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# Criteria for equality in two entropic inequalities

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**Abstract.** We obtain a simple criterion for local equality between the constrained Holevo capacity and the quantum mutual information of a quantum channel. This shows that the set of all states for which this equality holds is determined by the kernel of the channel (as a linear map).

Applications to Bosonic Gaussian channels are considered. It is shown that for a Gaussian channel having no completely depolarizing components the above characteristics may coincide only at non-Gaussian mixed states and a criterion for the existence of such states is given.

All the obtained results may be reformulated as conditions for equality between the constrained Holevo capacity of a quantum channel and the input von Neumann entropy.

Bibliography: 20 titles.

**Keywords:** quantum state, quantum channel, von Neumann entropy, quantum mutual information, Holevo capacity of a quantum channel.

## § 1. Introduction

Quantum information theory is a new scientific direction that has developed rapidly in the last two decades. It has generated a whole variety of interesting mathematical problems, whose formulation and analysis require methods from the theory of operators in Hilbert spaces, convex analysis and measure theory.

A basic role is played in quantum information theory by the notion of a quantum channel — a linear trace-preserving completely positive map between Banach spaces of trace class operators (Schatten classes of order 1). Generally speaking, such maps describe the irreversible dynamics of open quantum systems (see [1], Ch. 6).

Properties of quantum channels are described by many numerical and functional characteristics, in particular, by different capacities related to the reversibility properties of a channel. On the one hand, these capacities are defined as characteristics of given protocols of (classical or quantum) information transmissions by a channel, on the other hand, they are related to the analytical characteristics of a channel as a completely positive map (see [2]).

In this paper we consider two important characteristics of a quantum channel: the constrained Holevo capacity[1] $\bar{C}(\Phi,\rho)$ (also called the $\chi$-function) and the

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[1] The informational sense of this quantity is considered in [1], Ch. 8.

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quantum mutual information $I(\Phi,\rho)$ (one of the noncommutative analogues of the mutual information of classical channels introduced by Shannon); see [1], [3], [4]. These nonnegative characteristics have the following upper bounds

$$\overline{C}(\Phi,\rho)\leqslant H(\rho),\qquad I(\Phi,\rho)\leqslant 2H(\rho),\tag{1.1}$$

where $H(\rho)$ is the von Neumann entropy of a state $\rho$, and are connected by the inequality

$$\overline{C}(\Phi,\rho)\leqslant I(\Phi,\rho).\tag{1.2}$$

It is well known that equality in the second inequality in (1.1) is equivalent to perfect reversibility of the channel $\Phi$ on the support of the state $\rho$ (see [1], [4]). In this paper we obtain necessary and sufficient conditions for equality in the first inequality in (1.1), and in (1.2), expressed in terms of the structure of the channel $\Phi$.

We show that these inequalities are connected via the notion of a complementary channel. Thus, we may analyse conditions for equality in (1.2) (which are important for applications) by studying conditions for equality in the first inequality in (1.1).

This approach and the relation obtained between equality in the first inequality in (1.1) and the particular reversibility property of the channel $\Phi$ make it possible to obtain a criterion for equality in (1.2) for an arbitrary infinite-dimensional channel $\Phi$ and a state $\rho$ with finite von Neumann entropy. This criterion shows that the set of all mixed states with finite entropy, for which we have equality in (1.2), is determined by the set $\ker\Phi$ (Theorem 3). It also makes it possible to prove that this equality holds for all states $\rho$ if and only if $\Phi$ is a completely depolarizing channel (a conjecture made in [5]).

The above criterion is used to analyse the attainability of equality in (1.2) for Bosonic Gaussian channels. In particular, we show that for an arbitrary nontrivial Gaussian channel $\Phi$, a strict inequality holds in (1.2) for all non-degenerate states with finite entropy, while its validity for all mixed states with finite entropy is equivalent to coincidence of the rank of the operator describing the transformation of canonical observables with the dimension of the input symplectic space.

A criterion for equality in the first inequality in (1.1) and its application to Bosonic Gaussian channels are presented in § 4.

## § 2. Preliminaries

Let $\mathscr{H}$ be a separable Hilbert space, and $\mathfrak{B}(\mathscr{H})$ and $\mathfrak{T}(\mathscr{H})$ be the Banach spaces of, respectively, all bounded operators in $\mathscr{H}$ with the operator norm $\|\cdot\|$, and all trace-class operators in $\mathscr{H}$ with the trace norm $\|\cdot\|_1=\operatorname{Tr}|\cdot|$ (see [1], [6]). Operators from $\mathfrak{B}(\mathscr{H})$ will be denoted by Latin letters $A,B,\dots$, and operators from $\mathfrak{T}(\mathscr{H})$ by Greek letters $\rho,\sigma,\dots$. The closed convex subset

$$\mathfrak{S}(\mathscr{H})=\{\rho\in\mathfrak{T}(\mathscr{H})\mid\rho\geqslant 0,\ \operatorname{Tr}\rho=1\}$$

of $\mathfrak{T}(\mathscr{H})$ is a complete separable metric space with the metric defined by the trace norm. Operators in $\mathfrak{S}(\mathscr{H})$ will be called *states* since any such operator $\rho$ determines a linear normal functional $A\mapsto\operatorname{Tr}A\rho$ with the unit norm on $\mathfrak{B}(\mathscr{H})$. *Pure states* are one-dimensional projectors, which are extreme points of $\mathfrak{S}(\mathscr{H})$. The support $\operatorname{supp}\rho$ of a state $\rho$ is the orthogonal complement to its kernel $\ker\rho$; its dimension is called the rank of this state: $\operatorname{rank}\rho = \dim\operatorname{supp}\rho$. A state $\rho$ such that $\ker\rho = \{0\}$ is called nondegenerate.

For vectors and 1-rank operators in a Hilbert space we will use the Dirac notations $|\varphi\rangle, |\chi\rangle\langle\psi|, \dots$ (in which the action of the operator $|\chi\rangle\langle\psi|$ on the vector $|\varphi\rangle$ is the vector $\langle\psi,\varphi\rangle|\chi\rangle$).

