rangle\langle\varphi_i|) = |\varphi_i\rangle\langle\varphi_i|$ for all i .

4.3. Arbitrary families of pure states. Consider the structure of any quantum channel reversible with respect to an arbitrary complete family $\mathfrak{S} = \{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ of pure states.

It is well known that a quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is reversible with respect to the family of all pure states in $\mathfrak{S}(\mathcal{H}_A)$ (which means it is reversible with respect to $\mathfrak{S}(\mathcal{H}_A)$) if and only if its complementary channel is completely depolarizing, that is, it maps all states in $\mathfrak{S}(\mathcal{H}_A)$ into a fixed state in $\mathfrak{S}(\mathcal{H}_E)$ (see [4], Proposition 10.1.2). This means that the channel Φ is isometrically equivalent to the channel

$$\Phi'(A) = A \otimes \sigma \tag{4.7}$$

from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_A \otimes \mathcal{H})$, where \mathcal{H} is a Hilbert space and σ is a given state in $\mathfrak{S}(\mathcal{H})$.

We first give a characterization of family $\mathfrak{S} = \{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda} \subset \mathfrak{S}(\mathcal{H}_A)$ such that if the channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is reversible with respect to \mathfrak{S} then it is reversible with respect to $\mathfrak{S}(\mathcal{H}_A)$.

Definition 7. A family $\{|\varphi_\lambda\rangle\}_{\lambda\in\Lambda}$ of vectors in \mathcal{H} (a family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda\in\Lambda}$ of pure states in $\mathfrak{S}(\mathcal{H})$) is called *orthogonally decomposable* if there is a proper subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that some vectors of this family lie in \mathcal{H}_0 and all the others in \mathcal{H}_0^\perp .

A family of pure states which is not orthogonally decomposable will be called *orthogonally nondecomposable*.

Proposition 2. Let $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda\in\Lambda}$ be a complete family of pure states in $\mathfrak{S}(\mathcal{H}_A)$. The following statements are equivalent:

- (i) the family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda\in\Lambda}$ is orthogonally nondecomposable;
- (ii) any channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ reversible with respect to the family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda\in\Lambda}$, is isometrically equivalent to the channel (4.7).

Proof. (i) \implies (ii). If $\Psi: \mathfrak{T}(\mathcal{H}_B) \rightarrow \mathfrak{T}(\mathcal{H}_A)$ is a reversing channel for Φ then Lemma 4 below shows that $\Psi \circ \Phi = \text{Id}_{\mathcal{H}_A}$. Thus the channel Φ is reversible with respect to $\mathfrak{S}(\mathcal{H}_A)$ and hence its complementary channel $\hat{\Phi}$ is completely depolarizing.

(ii) \implies (i). If \mathcal{H}_0 is a proper subspace of \mathcal{H}_A such that for each $\lambda \in \Lambda$ the vector $|\varphi_\lambda\rangle$ lies either in \mathcal{H}_0 or in \mathcal{H}_0^\perp , then the channel $A \mapsto P_0AP_0 + \bar{P}_0A\bar{P}_0$, where P_0 is the projector onto the subspace \mathcal{H}_0 and $\bar{P}_0 = I_{\mathcal{H}_A} - P_0$, is reversible with respect to the family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda\in\Lambda}$ (since no state of this family is changed under the action of this channel).

Lemma 4. Let $\Phi: \mathfrak{T}(\mathcal{H}) \rightarrow \mathfrak{T}(\mathcal{H})$ be a quantum channel ($\dim \mathcal{H} \leq +\infty$) and $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda\in\Lambda}$ an orthogonally nondecomposable family of pure states in $\mathfrak{S}(\mathcal{H})$. If $\Phi(|\varphi_\lambda\rangle\langle\varphi_\lambda|) = |\varphi_\lambda\rangle\langle\varphi_\lambda|$ for all $\lambda \in \Lambda$ then $\Phi|_{\mathfrak{T}(\mathcal{H}_0)} = \text{Id}_{\mathcal{H}_0}$, where \mathcal{H}_0 is the subspace generated by the family $\{|\varphi_\lambda\rangle\}_{\lambda\in\Lambda}$.

Proof. Let $\Phi(A) = \text{Tr}_{\mathcal{H}} VAV^*$ be the Stinespring representation of the channel Φ in which V is an isometry from \mathcal{H} into $\mathcal{H} \otimes \mathcal{H}$.

Using the standard arguments based on Zorn's lemma, we can show that any complete orthogonally nondecomposable family of pure states contains a countable complete orthogonally nondecomposable subfamily (Lemma 7 in the Appendix).

Let $\{|\varphi_i\rangle\langle\varphi_i|\}$ be a countable orthogonally nondecomposable subfamily of the family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda\in\Lambda}$ such that the vectors of the family $\{|\varphi_i\rangle\}$ generate the subspace \mathcal{H}_0 . The hypothesis of the lemma implies

$$V|\varphi_i\rangle = |\varphi_i\rangle \otimes |\psi_i\rangle \quad \forall i,$$

where $\{|\psi_i\rangle\}$ is a family of unit vectors in \mathcal{H} . Since V is an isometry, we have

$$\langle\varphi_i|\varphi_j\rangle = \langle V\varphi_i|V\varphi_j\rangle = \langle\varphi_i|\varphi_j\rangle\langle\psi_i|\psi_j\rangle \quad \forall i, j$$

and hence $\langle\varphi_i|\varphi_j\rangle \neq 0 \implies \langle\psi_i|\psi_j\rangle = 1$.

It follows that $|\psi_i\rangle = |\psi_j\rangle$ for all i, j . Otherwise, we could decompose the set of all indices into two subsets I and J such that $|\psi_i\rangle \neq |\psi_j\rangle$ for all $i \in I, j \in J$ and the above implication implies $\langle\varphi_i|\varphi_j\rangle = 0$ for all $i \in I, j \in J$, and this contradicts the fact that the family $\{|\varphi_i\rangle\langle\varphi_i|\}$ is orthogonally nondecomposable.