Denote by $I_{\mathscr{H}}$ and $\mathrm{Id}_{\mathscr{H}}$ the unit operator in a Hilbert space $\mathscr{H}$ and the identity transformation of the Banach space $\mathfrak{T}(\mathscr{H})$, respectively.

The von Neumann entropy of a state $\rho$ is defined as follows:[2]

$$H(\rho) = -\operatorname{Tr}\rho\log\rho = -\sum_{i=1}^{+\infty}\lambda_i\log\lambda_i,$$

where $\{\lambda_i\}$ is a set of eigenvalues of $\rho$ (see [1], [4]).

The quantum relative entropy of states $\rho$ and $\sigma$ is defined as follows:

$$H(\rho\|\sigma) = \sum_{i=1}^{+\infty}\langle\varphi_i|[\rho\log\rho - \rho\log\sigma]\varphi_i\rangle,$$

where $\{|\varphi_i\rangle\}_{i=1}^{+\infty}$ is an orthonormal basis of eigenvectors of the state $\rho$ (or $\sigma$) and it is assumed that $H(\rho\|\sigma) = +\infty$ if $\operatorname{supp}\rho \nsubseteq \operatorname{supp}\sigma$ (see [1], [4]).

A finite or countable set of states $\{\rho_i\}$ with the corresponding probability distribution $\{\pi_i\}$ is called an *ensemble* and is denoted by $\{\pi_i,\rho_i\}$; the state $\overline{\rho} = \sum_i\pi_i\rho_i$ is called the *average state* of this ensemble.

The $\chi$-quantity of an ensemble $\{\pi_i,\rho_i\}$ is defined as follows:

$$\chi(\{\pi_i,\rho_i\}) \doteq \sum_i\pi_i H(\rho_i\|\overline{\rho}) = H(\overline{\rho}) - \sum_i\pi_i H(\rho_i), \tag{2.1}$$

where the second formula is valid under the condition $H(\overline{\rho}) < +\infty$. In [7] it is proved that this quantity is the upper bound for the classical mutual information which can be extracted from the ensemble $\{\pi_i,\rho_i\}$ by quantum measurements (for further details, see [1], Ch. 5). The $\chi$-quantity plays a central role in the analysis of different protocols of transmission of classical information by quantum channel; it is involved in the expressions for the capacities of these protocols.

Let $\mathscr{H}_A$ and $\mathscr{H}_B$ be Hilbert spaces, referred to as the input and output spaces, respectively. Let $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ be a linear map, which is positive and trace-preserving ($\Phi(\rho)\geqslant 0$ and $\operatorname{Tr}\Phi(\rho) = \operatorname{Tr}\rho$ for any $\rho\geqslant 0$). The *dual* map $\Phi^*\colon\mathfrak{B}(\mathscr{H}_B)\to\mathfrak{B}(\mathscr{H}_A)$ (defined by the relation $\operatorname{Tr}\Phi(\rho)A = \operatorname{Tr}\rho\Phi^*(A)$, $\rho\in\mathfrak{T}(\mathscr{H}_A)$, $A\in\mathfrak{B}(\mathscr{H}_B)$) is a positive map such that $\Phi^*(I_{\mathscr{H}_B}) = I_{\mathscr{H}_A}$.

Let $\mathscr{H}_d$ be a Hilbert space with dimension $d\in\mathbb{N}$ (isomorphic to $\mathbb{C}^d$).

The linear map $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ is called *completely positive* if the map $\Phi\otimes\mathrm{Id}_{\mathscr{H}_d}$ from $\mathfrak{T}(\mathscr{H}_A\otimes\mathscr{H}_d)$ into $\mathfrak{T}(\mathscr{H}_B\otimes\mathscr{H}_d)$ is positive for all natural $d$ (equivalent definitions of complete positivity can be found in [1], Ch. 6, § 2).

**Definition 1.** A linear completely positive trace-preserving map $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ is called a *quantum channel*.

[2]Here log is the natural logarithm.

This definition of a quantum channel corresponds to the Schrödinger picture, in which the dynamics of a quantum system is described via the evolution of states. In the Heisenberg picture, a quantum channel is the dual map $\Phi^*\colon \mathfrak{B}(\mathscr{H}_B) \to \mathfrak{B}(\mathscr{H}_A)$, determining the evolution of quantum observables (see [1], Ch. 6).

By using Stinespring's theorem on representations of completely positive maps of $C^*$-algebras and properties of the algebra $\mathfrak{B}(\mathscr{H})$, one can obtain (see [1], [4]) the following representation of an arbitrary quantum channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$: there exists a Hilbert space $\mathscr{H}_E$ and an isometry $V\colon \mathscr{H}_A \to \mathscr{H}_B \otimes \mathscr{H}_E$ such that

$$\Phi(\rho) = \operatorname{Tr}_{\mathscr{H}_E} V\rho V^* \quad \forall\, \rho \in \mathfrak{T}(\mathscr{H}_A). \tag{2.2}$$

This representation is called the Stinespring representation of $\Phi$, while the operator $V$ is the Stinespring isometry.

The quantum channel

$$\mathfrak{T}(\mathscr{H}_A) \ni \rho \mapsto \widehat{\Phi}(\rho) = \operatorname{Tr}_{\mathscr{H}_B} V\rho V^* \in \mathfrak{T}(\mathscr{H}_E) \tag{2.3}$$

is said to be *complementary* to the channel $\Phi$ (see [1], Ch. 6, § 6 and [8]). The complementary channel is defined uniquely: if $\widehat{\Phi}'\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_{E'})$ is a channel defined by formula (2.3) via the Stinespring isometry $V'\colon \mathscr{H}_A \to \mathscr{H}_B \otimes \mathscr{H}_{E'}$, then the channels $\widehat{\Phi}$ and $\widehat{\Phi}'$ are isometrically equivalent in the sense of the following definition (see the Appendix in [8]).

**Definition 2.** The channels $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$ and $\Phi'\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}'_B)$ are called *isometrically equivalent* if there exists a partial isometry $W\colon \mathscr{H}_B \to \mathscr{H}_{B'}$ such that

$$\Phi'(\rho) = W\Phi(\rho)W^*, \quad \Phi(\rho) = W^*\Phi'(\rho)W, \qquad \rho \in \mathfrak{T}(\mathscr{H}_A). \tag{2.4}$$

The notion of isometrical equivalence is very close to the notion of unitary equivalence (see the remark after Definition 2 in [9]).

We will use the following natural definition.

**Definition 3.** The restriction of a channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$ to the subspace $\mathfrak{T}(\mathscr{H}_0)$, where $\mathscr{H}_0$ is a nontrivial subspace of $\mathscr{H}_A$, is called the *subchannel of* $\Phi$ *corresponding to the subspace* $\mathscr{H}_0$.

It follows from the definition that the channel complementary to the subchannel of $\Phi$ corresponding to the subspace $\mathscr{H}_0$ coincides with the subchannel of the complementary channel $\widehat{\Phi}$, corresponding to the subspace $\mathscr{H}_0$, i.e., $\widehat{\Psi} = \widehat{\Phi}|_{\mathfrak{T}(\mathscr{H}_0)}$, where $\Psi = \Phi|_{\mathfrak{T}(\mathscr{H}_0)}$.

The following class of quantum channels plays an essential role in this paper (see [1], [4]).

**Definition 4.** A quantum channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$ is called a *classical-quantum channel of discrete type* (briefly, *discrete c-q channel*) if it has the representation

$$\Phi(\rho) = \sum_{i=1}^{\dim \mathscr{H}_A} \langle\varphi_i|\rho\varphi_i\rangle\sigma_i, \qquad \rho \in \mathfrak{T}(\mathscr{H}_A), \tag{2.5}$$

in which $\{|\varphi_i\rangle\}$ is an orthonormal basis in $\mathscr{H}_A$ and $\{\sigma_i\}$ is a set of states from $\mathfrak{S}(\mathscr{H}_B)$.

Note that there exist classical-quantum nondiscrete channels (see the Appendix in [10]).

A channel (2.5) such that $\sigma_i = \sigma$ for all $i$ has the representation $\Phi(\rho) = [\operatorname{Tr}\rho]\sigma$ and is called *completely depolarizing* (see [1], [4]). We will use the following simple lemma.

**Lemma 1.** *A channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$ is completely depolarizing if and only if $\Phi(|\varphi\rangle\langle\psi|) = 0$ for any orthogonal vectors $\varphi, \psi \in \mathscr{H}_A$.*

For an arbitrary channel $\Phi$ and any ensemble $\{\pi_i, \rho_i\}$ of its input states, the $\chi$-quantity of the ensemble $\{\pi_i, \Phi(\rho_i)\}$ will be denoted by $\chi_\Phi(\{\pi_i, \rho_i\})$.

The constrained Holevo capacity of a channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$ at a state $\rho \in \mathfrak{S}(\mathscr{H}_A)$ is defined as follows:[3]

$$\overline{C}(\Phi, \rho) = \sup_{\sum_i \pi_i\rho_i = \rho} \chi_\Phi(\{\pi_i, \rho_i\}), \tag{2.6}$$

where the supremum is over all finite or countable ensembles $\{\pi_i, \rho_i\}$ with the average state $\rho$ (see [1], [11]). If $H(\Phi(\rho)) < +\infty$, then

$$\overline{C}(\Phi, \rho) = H(\Phi(\rho)) - \widehat{H}_\Phi(\rho), \tag{2.7}$$

where $\widehat{H}_\Phi(\rho) = \inf_{\sum_i \pi_i\rho_i=\rho} \sum_i \pi_i H(\Phi(\rho_i))$ is the $\sigma$-convex hull of the concave function $\rho \mapsto H(\Phi(\rho))$. Note that the supremum in (2.6) and the infimum in (2.7) can be taken over ensembles of pure states (due to the convexity and concavity of the corresponding functions).

By monotonicity of the quantum relative entropy,

$$\chi_\Phi(\{\pi_i, \rho_i\}) \leqslant \chi(\{\pi_i, \rho_i\}) \leqslant H(\rho) \tag{2.8}$$

for any ensemble $\{\pi_i, \rho_i\}$ with the average state $\rho$. Hence for an arbitrary quantum channel $\Phi$ and any state $\rho$ the following inequality holds

$$\overline{C}(\Phi, \rho) \leqslant H(\rho). \tag{2.9}$$

The quantum mutual information of a finite-dimensional channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$ at a state $\rho \in \mathfrak{S}(\mathscr{H}_A)$ is defined by the expression

$$I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\Phi, \rho), \tag{2.10}$$

where $H(\Phi, \rho)$ is the entropy exchange of $\Phi$ at $\rho$ (see [1], [3], [4]).

Using the notion of a complementary channel, this expression can be rewritten as follows:

$$I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\widehat{\Phi}(\rho)) \tag{2.11}$$

(since it is easy to check that $H(\widehat{\Phi}(\rho)) = H(\Phi, \rho)$; see [1], Ch. 7]).

To avoid the uncertainty $\infty - \infty$ in (2.10) and in (2.11) it is reasonable in the infinite-dimensional case to define the quantum mutual information by the expression

$$I(\Phi, \rho) = H\big(\Phi \otimes \mathrm{Id}_{\mathscr{H}_R}(\widehat{\rho}) \| \Phi \otimes \mathrm{Id}_{\mathscr{H}_R}(\rho \otimes \varrho)\big),$$

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[3] In [5], [11] the constrained Holevo capacity $\rho \mapsto \overline{C}(\Phi, \rho)$ is denoted by $\chi_\Phi(\rho)$ and called the $\chi$-function of the channel $\Phi$.

where $\mathscr{H}_R$ is a Hilbert space isomorphic to $\mathscr{H}_A$, $\widehat{\rho}$ is a purification[4] of the state $\rho$ in $\mathscr{H}_A\otimes\mathscr{H}_R$, and $\varrho=\operatorname{Tr}_{\mathscr{H}_A}\widehat{\rho}$ is a state in $\mathfrak{S}(\mathscr{H}_R)$ isomorphic to $\rho$ (see [1], [10]).