Thus $V|\varphi_i\rangle = |\varphi_i\rangle \otimes |\psi\rangle$ for all i and hence $V|\varphi\rangle = |\varphi\rangle \otimes |\psi\rangle$ for all $|\varphi\rangle \in \mathcal{H}_0$ since the family of vectors $\{|\varphi_i\rangle\}$ generates the subspace \mathcal{H}_0 . Hence $\Phi(A) = A$ for any operator $A \in \mathfrak{T}(\mathcal{H}_0)$.

In analysing the reversibility of a channel with respect to orthogonally decomposable families of pure states the following simple observation plays an essential role.

Lemma 5. *An arbitrary family \mathfrak{S} of pure states in $\mathfrak{S}(\mathcal{H})$ can be decomposed as $\mathfrak{S} = \bigcup_k \mathfrak{S}_k$, where $\{\mathfrak{S}_k\}$ is a finite or countable collection of disjoint orthogonally nondecomposable subfamilies of \mathfrak{S} such that $\rho \perp \rho'$ for all $\rho \in \mathfrak{S}_k, \rho' \in \mathfrak{S}_{k'}, k \neq k'$. This decomposition is unique (up to permutations of the subfamilies).*

Proof. For a given state $\rho \in \mathfrak{S}$ consider the monotonic sequence $\{\mathfrak{C}_n^\rho\}$ of subfamilies of \mathfrak{S} constructed as follows. Let $\mathfrak{C}_1^\rho = \{\rho\}$, let \mathfrak{C}_2^ρ be the family of all states in \mathfrak{S} nonorthogonal to the state ρ , \mathfrak{C}_{n+1}^ρ the family of all states in \mathfrak{S} nonorthogonal to at least one state in $\mathfrak{C}_n^\rho, n = 2, 3, \dots$. Let $\mathfrak{C}_*^\rho = \bigcup_n \mathfrak{C}_n^\rho$. It is easy to verify by induction that \mathfrak{C}_n^ρ is an orthogonally nondecomposable family for each n and hence \mathfrak{C}_*^ρ is an orthogonally nondecomposable family. Note that any state in \mathfrak{C}_*^ρ is orthogonal to any state in $\mathfrak{S} \setminus \mathfrak{C}_*^\rho$. Indeed, if $\rho \in \mathfrak{C}_*^\rho$ then $\rho \in \mathfrak{C}_n^\rho$ for some n . So, if a state σ is nonorthogonal to the state ρ then it lies in $\mathfrak{C}_{n+1}^\rho \subseteq \mathfrak{C}_*^\rho$.

It is easy to see that the families \mathfrak{C}_*^ρ and $\mathfrak{C}_*^{\rho'}, \rho, \rho' \in \mathfrak{S}$ either coincide or have empty intersection. Since the Hilbert space \mathcal{H} is separable and each family \mathfrak{C}_*^ρ corresponds to a nontrivial subspace of \mathcal{H} , the collection $\{\mathfrak{C}_*^\rho\}_{\rho \in \mathfrak{S}}$ contains either a finite or countable number of different families. These families form the required decomposition.

The above decomposition of a complete family \mathfrak{S} of pure states provides a description of the class of all channels which are reversible with respect to \mathfrak{S} .

Theorem 4. *Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel, \mathfrak{S} a complete family of pure states in $\mathfrak{S}(\mathcal{H}_A)$ and let $m = \min\{\dim \ker \Phi^* + 1, \dim \mathcal{H}_B\}$. Let $\mathfrak{S} = \bigcup_k \mathfrak{S}_k$ be the decomposition of \mathfrak{S} into orthogonally nondecomposable subfamilies (from Lemma 5) and P_k the projector onto the subspace generated by the states in \mathfrak{S}_k . The following statements are equivalent:*

- (i) *the channel Φ is reversible with respect to the family \mathfrak{S} ;*
- (ii) *the channel Φ is reversible with respect to the family*

$$\widehat{\mathfrak{S}} = \left\{ \rho \in \mathfrak{S}(\mathcal{H}_A) \mid \rho = \sum_k P_k \rho P_k \right\};$$

- (iii) *$\widehat{\Phi}$ is a c - q channel with the representation $\widehat{\Phi}(A) = \sum_k [\text{Tr} A P_k] \sigma_k$, where $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_E)$ such that $\text{rank } \sigma_k \leq m$ for all k ;*
- (iv) *the channel Φ is isometrically equivalent to the channel*

$$\Phi'(A) = \sum_{k,l} P_k A P_l \otimes \sum_{p,t=1}^m \langle \psi_t^l | \psi_p^k \rangle |p\rangle \langle t|$$

from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_A \otimes \mathcal{H}_m)$, where $\{|\psi_p^k\rangle\}$ is a set of vectors in a separable Hilbert space such that $\sum_{p=1}^m \|\psi_p^k\|^2 = 1$ and $\langle \psi_t^k | \psi_p^k \rangle = 0$ for all $p \neq t$ for each k and $\{|p\rangle\}$ is an orthonormal basis in \mathcal{H}_m .

Proof. (i) \implies (ii). Let Ψ be a reversing channel for the channel Φ with respect to the family \mathfrak{S} and \mathcal{H}_k the subspace of \mathcal{H} generated by the states in \mathfrak{S}_k . Since \mathfrak{S}_k is an orthogonally nondecomposable family, Lemma 4 shows that $\Psi \circ \Phi|_{\mathfrak{T}(\mathcal{H}_k)} = \text{Id}_{\mathcal{H}_k}$ for all k .

(ii) \implies (iii). Let $\{|\phi_i\rangle\}$ be an orthonormal basis corresponding to the decomposition $\mathcal{H}_A = \bigoplus_k \mathcal{H}_k$ (such that each vector $|\phi_i\rangle$ lies in some \mathcal{H}_k). Let I_k be the set of all i such that $|\phi_i\rangle \in \mathcal{H}_k$. Since $|\phi_i\rangle\langle\phi_i| \in \mathfrak{S}$ for all i , the channel Φ is reversible with respect to the family $\{|\phi_i\rangle\langle\phi_i|\}$. By Proposition 1 we have

$$\widehat{\Phi}(A) = \sum_k \sum_{i \in I_k} \langle\phi_i|A\phi_i\rangle\sigma_i,$$

where $\{\sigma_i\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_E)$ such that $\text{rank}\sigma_i \leq m$ for all i . Since \mathfrak{S}_k is an orthogonally nondecomposable family, Proposition 2 shows that the restriction of $\widehat{\Phi}$ to the set $\widehat{\mathfrak{S}}(\mathcal{H}_k)$ is a completely depolarizing channel. Thus $\sigma_i = \bar{\sigma}_k$ for all $i \in I_k$ and hence $\widehat{\Phi}(A) = \sum_k [\text{Tr}AP_k]\bar{\sigma}_k$.