We will use the following important expression

$$I(\Phi,\rho)=H(\rho)+\overline{C}(\Phi,\rho)-\overline{C}(\widehat{\Phi},\rho), \tag{2.12}$$

valid for an arbitrary quantum channel $\Phi$ and any state $\rho$ with finite entropy (the condition $H(\rho)<+\infty$ guarantees the finiteness of all terms in (2.12) by inequality (2.9)). If $H(\Phi(\rho))<+\infty$ and $H(\widehat{\Phi}(\rho))<+\infty$, then this expression is easily derived by using (2.7) and noting that $\widehat{H}_\Phi\equiv\widehat{H}_{\widehat{\Phi}}$ (this follows from the coincidence of the functions $\rho\mapsto H(\Phi(\rho))$ and $\rho\mapsto H(\widehat{\Phi}(\rho))$ on the set of pure states; see [8]). In the general case, expression (2.12) is proved by using Proposition 4 in [10].

The constrained Holevo capacity and the quantum mutual information of an arbitrary channel $\Phi$ at a state $\rho$ are related by the inequality

$$\overline{C}(\Phi,\rho)\leqslant I(\Phi,\rho). \tag{2.13}$$

If $H(\rho)<+\infty$, then this inequality follows directly from (2.9) and (2.12). For an arbitrary state $\rho$ it can be proved by using the sequence of finite-rank states $\rho_n=[\operatorname{Tr}P_n\rho]^{-1}P_n\rho$, where $P_n$ is the spectral projector of $\rho$ corresponding to its $n$ maximal eigenvalues. The concavity and lower semicontinuity of the functions $\rho\mapsto\overline{C}(\Phi,\rho)$ and $\rho\mapsto I(\Phi,\rho)$ imply

$$\lim_{n\to+\infty}\overline{C}(\Phi,\rho_n)=\overline{C}(\Phi,\rho)\leqslant+\infty,\qquad \lim_{n\to+\infty}I(\Phi,\rho_n)=I(\Phi,\rho)\leqslant+\infty. \tag{2.14}$$

Hence inequality (2.13) for $\rho$ follows from the validity of this inequality for each state of the sequence $\{\rho_n\}$.

Expression (2.12) shows that inequalities (2.9) and (2.13) are, roughly speaking, mutually complementary under the condition $H(\rho)<+\infty$, in particular

$$\{\overline{C}(\Phi,\rho)=I(\Phi,\rho)\}\quad\Longleftrightarrow\quad\{\overline{C}(\widehat{\Phi},\rho)=H(\rho)\}, \tag{2.15}$$

and hence one can obtain conditions for equality in (2.9) by studying conditions for equality in (2.13) and vice versa.

The following result (proved by using (2.15)) is essential for the present paper.

**Lemma 2.** *Let $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ and $\Psi\colon\mathfrak{T}(\mathscr{H}_B)\to\mathfrak{T}(\mathscr{H}_C)$ be quantum channels and $\rho$ be a state in $\mathfrak{S}(\mathscr{H}_A)$ such that $H(\rho)<+\infty$. Then*

$$\overline{C}(\Phi,\rho)=I(\Phi,\rho)\quad\Longrightarrow\quad\overline{C}(\Psi\circ\Phi,\rho)=I(\Psi\circ\Phi,\rho).$$

*Proof.* In the proof of Lemma 17 in [12] the existence of a channel $\Theta$ such that $\widehat{\Phi}=\Theta\circ\widehat{\Psi\circ\Phi}$ was shown.[5] Thus, the chain rule for the Holevo capacity and (2.9) imply that

$$\overline{C}(\widehat{\Phi},\rho)=H(\rho)\quad\Longrightarrow\quad\overline{C}(\widehat{\Psi\circ\Phi},\rho)=H(\rho).$$

By (2.15), this implication is equivalent to the assertion of the lemma.

[4]This means that $\widehat{\rho}$ is a pure state in $\mathfrak{S}(\mathscr{H}_A\otimes\mathscr{H}_R)$ such that $\operatorname{Tr}_{\mathscr{H}_R}\widehat{\rho}=\rho$.

[5]This can also be proved using the representation of a complementary channel via the Kraus operators of the initial channel (see [8], formula (11)).

We will study the case of equality in (2.9) by using the notion of the reversibility[6] of a quantum channel with respect to a family of input states (see [13], [14]).

**Definition 5.** A quantum channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$ is *reversible* with respect to a family $\mathfrak{S} \subseteq \mathfrak{S}(\mathscr{H}_A)$ if there exists a quantum channel $\Psi\colon \mathfrak{T}(\mathscr{H}_B) \to \mathfrak{T}(\mathscr{H}_A)$ such that $\rho = \Psi \circ \Phi(\rho)$ for all $\rho \in \mathfrak{S}$.

Note that a quantum channel is reversible with respect to the family of all input states if and only if this channel is noiseless (see [1], Ch. 9).

**Definition 6.** A quantum channel $\Phi$ is called *noiseless* if it is unitary equivalent to the channel $\rho \mapsto \rho \otimes \sigma$, where $\sigma$ is a given state.

A general criterion for the reversibility of a quantum channel with respect to families of input states was obtained in [13]. It gives the following criterion for equality in the first inequality in (2.8).

**Theorem 1.** *Let $\mathfrak{S} = \{\rho_i\}$ be a family of states in $\mathfrak{S}(\mathscr{H}_A)$ and $\{\pi_i\}$ be a non-degenerate probability distribution such that $\chi(\{\pi_i, \rho_i\}) < +\infty$. A quantum channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$ is reversible with respect to the family $\mathfrak{S}$ if and only if*

$$\chi_\Phi(\{\pi_i, \rho_i\}) = \chi(\{\pi_i, \rho_i\}). \tag{2.16}$$

*Remark* 1. The nontrivial part of Theorem 1 is the assertion that (2.16) implies the reversibility of $\Phi$. The converse implication is immediately deduced from the first inequality in (2.8) and Definition 5.

By this implication, (2.6) and the second formula in (2.1) show that the reversibility of $\Phi$ with respect to some family $\{\rho_i\}$ of pure states implies that $\overline{C}(\Phi, \overline{\rho}) = H(\overline{\rho})$, where $\overline{\rho} = \sum_i \pi_i \rho_i$, for any probability distribution $\{\pi_i\}$. We will prove the strong converse of this assertion below: *if $\overline{C}(\Phi, \rho) = H(\rho)$ for a mixed state $\rho$, then $\Phi$ is reversible with respect to the family of orthogonal pure states corresponding to a particular basis of eigenvectors of $\rho$* (see the proof of Theorem 3).

Necessary and sufficient conditions for the reversibility of a quantum channel with respect to complete families of pure states were obtained in [9], where the following natural definition was used.

**Definition 7.** A family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda\in\Lambda}$ of pure states in $\mathfrak{S}(\mathscr{H})$ is called *orthogonally-indecomposable* if there is no subspace $\mathscr{H}_0 \subset \mathscr{H}$ such that some vectors of the family $\{|\varphi_\lambda\rangle\}_{\lambda\in\Lambda}$ lie in $\mathscr{H}_0$, while all the others are in $\mathscr{H}_0^\perp$.

It is easy to show that an arbitrary family $\mathfrak{S}$ of pure states in $\mathfrak{S}(\mathscr{H})$ is represented as $\mathfrak{S} = \bigcup_k \mathfrak{S}_k$, where $\{\mathfrak{S}_k\}$ is a finite or countable set of orthogonally-indecomposable subfamilies of $\mathfrak{S}$ such that $\rho \perp \sigma$ for all $\rho \in \mathfrak{S}_k$, $\sigma \in \mathfrak{S}_l$, $k \neq l$ (see [9], Lemma 4.3). This decomposition is unique (up to permutation of subfamilies).

By using the notion of a subchannel (Definition 3) and the remark after this definition, one can deduce from [9] (Proposition 1 and Theorem 4) the following two statements.

[6]In [13] this property is called the sufficiency of the channel $\Phi$ with respect to a family $\mathfrak{S}$.

**Theorem 2.** *Let $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ be a quantum channel and $\mathfrak{S}=\{|\varphi_i\rangle\langle\varphi_i|\}$ be a family of pure states in $\mathfrak{S}(\mathscr{H}_A)$.*

A) *If $\mathfrak{S}$ consists of orthogonal states, then $\Phi$ is reversible with respect to $\mathfrak{S}$ if and only if*

$$\widehat{\Phi}(\rho)=\sum_{i=1}^{\dim\mathscr{H}_{\mathfrak{S}}}\langle\varphi_i|\rho\varphi_i\rangle\sigma_i\quad\forall\,\rho\in\mathfrak{T}(\mathscr{H}_{\mathfrak{S}}),$$

*where $\mathscr{H}_{\mathfrak{S}}$ is a subspace of $\mathscr{H}_A$ generated by the family $\{|\varphi_i\rangle\}$ and $\{\sigma_i\}$ is a collection of states in $\mathfrak{S}(\mathscr{H}_E)$.*

B) *Let $\mathfrak{S}=\bigcup_k\mathfrak{S}_k$ be a decomposition of $\mathfrak{S}$ into orthogonally-indecomposable subfamilies and $P_k$ be the projector on the subspace generated by all the states in $\mathfrak{S}_k$. If $\Phi$ is reversible with respect to $\mathfrak{S}$, then it is reversible with respect to the family*

$$\widehat{\mathfrak{S}}=\biggl\{\rho\in\mathfrak{S}(\mathscr{H}_A)\biggm|\rho=\sum_k P_k\rho P_k\biggr\}.$$

Theorem 2 shows in particular that the reversibility of a quantum channel with respect to at least one family of pure states is equivalent to the existence of at least one discrete $c$-$q$ subchannel of the complementary channel. A simple criterion for the last property is given by the following lemma.

**Lemma 3.** *A channel $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ has a discrete $c$-$q$ subchannel if and only if there exists an orthogonal family $\{|\varphi_i\rangle\}$ of unit vectors in $\mathscr{H}_A$ such that $\Phi(|\varphi_i\rangle\langle\varphi_j|)=0$ for all $i\neq j$. In this case the subchannel $\Phi|_{\mathfrak{T}(\mathscr{H}_0)}$ has representation (2.5) with $\mathscr{H}_0=\overline{\operatorname{lin}}\{|\varphi_i\rangle\}$ instead of $\mathscr{H}_A$.*

To prove this lemma it suffices to note that $\rho=\sum_{i,j}\langle\varphi_i|\rho\varphi_j\rangle|\varphi_i\rangle\langle\varphi_j|$ for all $\rho$ in $\mathfrak{T}(\mathscr{H}_0)$.

## § 3. The main results

Using Theorem 1 and equivalence relation (2.15) it was shown in [5] that

$$\overline{C}(\Phi,\rho)=I(\Phi,\rho)\quad\Longrightarrow\quad\Phi|_{\mathfrak{T}(\mathscr{H}_\rho)}\text{ is a discrete }c\text{-}q\text{ channel}$$

for a finite-dimensional channel $\Phi$, where $\mathscr{H}_\rho$ is the support of the state $\rho$. Theorem 2 makes it possible to strengthen this result by showing that $\Phi|_{\mathfrak{T}(\mathscr{H}_\rho)}$ is a discrete $c$-$q$ channel determined by a particular basis of eigenvectors of $\rho$.[7] This gives the following criterion for the equality $\overline{C}(\Phi,\rho)=I(\Phi,\rho)$.