(iii) \implies (iv). Let $k(i)$ be the index of the set I_k containing i , that is, $i \in I_{k(i)}$ for all i . If $\sigma_k = \sum_{p=1}^m |\psi_p^k\rangle\langle\psi_p^k|$ then $\widehat{\Phi}(A) = \sum_{i,p} W_{ip}AW_{ip}^*$, where $W_{ip} = |\psi_p^{k(i)}\rangle\langle\phi_i|$ and hence the representation (2.6) implies

$$\begin{aligned} \widehat{\Phi}(A) &= \sum_{i,j,p,t} [\text{Tr}W_{ip}AW_{jt}^*]|\phi_i\rangle\langle\phi_j| \otimes |p\rangle\langle t| \\ &= \sum_{k,l,p,t} \sum_{i \in I_k, j \in I_l} \langle\phi_i|A\phi_j\rangle|\phi_i\rangle\langle\phi_j| \otimes \langle\psi_t^l|\psi_p^k\rangle|p\rangle\langle t| = \sum_{k,l} P_kAP_l \otimes \sum_{p,t} \langle\psi_t^l|\psi_p^k\rangle|p\rangle\langle t|, \end{aligned}$$

where $\{|p\rangle\}$ is an orthonormal basis in \mathcal{H}_m .

(iv) \implies (i) follows from Lemma 1, since $\Psi(\cdot) = \text{Tr}_{\mathcal{H}_m}(\cdot)$ is a reversing channel for the channel Φ' with respect to the family \mathfrak{S} .

We can deduce the following useful observation from Theorem 4.

Corollary 2. *If a quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is reversible with respect to a complete family \mathfrak{S} of pure states in $\mathfrak{S}(\mathcal{H}_A)$ then it is reversible with respect to some complete family of orthogonal pure states in $\mathfrak{S}(\mathcal{H}_A)$.*

Remark 1. If a complete family of pure states \mathfrak{S} contains a subfamily $\mathfrak{S}_0 = \{|\varphi_i\rangle\langle\varphi_i|\}$ such that $\{|\varphi_i\rangle\}$ is a basis in \mathcal{H}_A (in the sense that any vector $|\psi\rangle$ in \mathcal{H}_A has a unique decomposition $|\psi\rangle = \sum_i c_i|\varphi_i\rangle$),⁶ then the family of orthogonal pure states mentioned in Corollary 2 is given explicitly by Theorem 3. Indeed, by Lemma 8 in the Appendix $\{|\phi_i\rangle\}$, the set of vectors defined in (4.2) by means of an arbitrary nondegenerate probability distribution $\{\pi_i\}$, forms an orthonormal basis in \mathcal{H}_A . It is easy to see that the channel Φ' defined by (4.3) is reversible with respect to the family $\{|\phi_i\rangle\langle\phi_i|\}$. By Theorem 3 and Lemma 1 the same reversibility holds for the channel Φ .

⁶The existence of this subfamily \mathfrak{S}_0 is obvious if \mathcal{H}_A is finite-dimensional. Conditions which guarantee that a complete countable family of unit vectors in an infinite-dimensional Hilbert space forms a basis can be found in [10], Ch. 1.

Theorem 4 gives the following description of the class of all reversible channels between finite-dimensional quantum systems of the same dimensions.

Corollary 3. *Let \mathcal{H} be a finite-dimensional Hilbert space and \mathfrak{S} a complete family of pure states in $\mathfrak{S}(\mathcal{H})$. Let $\mathfrak{S} = \bigcup_k \mathfrak{S}_k$ be the decomposition of \mathfrak{S} into orthogonally nondecomposable subfamilies (from Lemma 5) and P_k the projector onto the subspace generated by the states in \mathfrak{S}_k .*

A quantum channel $\Phi: \mathfrak{T}(\mathcal{H}) \rightarrow \mathfrak{T}(\mathcal{H})$ is reversible with respect to the family \mathfrak{S} if and only if it is unitarily equivalent to the channel

$$\Phi'(A) = \sum_{k,l} c_{kl} P_k A P_l, \quad A \in \mathfrak{T}(\mathcal{H}),$$

where $\|c_{kl}\|$ is the Gram matrix of a set of unit vectors in a finite-dimensional Hilbert space.

Proof. Since $\Phi'(\rho) = \rho$ for all $\rho \in \mathfrak{S}$, if the channels Φ and Φ' are unitarily equivalent then Φ is reversible with respect to the family \mathfrak{S} .

By Corollary 2 if Φ is reversible with respect to \mathfrak{S} it is reversible with respect to some family $\{\rho_i\}_{i=1}^n$ of orthogonal pure states in $\mathfrak{S}(\mathcal{H})$ (where $n = \dim \mathcal{H}$). As relative entropy is monotonic we have

$$\frac{1}{n} \sum_{i=1}^n H(\Phi(\rho_i) \| \Phi(\bar{\rho})) = \frac{1}{n} \sum_{i=1}^n H(\rho_i \| \bar{\rho}) = \log n,$$

where $\bar{\rho} = n^{-1} \sum_{i=1}^n \rho_i = n^{-1} I_{\mathcal{H}}$. The left-hand side of this equality is the χ -quantity of the family of states $\{\Phi(\rho_i)\}_{i=1}^n$ with the uniform probability distribution (see § 5). It follows from the general properties of the χ -quantity (see [4], Ch. 5) that this family consists of orthogonal pure states and that $\Phi(I_{\mathcal{H}}) = I_{\mathcal{H}}$.