**Theorem 3.** *Let $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ be a quantum channel and $\Pi(\Phi)$ be the set of all orthogonal families $\{|\varphi_i\rangle\}$ of unit vectors in $\mathscr{H}_A$ such that $\Phi(|\varphi_i\rangle\langle\varphi_j|)=0$ for all $i\neq j$.*

A) *Let $\rho$ be a mixed state in $\mathfrak{S}(\mathscr{H}_A)$ such that $H(\rho)<+\infty$. The following statements are equivalent*:

(i) $\overline{C}(\Phi,\rho)=I(\Phi,\rho)$;

[7] Here and in what follows we assume that $\rho$ is a mixed state (since $\overline{C}(\Phi,\rho)=I(\Phi,\rho)=0$ for any pure state $\rho$). Speaking about a basis of eigenvectors of a state $\rho$, we keep in mind a basis in the support $\mathscr{H}_\rho$ of this state.

(ii) $\Pi(\Phi)$ *contains at least one basis of eigenvectors of* $\rho$;

(iii) $\Phi(\varrho)=\sum_i\langle\varphi_i|\varrho\varphi_i\rangle\sigma_i$ *for any* $\varrho\in\mathfrak{T}(\mathscr{H}_\rho)$, *where* $\mathscr{H}_\rho=\operatorname{supp}\rho$, $\{|\varphi_i\rangle\}$ *is a particular orthonormal basis of eigenvectors of* $\rho$ *and* $\{\sigma_i\}$ *is a collection of states in* $\mathfrak{S}(\mathscr{H}_B)$.

*For a state* $\rho$ *with* $H(\rho)=+\infty$, *these statements are related as follows*: (iii) $\iff$ (ii) $\Longrightarrow$ (i) (*with possible values* $+\infty$ *in both parts of the equality in* (i)).

B) *The set* $\mathfrak{S}_\Phi^{=}$ *of all mixed states* $\rho$ *in* $\mathfrak{S}(\mathscr{H}_A)$ *with finite entropy, for which* (i) *holds, can be represented as follows*

$$\mathfrak{S}_\Phi^{=}=\bigcup_{\{|\varphi_i\rangle\}\in\Pi(\Phi)}\biggl\{\rho=\sum_i\pi_i|\varphi_i\rangle\langle\varphi_i|\biggm|\{\pi_i\}\in\mathfrak{P}_{\mathrm{f}}\biggr\},\tag{3.1}$$

*where* $\mathfrak{P}_{\mathrm{f}}$ *is the set of all probability distributions with finite Shannon entropy.*

*Remark* 2. Statement (i) in Theorem 3 does not imply that $\Pi(\Phi)$ contains any orthonormal basis of eigenvectors of $\rho$. To show this, consider the channel

$$\Phi(\rho)=\langle\varphi|\rho\varphi\rangle|\varphi\rangle\langle\varphi|+\langle\psi|\rho\psi\rangle|\psi\rangle\langle\psi|$$

from $\mathfrak{T}(\mathscr{H}_2)$ into itself, where $\{|\varphi\rangle,|\psi\rangle\}$ is an orthonormal basis in $\mathscr{H}_2$. It is easy to see that $\widehat{\Phi}=\Phi$ and hence $\overline{C}(\Phi,\rho_c)=I(\Phi,\rho_c)=H(\rho_c)=\log 2$, where $\rho_c=\frac{1}{2}I_{\mathscr{H}_2}$. The set $\Pi(\Phi)$ contains only one basis of eigenvectors of the state $\rho_c$, the basis $\{|\varphi\rangle,|\psi\rangle\}$.

*Remark* 3. By Theorem 3, $\mathfrak{S}_\Phi^{=}$ is completely determined by the set $\ker\Phi$ (since this set determines $\Pi(\Phi)$). Examples of channels $\Phi$ for which $\Pi(\Phi)$ contains infinitely many different incomplete families of vectors are considered in § 4 (where a complete description is given of $\Pi(\Phi)$ for Bosonic Gaussian channels).

*Remark* 4. Although the condition $H(\rho)<+\infty$ is essentially used in the proof of the implication (i) $\Longrightarrow$ (ii), (iii) in Theorem 3 (since it is based on relation (2.15)), it seems technical. The question about the validity of this implication and of Corollary 1 below for states with infinite entropy remains open.

*Proof of Theorem* 3. A) Lemma 3 shows directly that (ii) $\iff$ (iii).

(iii) $\Longrightarrow$ (i). Note that the channels $\Phi$ and $\widehat{\widehat{\Phi}}$ are isometrically equivalent (see Definition 2 and the remark before). So, (iii) holds for $\Phi$ if and only if (iii) holds for $\widehat{\widehat{\Phi}}$ (with the same basis $\{|\varphi_i\rangle\}$).

By this remark and part A) of Theorem 2, (iii) implies the reversibility of the channel $\widehat{\Phi}$ with respect to the family $\{|\varphi_i\rangle\langle\varphi_i|\}$. By Remark 1, this reversibility shows that $\overline{C}(\widehat{\Phi},\rho)=H(\rho)$. So, if $H(\rho)<+\infty$, then (i) follows from (2.15).

If $H(\rho)=+\infty$, then the above reversibility shows that $\overline{C}(\widehat{\Phi},\rho_n)=H(\rho_n)$ and hence $\overline{C}(\Phi,\rho_n)=I(\Phi,\rho_n)$, where $\rho_n=[\operatorname{Tr}P_n\rho]^{-1}P_n\rho$, $P_n=\sum_{i=1}^n|\varphi_i\rangle\langle\varphi_i|$. It follows from (2.14) that $\overline{C}(\Phi,\rho)=I(\Phi,\rho)\leqslant+\infty$.

(i) $\Longrightarrow$ (iii). Here we will prove this implication, assuming that $\Phi$ is a finite-dimensional channel ($\dim\mathscr{H}_A,\dim\mathscr{H}_B<+\infty$). This assumption makes it possible to show the basis idea of the proof without technical difficulties which inevitably appear in the analysis of infinite-dimensional channels. The general proof of this implication is presented in § 6.1.