By (2.2), which gives the definition of the complementary channel, $\{\widehat{\Phi}(\rho_i)\}_{i=1}^n$ is a family of pure states. Theorem 4 shows that $\widehat{\Phi}(A) = \sum_k [\text{Tr} A P_k] |\psi_k\rangle \langle \psi_k|$, where $\{|\psi_k\rangle\}$ is a set of unit vectors in \mathcal{H}_E . Hence the channel Φ is isometrically equivalent to the channel $\widehat{\Phi} = \Phi'$ with matrix $c_{kl} = \langle \psi_l | \psi_k \rangle$. Since $\Phi(I_{\mathcal{H}}) = \Phi'(I_{\mathcal{H}}) = I_{\mathcal{H}}$, if Φ and Φ' are isometrically equivalent they are unitarily equivalent.

Remark 2. Corollary 3 shows that if $\dim \mathcal{H}_A = \dim \mathcal{H}_B < +\infty$ then a quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is reversible with respect to a complete family \mathfrak{S} of pure states if and only if $\Phi(\rho) = U \rho U^*$ for all $\rho \in \mathfrak{S}$, where U is a unitary operator; that is, Φ being reversible with respect to a complete family of pure states is equivalent to all the states of this family being preserved by Φ (up to a unitary transformation).

§ 5. A condition for the χ -quantity to be preserved and some corollaries

We consider some applications of the results obtained in § 4 to quantum information theory.

A finite or countable set $\{\rho_i\}$ of states in $\mathfrak{S}(\mathcal{H})$ with the corresponding probability distribution $\{\pi_i\}$ is called an *ensemble* and denoted by $\{\pi_i, \rho_i\}$, the state $\bar{\rho} = \sum_i \pi_i \rho_i$ is called the *average state* of this ensemble.

The χ -quantity of an ensemble $\{\pi_i, \rho_i\}$ is defined as follows:

$$\chi(\{\pi_i, \rho_i\}) \doteq \sum_i \pi_i H(\rho_i \| \bar{\rho}) = H(\bar{\rho}) - \sum_i \pi_i H(\rho_i),$$

where the second formula is valid provided that $H(\bar{\rho}) < +\infty$. It is proved in [18] that this quantity provides an upper bound for the accessible classical information which can be obtained by distinguishing the set of states $\{\rho_i\}$ using quantum measurements (see details in [4], Ch. 5). The χ -quantity plays a central role in the analysis of different protocols of classical information transmissions through a quantum channel and is involved in the expressions for the capacities of these protocols.

Let Φ be a quantum channel from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_B)$. Since the relative entropy is monotonic, for an arbitrary ensemble $\{\pi_i, \rho_i\}$ of states in $\mathfrak{S}(\mathcal{H}_A)$ the following inequality holds

$$\chi(\{\pi_i, \Phi(\rho_i)\}) \leq \chi(\{\pi_i, \rho_i\}). \tag{5.1}$$

Remark 3. If $H(\bar{\rho}) < +\infty$ and $H(\Phi(\bar{\rho})) < +\infty$ then (5.1) means that the function $\rho \mapsto H(\Phi(\rho)) - H(\rho)$, the entropy gain of the channel Φ , is convex.

Using the monotonicity of the relative entropy and Theorem 1 equality in (5.1) (under the condition that the right-hand side is finite) is equivalent to the channel Φ being reversible with respect to the family $\{\rho_i\}$. So, using the results from §4 we can obtain conditions for equality, and this can be interpreted as preserving the χ -quantity of the ensemble $\{\pi_i, \rho_i\}$ under the action of Φ .

When we analyse infinite-dimensional quantum systems and channels we have to consider *generalized* (or *continuous*) ensembles, defined as Borel probability measures on the set of quantum states (from this point of view the ensemble $\{\pi_i, \rho_i\}$ is the purely atomic measure $\sum_i \pi_i \delta_{\rho_i}$, where δ_ρ is the Dirac measure concentrated at the state ρ) [4], [19].

The set of all Borel probability measures on $\mathfrak{S}(\mathcal{H})$ whose support lies in a closed subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ will be denoted by $\mathcal{P}(\mathcal{A})$. We will call such measures generalized ensembles of states in \mathcal{A} .

The average state of a generalized ensemble μ is the barycenter of μ defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \rho \mu(d\rho).$$

The χ -quantity of a generalized ensemble μ is defined as follows:

$$\chi(\mu) = \int_{\mathfrak{S}(\mathcal{H})} H(\rho \| \bar{\rho}(\mu)) \mu(d\rho) = H(\bar{\rho}(\mu)) - \int_{\mathfrak{S}(\mathcal{H})} H(\rho) \mu(d\rho), \tag{5.2}$$

where the second formula is valid provided that $H(\bar{\rho}(\mu)) < +\infty$ [19].

The image of a generalized ensemble $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ under the action of a quantum channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is the generalized ensemble

$$\Phi(\mu) \doteq \mu \circ \Phi^{-1} \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_B)).$$

This has χ -quantity equal to

$$\chi(\Phi(\mu)) = \int_{\mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho)\|\Phi(\bar{\rho}(\mu)))\mu(d\rho) = H(\Phi(\bar{\rho}(\mu))) - \int_{\mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho))\mu(d\rho), \tag{5.3}$$

where the second formula is valid provided that $H(\Phi(\bar{\rho}(\mu))) < +\infty$.

Similarly to the discrete case, as the relative entropy is monotonic this implies the χ -quantity for generalized ensembles is too:

$$\chi(\Phi(\mu)) \leq \chi(\mu). \tag{5.4}$$

Theorem 1 gives the following criterion for equality in (5.4), which is a modification of Theorem 3 in [14] for the von Neumann algebra $\mathcal{M} = \mathfrak{B}(\mathcal{H})$.

Proposition 3. *Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel and μ a generalized ensemble of states in $\mathfrak{S}(\mathcal{H}_A)$ with the nondegenerate average state $\bar{\rho}(\mu)$ such that $\chi(\mu) < +\infty$. The following statements are equivalent:*

- (i) $\chi(\Phi(\mu)) = \chi(\mu)$;
- (ii) $H(\Phi(\rho)\|\Phi(\bar{\rho}(\mu))) = H(\rho\|\bar{\rho}(\mu))$ for μ -almost all ρ in $\mathfrak{S}(\mathcal{H}_A)$;
- (iii) $\rho = \Theta_{\bar{\rho}(\mu)}(\Phi(\rho))$ for μ -almost all ρ in $\mathfrak{S}(\mathcal{H}_A)$;
- (iv) the channel Φ is reversible with respect to μ -almost all ρ in $\mathfrak{S}(\mathcal{H}_A)$.

By contrast with Theorem 3 in [14], in Proposition 3 it is not assumed that the state $\bar{\rho}(\mu)$ is a countable convex mixture of states of the ensemble μ .

Using this proposition, Theorem 3, Corollary 2 and Proposition 1 (taking [14], Lemma 2 into account) we obtain the following necessary conditions for equality in (5.4).

Theorem 5. *Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel. If there exists a generalized ensemble $\mu \in \mathcal{P}(\mathfrak{S}^r)$, where $\mathfrak{S}^r = \{\rho \in \mathfrak{S}(\mathcal{H}_A) \mid \text{rank } \rho \leq r\}$, with the nondegenerate average state $\bar{\rho}(\mu)$ such that*

$$\chi(\Phi(\mu)) = \chi(\mu) < +\infty, \tag{5.5}$$

then the complementary channel $\hat{\Phi}$ has a Kraus representation (2.5) with the number of summands $\dim \mathcal{H}_A \cdot \min\{\dim \ker \Phi^* + r^2, \dim \mathcal{H}_B\}$, such that $\text{rank } V_k \leq r$ for all k , and hence it is a PEB-channel of order r .

If this condition holds with $r = 1$ then assertions (i)–(iii) of Proposition 1 are valid for the channel Φ with a particular orthonormal basis $\{|\varphi_i\rangle\}$ for the space \mathcal{H}_A .

We consider some corollaries of this theorem below, which are related to different characteristics of quantum systems and channels.

5.1. The Holevo capacity and the minimal output entropy of a finite-dimensional quantum channel. Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a channel between finite-dimensional quantum systems ($\dim \mathcal{H}_A, \dim \mathcal{H}_B < +\infty$).

The Holevo capacity of the channel Φ (which is closely related to its classical capacity [4], Ch. 8) is defined as follows:

$$\bar{C}(\Phi) = \max_{\{\pi_i, \rho_i\}} \chi(\{\pi_i, \Phi(\rho_i)\}), \tag{5.6}$$

where the maximum is taken over all ensembles of states in $\mathfrak{S}(\mathcal{H}_A)$.

Using inequality (5.1), one can show that

$$\overline{C}(\Phi) \leq \log \dim \mathcal{H}_A \tag{5.7}$$

for any channel Φ . Theorem 5 gives the following criterion for equality to hold in this inequality.

Corollary 4. *Let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel such that*

$$\dim \mathcal{H}_A, \dim \mathcal{H}_B < +\infty.$$

A) *Equality holds in (5.7) if and only if the equivalent statements (i)–(iii) in Proposition 1 hold for Φ for some particular orthonormal basis $\{|\varphi_i\rangle\}$ of the space \mathcal{H}_A .*

B) *If $\mathcal{H}_B = \mathcal{H}_A$ then equality holds in (5.7) if and only if the channel Φ is unitarily equivalent to the channel Φ' described in Corollary 3 with a particular collection $\{P_k\}$ of mutually orthogonal projectors such that $\sum_k P_k = I_{\mathcal{H}_A}$.*

Another important characteristic of a quantum channel Φ is its minimal output entropy

$$H_{\min}(\Phi) = \min_{\rho \in \mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho)).$$

(The role of this characteristic in analysing the informational properties of quantum channels is considered in [4], Ch. 8.)

Corollary 4 leads to the following criterion for the minimal output entropy of covariant channels to take the value zero.

Corollary 5. *Let $\mathcal{H}_B = \mathcal{H}_A$ be a finite-dimensional space and let $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel covariant with respect to some irreducible representation $\{V_g\}_{g \in G}$ of a compact group G (in the sense that $\Phi(V_g A V_g^*) = V_g \Phi(A) V_g^*$ for all $g \in G$ and all $A \in \mathfrak{T}(\mathcal{H}_A)$).*

The equality $H_{\min}(\Phi) = 0$ holds if and only if Φ is unitarily equivalent to the channel Φ' described in Corollary 3 with a particular collection $\{P_k\}$ of mutually orthogonal projectors such that $\sum_k P_k = I_{\mathcal{H}_A}$.

To derive this assertion from Corollary 4 it suffices to note that the covariance condition implies $\overline{C}(\Phi) = \log \dim \mathcal{H}_B - H_{\min}(\Phi)$ (see [4], Ch. 6).

All the assumptions of Corollary 5 hold for any unital qubit channel Φ , that is, a channel $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ such that $\dim \mathcal{H}_A = \dim \mathcal{H}_B = 2$ and $\Phi(I_{\mathcal{H}_A}) = I_{\mathcal{H}_B}$ (see [4], Ch. 6).

5.2. On the strict decrease of the χ -quantity under partial trace and the strict concavity of the quantum conditional entropy. The operation of partial trace $\mathfrak{T}(\mathcal{H} \otimes \mathcal{K}) \ni A \mapsto \text{Tr}_{\mathcal{K}} A \in \mathfrak{T}(\mathcal{H})$ is a noncommutative analogue of the transition from a joint probability distribution to their partial distributions. This operation can be considered as a quantum channel from $\mathfrak{T}(\mathcal{H} \otimes \mathcal{K})$ into $\mathfrak{T}(\mathcal{H})$. Its complementary channel is the operation of partial trace $A \mapsto \text{Tr}_{\mathcal{H}} A$ over the other space.

Noting that the map $A \mapsto \text{Tr}_{\mathcal{K}} A$ is not a r -PEB channel for any $r < \dim \mathcal{K}$, from Theorem 5 we obtain the following assertion.