If $\Phi$ is a finite-dimensional channel, then $\widehat{\Phi}$ is also finite-dimensional (see [8]) and hence for any state $\rho\in\mathfrak{S}(\mathscr{H}_A)$ the supremum in the expression for $\overline{C}(\widehat{\Phi},\rho)$ (expression (2.6) with $\Phi$ replaced by $\widehat{\Phi}$) is attained in some finite ensemble $\{\pi_i,\rho_i\}$ of pure states, that is,

$$\overline{C}(\widehat{\Phi},\rho)=\chi_{\widehat{\Phi}}(\{\pi_i,\rho_i\}),\qquad \sum_i\pi_i\rho_i=\rho$$

(the existence of such an ensemble is proved by the arguments in [15]).

If (i) holds, then (2.15) implies $\overline{C}(\widehat{\Phi},\rho)=H(\rho)$. Since $H(\rho)=\chi(\{\pi_i,\rho_i\})$ by the second formula in (2.1), this means that

$$\chi_{\widehat{\Phi}}(\{\pi_i,\rho_i\})=\chi(\{\pi_i,\rho_i\}).$$

By Theorem 1, this is equivalent to the reversibility of $\widehat{\Phi}$ with respect to the family $\mathfrak{S}=\{\rho_i\}$. Let $\mathfrak{S}=\bigcup_k\mathfrak{S}_k$ be the decomposition of $\mathfrak{S}$ into orthogonally-indecomposable subfamilies (see the paragraph before Theorem 2). Denote by $I_k$ the set of all $i$ such that $\rho_i\in\mathfrak{S}_k$. Let $\{|\varphi^i_k\rangle\}_i$ be an orthonormal basis of eigenvectors of the positive operator $\rho_k=\sum_{i\in I_k}\pi_i\rho_i$. Since $\rho=\sum_k\rho_k$ and $\operatorname{supp}\rho_k\perp\operatorname{supp}\rho_l$ for all $k\neq l$, $\{|\varphi^i_k\rangle\}_{ik}$ is an orthonormal basis of eigenvectors of $\rho$.

By part B) of Theorem 2, the reversibility of $\widehat{\Phi}$ with respect to $\mathfrak{S}$ implies the reversibility of this channel with respect to the orthogonal family $\{|\varphi^i_k\rangle\langle\varphi^i_k|\}_{ik}$ (contained in the family $\widehat{\mathfrak{S}}$). Part A) of Theorem 2 implies the validity of (iii) with the basis $\{|\varphi^i_k\rangle\}_{ik}$ for the channel $\widehat{\widehat{\Phi}}$ and hence for $\Phi$ (by the remark at the begin of the proof of the implication (iii)$\Longrightarrow$ (i)).

B) Representation (3.1) follows directly from part A) of the theorem.

Theorem 3 gives sufficient conditions for strict inequality in (2.13).

**Corollary 1.** *Let $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ be a quantum channel.*

A) *If $\Phi$ is not a discrete c-q channel, then $\overline{C}(\Phi,\rho)<I(\Phi,\rho)$ for any nondegenerate state $\rho$ in $\mathfrak{S}(\mathscr{H}_A)$ with finite entropy.*

B) *If the set $\ker\Phi$ does not contain 1-rank operators, then $\overline{C}(\Phi,\rho)<I(\Phi,\rho)$ for all mixed states $\rho$ in $\mathfrak{S}(\mathscr{H}_A)$ with finite entropy.*

Examples of channels for which the condition of part B) of Corollary 1 holds are considered in § 4 (Proposition 1 and Example 1).

The following result is obtained in the proof of the implication (i) $\Longrightarrow$ (iii) of Theorem 3 (its generalized version is presented in § 6.1).

**Corollary 2.** *Let $\Phi$ be a channel that is reversible with respect to the family $\{\rho_i\}$ of pure states and $\{\pi_i\}$ be an arbitrary probability distribution. Then $\Phi$ is reversible with respect to the family of orthogonal pure states corresponding to a particular basis of eigenvectors of the state $\overline{\rho}\doteq\sum_i\pi_i\rho_i$.*

Now we consider a criterion for global equality in (2.13) and prove the strengthened version of the conjecture mentioned in [5].

**Corollary 3.** *If $\overline{C}(\Phi,\rho)=I(\Phi,\rho)$ for any state $\rho$ of rank 2, then $\Phi$ is a completely depolarizing channel and hence $\overline{C}(\Phi,\rho)=I(\Phi,\rho)=0$ for any state $\rho$.*

*Proof.* By Lemma 1, it suffices to show that $\Phi(|\varphi\rangle\langle\psi|)=0$ for any orthogonal unit vectors $\varphi,\psi\in\mathscr{H}_A$.

Let $\rho=0.3|\varphi\rangle\langle\varphi|+0.7|\psi\rangle\langle\psi|$ be a state in $\mathfrak{S}(\mathscr{H}_A)$ of rank 2. By assumption, $\overline{C}(\Phi,\rho)=I(\Phi,\rho)$, and Theorem 3 implies that $\Phi(|\varphi\rangle\langle\psi|)=0$, since $\{|\varphi\rangle,|\psi\rangle\}$ is the only basis of eigenvectors of $\rho$.

*Remark* 5. Corollary 3 shows that for any nontrivial channel $\Phi$ the concave nonzero functions $\rho\mapsto\overline{C}(\Phi,\rho)$ and $\rho\mapsto I(\Phi,\rho)$ (equal to zero on the set of 1-rank states) are always separated by a particular 2-rank state.