Proposition 4. Let $\mathcal{H}_A = \mathcal{H}_B \otimes \mathcal{H}_E$ and $\Phi(A) = \text{Tr}_{\mathcal{H}_E} A$, $A \in \mathfrak{T}(\mathcal{H}_A)$.

A) $\chi(\{\pi_i, \Phi(\rho_i)\}) < \chi(\{\pi_i, \rho_i\})$ for any ensemble $\{\pi_i, \rho_i\}$ of states in $\mathfrak{S}(\mathcal{H}_A)$ with nondegenerate average state such that $\sup_i \text{rank } \rho_i < \dim \mathcal{H}_E$ and $\chi(\{\pi_i, \rho_i\}) < +\infty$.

B) $\chi(\Phi(\mu)) < \chi(\mu)$ for any generalized ensemble μ in $\mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ with nondegenerate average state such that $\sup_{\rho \in \text{supp } \mu} \text{rank } \rho < \dim \mathcal{H}_E$ and $\chi(\mu) < +\infty$.

Remark 4. By the Stinespring representation (2.1) every quantum channel is unitarily equivalent to a restriction of the channel $\Phi(A) = \text{Tr}_{\mathcal{H}_E} A$ to the set of states supported by some subspace of $\mathcal{H}_B \otimes \mathcal{H}_E$. Since the χ -quantity does not necessarily strictly decrease under the action of the quantum channel, the nondegeneracy condition for the average state is necessary in Proposition 4 and hence also in Theorem 5.

The quantum conditional entropy of a state ρ of a composite system AB is defined as follows:

$$H_{A|B}(\rho) \doteq H(\rho) - H(\text{Tr}_{\mathcal{H}_A} \rho)$$

provided that

$$H(\rho) < +\infty \quad \text{and} \quad H(\text{Tr}_{\mathcal{H}_A} \rho) < +\infty. \tag{5.8}$$

By Remark 3 the concavity of the function $\rho \mapsto H_{A|B}(\rho)$ on the convex set defined by (5.8) is equivalent to the χ -quantity being nonincreasing under partial trace. Proposition 4, A) gives the following sufficient condition for the conditional entropy to be strictly concave.

Corollary 6. Let ρ be a nondegenerate state in $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ satisfying condition (5.8). Then

$$H_{A|B}(\rho) > \sum_i \pi_i H_{A|B}(\rho_i)$$

for any ensemble $\{\pi_i, \rho_i\}$ with the average state ρ such that $\sup_i \text{rank } \rho_i < \dim \mathcal{H}_A$.

Using Proposition 4, B) one can obtain a continuous (integral) version of Corollary 6.

It is easy to construct examples showing that the property of strict concavity for the conditional entropy does not hold for any convex decomposition of an arbitrary state ρ .

An important consequence of Theorem 5 is its use in the proof of the criterion for the capacities of two protocols for the transmission of classical information through a quantum channel to coincide, which was considered in [7].

§ 6. Appendix

6.1. The proof of Theorem 1. It suffices to show that (i) \implies (ii). Consider the ensemble of two states ρ and σ with probabilities t and $1 - t$ where $t \in (0, 1)$. Let $\sigma_t = t\rho + (1 - t)\sigma$. By Donald's identity (see [3], Proposition 5.22) we have

$$tH(\rho\|\sigma) + (1 - t)H(\sigma\|\sigma) = tH(\rho\|\sigma_t) + (1 - t)H(\sigma\|\sigma_t) + H(\sigma_t\|\sigma), \tag{6.1}$$

$$\begin{aligned} tH(\Phi(\rho)\|\Phi(\sigma)) + (1 - t)H(\Phi(\sigma)\|\Phi(\sigma)) \\ = tH(\Phi(\rho)\|\Phi(\sigma_t)) + (1 - t)H(\Phi(\sigma)\|\Phi(\sigma_t)) + H(\Phi(\sigma_t)\|\Phi(\sigma)), \end{aligned} \tag{6.2}$$

where the left-hand sides are finite and, by assumption, they coincide. Since all the terms on the right-hand side of (6.1) are no smaller than the corresponding terms on the right-hand side of (6.2), as the relative entropy is monotonic

$$H(\Phi(\rho)\|\Phi(\sigma_t)) = H(\rho\|\sigma_t), \quad H(\Phi(\sigma)\|\Phi(\sigma_t)) = H(\sigma\|\sigma_t). \tag{6.3}$$

It follows from [14], Theorem 3 and Proposition 4 that $\rho = \Theta_t(\Phi(\rho))$ for all $t \in (0, 1)$ where

$$\Theta_t(B) = [\sigma_t]^{1/2}\Phi^*([\Phi(\sigma_t)]^{-1/2}B[\Phi(\sigma_t)]^{-1/2})[\sigma_t]^{1/2}, \quad B \in \mathfrak{B}(\mathcal{H}_B).$$

To complete the proof it suffices to show that

$$\lim_{t \rightarrow +0} \Theta_t = \Theta_\sigma \tag{6.4}$$

in the strong convergence topology on the set of quantum channels; in this topology (see [20]) $\Phi_n \rightarrow \Phi$ means that $\Phi_n(\rho) \rightarrow \Phi(\rho)$ for all ρ , since (6.4) implies that

$$\rho = \lim_{t \rightarrow +0} \Theta_t(\Phi(\rho)) = \Theta_\sigma(\Phi(\rho)).$$

Now, $\Theta_t(\Phi(\sigma)) = \sigma$ for all $t \in (0, 1)$ and so the set of channels $\{\Theta_t\}_{t \in (0,1)}$ is relatively compact in the topology of strong convergence by [20], Corollary 2. Hence there exists a sequence $\{t_n\}$ converging to zero such that

$$\lim_{n \rightarrow +\infty} \Theta_{t_n} = \Theta_0, \tag{6.5}$$

where Θ_0 is a particular channel. We will show that $\Theta_0 = \Theta_\sigma$.

Note that (6.5) means the sequence $\{\Theta_{t_n}^*(A)\}$ converges to the operator $\Theta_0^*(A)$ in the weak operator topology for any positive operator $A \in \mathfrak{B}(\mathcal{H}_A)$.⁷ By Lemma 6 below we have

$$\lim_{n \rightarrow +\infty} [\Phi(\sigma_{t_n})]^{1/2}\Theta_{t_n}^*(A)[\Phi(\sigma_{t_n})]^{1/2} = [\Phi(\sigma)]^{1/2}\Theta_0^*(A)[\Phi(\sigma)]^{1/2} \tag{6.6}$$

in the Hilbert-Schmidt norm. The explicit form of the map $\Theta_{t_n}^*$ shows that

$$[\Phi(\sigma_{t_n})]^{1/2}\Theta_{t_n}^*(A)[\Phi(\sigma_{t_n})]^{1/2} = \Phi([\sigma_{t_n}]^{1/2}A[\sigma_{t_n}]^{1/2})$$

and, since $\lim_{n \rightarrow +\infty} [\sigma_{t_n}]^{1/2}A[\sigma_{t_n}]^{1/2} = [\sigma]^{1/2}A[\sigma]^{1/2}$ in the trace norm, the limit in (6.6) coincides with $\Phi([\sigma]^{1/2}A[\sigma]^{1/2})$. Thus, $\Theta_0^*(A) = \Theta_\sigma^*(A)$ for all A in $\mathfrak{B}(\mathcal{H}_A)$ and hence $\Theta_0 = \Theta_\sigma$.

The above arguments show that for any sequence $\{t_n\}$ converging to zero any partial limit of the sequence $\{\Theta_{t_n}\}$ coincides with Θ_σ . This is equivalent to (6.4).

Lemma 6. *Let $\{\rho_n\}$ be a sequence of states in $\mathfrak{S}(\mathcal{H})$ converging to a state ρ_0 and let $\{A_n\}$ be a sequence of operators from the unit ball of $\mathfrak{B}(\mathcal{H})$ converging to an operator A_0 in the weak operator topology. Then the sequence $\{\sqrt{\rho_n}A_n\sqrt{\rho_n}\}$ converges to the operator $\sqrt{\rho_0}A_0\sqrt{\rho_0}$ in the Hilbert-Schmidt norm.*

⁷This is because this topology coincides with the σ -weak operator topology on the unit ball of the space $\mathfrak{B}(\mathcal{H}_B)$ (see [11]).

Proof. Since $\{\rho_n\}_{n \geq 0}$ is a compact set, the compactness criterion for subsets of $\mathfrak{S}(\mathcal{H})$ (see [19], the Appendix) shows that for any $\varepsilon > 0$ there exists a projector P_ε of finite rank such that $\text{Tr } \overline{P}_\varepsilon \rho_n < \varepsilon$ for all $n \geq 0$, where $\overline{P}_\varepsilon = I_{\mathcal{H}} - P_\varepsilon$. We have

$$\begin{aligned} \sqrt{\rho_n} A_n \sqrt{\rho_n} &= \sqrt{\rho_n} P_\varepsilon A_n P_\varepsilon \sqrt{\rho_n} + \sqrt{\rho_n} P_\varepsilon A_n \overline{P}_\varepsilon \sqrt{\rho_n} \\ &\quad + \sqrt{\rho_n} \overline{P}_\varepsilon A_n P_\varepsilon \sqrt{\rho_n} + \sqrt{\rho_n} \overline{P}_\varepsilon A_n \overline{P}_\varepsilon \sqrt{\rho_n}, \quad n \geq 0. \end{aligned} \tag{6.7}$$

Since P_ε is a finite rank projector, the sequence $\{P_\varepsilon A_n P_\varepsilon\}$ converges to the operator $P_\varepsilon A_0 P_\varepsilon$ in the norm of the space $\mathfrak{B}(\mathcal{H})$ and hence the sequence $\{\sqrt{\rho_n} P_\varepsilon A_n P_\varepsilon \sqrt{\rho_n}\}$ converges to the operator $\sqrt{\rho_0} P_\varepsilon A_0 P_\varepsilon \sqrt{\rho_0}$ in the trace norm. Further, it is easy to see that the Hilbert-Schmidt norm of the other terms in the right-hand side of (6.7) tends to zero as $\varepsilon \rightarrow 0$ uniformly in n .

6.2. Some auxiliary results.

Lemma 7. *An arbitrary complete orthogonally nondecomposable family of pure states in a separable Hilbert space \mathcal{H} contains a countable complete orthogonally nondecomposable subfamily.*

Proof. Let \mathfrak{H} be the set of all subspaces of \mathcal{H} generated by countable orthogonally nondecomposable subfamilies of \mathfrak{S} endowed with partial ordering by inclusion. Let \mathfrak{H}_0 be a chain in \mathfrak{H} and $\mathcal{H}_0 = \overline{\bigcup_{\mathcal{K} \in \mathfrak{H}_0} \mathcal{K}}$. Since there is a countable chain $\{\mathcal{H}_k\}$ in \mathfrak{H} such that $\mathcal{H}_0 = \overline{\bigcup_k \mathcal{H}_k}$, and a countable union of countable orthogonally nondecomposable subfamilies is a countable orthogonally nondecomposable subfamily, the subspace \mathcal{H}_0 lies in \mathfrak{H} . Hence \mathcal{H}_0 is an upper bound for the chain \mathfrak{H}_0 , and Zorn’s lemma implies the existence of a maximal element \mathcal{H}_m in \mathfrak{H} . Suppose that $\mathcal{H}_m \subsetneq \mathcal{H}$. Since the family \mathfrak{S} is complete and orthogonally nondecomposable, it contains a pure state $|\varphi\rangle\langle\varphi|$ such that the vector $|\varphi\rangle$ lies neither in \mathcal{H}_m nor in \mathcal{H}_m^\perp . By adding the state $|\varphi\rangle\langle\varphi|$ to the countable orthogonally nondecomposable subfamily corresponding to the subspace \mathcal{H}_m we obtain a countable orthogonally nondecomposable subfamily. Hence $\mathcal{H}_m \vee \{\lambda|\varphi\rangle\} \in \mathfrak{H}$, contradicting the maximality of \mathcal{H}_m .

Lemma 8. *Let $\{|\varphi_i\rangle\}$ be a basis in a separable Hilbert space \mathcal{H} (in the sense that any vector $|\psi\rangle$ in \mathcal{H} has a unique decomposition $|\psi\rangle = \sum_i c_i |\varphi_i\rangle$). The set of vectors $\{|\phi_i\rangle\}$, defined in (4.2) by means of any nondegenerate probability distribution $\{\pi_i\}$, forms an orthonormal basis in \mathcal{H} .*

Proof. Since $\sum_i |\phi_i\rangle\langle\phi_i| = I_{\mathcal{H}}$ for any j , we have

$$|\phi_j\rangle = \sum_i \langle\phi_i|\phi_j\rangle |\phi_i\rangle$$

and hence

$$(\|\phi_j\|^2 - 1)|\phi_j\rangle + \sum_{i \neq j} \langle\phi_i|\phi_j\rangle |\phi_i\rangle = 0.$$

By applying the operator $\sqrt{\pi_j}$ (defined in (4.2)) to all the terms of this vector equality we obtain

$$\sqrt{\pi_j}(\|\phi_j\|^2 - 1)|\varphi_j\rangle + \sum_{i \neq j} \sqrt{\pi_i} \langle\phi_i|\phi_j\rangle |\varphi_i\rangle = 0.$$

Since $\{|\varphi_i\rangle\}$ is a basis and $\pi_i > 0$ for all i , we have $\|\phi_j\|^2 = 1$ and $\langle\phi_i|\phi_j\rangle = 0$ for all $i \neq j$. So, $\{|\phi_i\rangle\}$ is an orthonormal set of vectors in \mathcal{H} . This set is a basis since $\sum_i |\phi_i\rangle\langle\phi_i| = I_{\mathcal{H}}$.

The author is grateful to A. S. Holevo and the participants in his seminar “Quantum probability, statistics and information” (the Steklov Mathematical Institute) for their interest in this work and for some useful discussions. The author is also grateful to A. Jenčová for her help in solving a particular problem, and to the referee for useful suggestions for improving the paper and for pointing out some misprints.

Bibliography

- [1] A. S. Holevo, *Statistical structure of quantum theory*, IKI, Moscow–Izhevsk 2003; English transl., Lect. Notes Phys. Monogr., vol. 67, Springer-Verlag, Berlin 2001.
- [2] D. Petz, “Sufficiency of channels over von Neumann algebras”, *Quart. J. Math.* **39**:1 (1988), 97–108.
- [3] M. Ohya and D. Petz, *Quantum entropy and its use*, Texts Monogr. Phys., Springer-Verlag, Berlin 1993.
- [4] A. S. Holevo, *Quantum systems, channels, information. A mathematical introduction*, Moscow Center for Continuous Mathematical Education, Moscow 2010; English transl., De Gruyter Stud. Math. Phys., vol. 16, de Gruyter, Berlin 2012.
- [5] F. Hiai, M. Mosonyi, D. Petz and C. Beny, “Quantum f -divergences and error correction”, *Rev. Math. Phys.* **23**:7 (2011), 691–747.
- [6] A. Jenčová, “Reversibility conditions for quantum operations”, *Rev. Math. Phys.* **24**:7 (2012), 1250016.
- [7] M. E. Shirokov, *A criterion for coincidence of the entanglement-assisted classical capacity and the Holevo capacity of a quantum channel*, arXiv: 1202.3449.
- [8] T. Ogawa, A. Sasaki, M. Iwamoto and H. Yamamoto, “Quantum secret sharing schemes and reversibility of quantum operations”, *Phys. Rev. A* **72** (2005), 032318.
- [9] A. S. Holevo, “Complementary channels and the additivity problem”, *Teoriya Veroyatn. i Prilozhen.* **51**:1 (2006), 133–143; English transl. in *Theory Probab. Appl.* **51**:1 (2007), 92–100.
- [10] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Vyshcha Shkola, Kiev 1977; English transl., Pitman, Boston–London 1981.
- [11] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics 1. C^* - and W^* -algebras, symmetry groups, decomposition of states*, Texts and Monographs in Physics, Springer-Verlag, New York–Heidelberg–Berlin 1979.
- [12] C. King, K. Matsumoto, M. Nathanson and M. B. Ruskai, “Properties of conjugate channels with applications to additivity and multiplicativity”, *Markov Process. Related Fields* **13**:2 (2007), 391–423.
- [13] M. Horodecki, P. W. Shor and M. B. Ruskai, “Entanglement breaking channels”, *Rev. Math. Phys.* **15**:6 (2003), 629–641.
- [14] A. Jencova and D. Petz, “Sufficiency in quantum statistical inference”, *Comm. Math. Phys.* **263**:1 (2006), 259–276.
- [15] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*, Cambridge Univ. Press, Cambridge 2000.
- [16] B. M. Terhal and P. Horodecki, “Schmidt number for density matrices”, *Phys. Rev. A* **61**:4 (2000), 040301.

- [17] M. E. Shirokov, *The Schmidt number and partially entanglement breaking channels in infinite dimensions*, [arXiv: 1110.4363](#).
- [18] A. S. Holevo, “Some estimates for the amount of information transmittable by a quantum communication channel”, *Probl. Peredachi Informatsii* **9**:3 (1973), 3–11; English transl. in *Problems Inform. Transmission* **9**:3 (1973), 177–183.
- [19] A. S. Holevo and M. E. Shirokov, “Continuous ensembles and the capacity of infinite-dimensional quantum channels”, *Teoriya Veroyatn. i Prilozhen.* **50**:1 (2005), 98–114; English transl. in *Theory Probab. Appl.* **50**:1 (2005), 86–98.
- [20] A. S. Holevo and M. E. Shirokov, “On approximation of infinite-dimensional quantum channels”, *Probl. Peredachi Informatsii* **44**:2 (2008), 3–22; English transl. in *Problems Inform. Transmission* **44**:2 (2008), 73–90.

Maksim E. Shirokov
Steklov Mathematical Institute,
Russian Academy of Sciences, Moscow
E-mail: msh@mi.ras.ru

Received 18/JUL/12 and 18/MAR/13
Translated by M. SHIROKOV