It was shown in [5] that for a finite-dimensional channel $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ the following relation holds

$$D(\Phi)\doteq\max_{\rho\in\mathfrak{S}(\mathscr{H}_A)}\bigl[I(\Phi,\rho)-\overline{C}(\Phi,\rho)\bigr]=\sup_{H,h}\bigl[C_{\mathrm{ea}}(\Phi,H,h)-\overline{C}(\Phi,H,h)\bigr],$$

where $C_{\mathrm{ea}}(\Phi,H,h)$ and $\overline{C}(\Phi,H,h)$ are, respectively, the classical entanglement-assisted capacity and the Holevo capacity of $\Phi$ with the linear constraint determined by the inequality $\operatorname{Tr}H\rho\leqslant h$, and the supremum is taken over all pairs (positive operator $H$, positive number $h$). Corollary 3 shows that $D(\Phi)>0$ if $\Phi$ is not completely depolarizing. This completes the proof (started in [5]) of the following list of properties of the parameter $D(\Phi)$ (showing that it can be considered as one of the characteristics of $\Phi$ describing its 'level of noise'):

- $D(\Psi\circ\Phi)\leqslant D(\Phi)$ for any channel $\Psi\colon\mathfrak{T}(\mathscr{H}_B)\to\mathfrak{T}(\mathscr{H}_C)$;
- $D(\Phi)\in[0,\log\dim\mathscr{H}_A]$;
- $D(\Phi)=\log\dim\mathscr{H}_A$ if and only if $\Phi$ is a noiseless channel (see Definition 6);
- $D(\Phi)=0$ if and only if $\Phi$ is a completely depolarizing channel.

## § 4. Condition for equality for Bosonic Gaussian channels

Now we use Theorem 3 for the analysis of Bosonic Gaussian channels, the important role of which in quantum information theory is justified by their physical applications (for example, in quantum optics); see [1], [16], [17].

Let $\mathscr{H}_X$ $(X=A,B,\dots)$ be a space of irreducible representations of the Canonical Commutation Relations (CCR)

$$W_X(z)W_X(z')=\exp\biggl[-\frac{\mathrm{i}}{2}\Delta_X(z,z')\biggr]W_X(z'+z), \tag{4.1}$$

where $(Z_X,\Delta_X)$ is a symplectic space and $W_X(z)$ are Weyl operators (see [1], Ch. 11). Denote by $s_X$ the number of modes of the system $X$, i.e., $2s_X=\dim Z_X$.

A Bosonic Gaussian channel $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ is defined via the action of its dual map $\Phi^*\colon\mathfrak{B}(\mathscr{H}_B)\to\mathfrak{B}(\mathscr{H}_A)$ on the Weyl operators:

$$\Phi^*(W_B(z))=W_A(Kz)\exp\biggl[\mathrm{i}lz-\frac{1}{2}z^\top\alpha z\biggr],\qquad z\in Z_B, \tag{4.2}$$

where $K$ is a linear operator $Z_B\to Z_A$, $l$ is a $2s_B$-dimensional real row and $\alpha$ is a $(2s_B\times 2s_B)$-dimensional real symmetric matrix satisfying the inequality

$$\alpha\geqslant\frac{\mathrm{i}}{2}[\Delta_B-K^\top\Delta_AK]. \tag{4.3}$$

By means of unitary displacement operators, any Gaussian channel can be transformed into the Gaussian channel with $l=0$ and the same matrices $K$ and $\alpha$ (this was shown in [17]). Such a channel is called *centred* and will be denoted by $\Phi_{K,\alpha}$. Therefore, in the study of relations between constrained Holevo capacity and quantum mutual information we may (and will) assume that all Gaussian channels are centred.

In [10], Proposition 5, it was shown that $\Phi_{K,\alpha}$ is a discrete *c-q* channel if and only if $K=0$ (that is, if and only if $\Phi_{K,\alpha}$ is a completely depolarizing channel). This means, by Lemma 3, that for any nontrivial Gaussian channel $\Phi_{K,\alpha}$, $K\neq 0$, the set $\Pi(\Phi_{K,\alpha})$ (introduced in Theorem 3) does not contain a complete family of vectors in $\mathscr{H}_A$.

Lemma 3 and the following Lemma 4 show that $\Phi_{K,\alpha}$ has discrete *c-q* subchannels if and only if $\operatorname{Ran} K\neq Z_A$ (that is, if and only if $\operatorname{rank} K<\dim Z_A$).

**Lemma 4.** *The set* $\Pi(\Phi_{K,\alpha})$ *is nonempty if and only if* $\operatorname{Ran} K\neq Z_A$, *and consists of all orthonormal families* $\{|\varphi_i\rangle\}\subset\mathscr{H}_A$ *such that*

$$\langle\varphi_i|W_A(Kz)\varphi_j\rangle=0\quad\forall\, z\in Z_B,\quad\forall\, i\neq j. \tag{4.4}$$

*Proof*. Since the family $\{W_B(z)\}_{z\in Z_B}$ generates the algebra $\mathfrak{B}(\mathscr{H}_B)$, the equality $\Phi_{K,\alpha}(|\varphi_i\rangle\langle\varphi_j|)=0$ is equivalent to condition (4.4).

If $\operatorname{Ran} K=Z_A$, then the family $\{W_A(Kz)\}_{z\in Z_B}$ of Weyl operators in $\mathscr{H}_A$ is irreducible. So condition (4.4) cannot be valid.

If $\operatorname{Ran} K\neq Z_A$, then the commutant of the family $\{W_A(Kz)\}_{z\in Z_B}$ contains the Weyl operators $W_A(z)$, $z\in[\operatorname{Ran} K]^{\perp}$. Hence there exists a nontrivial subspace in $\mathscr{H}_A$, invariant with respect to this family.[8] This guarantees the existence of at least two orthogonal unit vectors satisfying (4.4).

Let $\Phi_{K,\alpha}$ be a nontrivial Gaussian channel ($K\neq 0$). By Lemma 4 and the remark before it, Theorem 3 shows that the strict inequality $\overline{C}(\Phi_{K,\alpha},\rho)<I(\Phi_{K,\alpha},\rho)$ is valid for any nondegenerate state $\rho$ with finite entropy, while mixed degenerate states, for which we have equality in this inequality, exist if and only if $\operatorname{Ran} K\neq Z_A$. This condition holds in the following cases:[9]

A) $[\operatorname{Ran} K]^{\perp}$ is a nontrivial isotropic subspace in $Z_A$;

B) $[\operatorname{Ran} K]^{\perp}$ contains a nontrivial symplectic subspace.

The proof of Proposition 1 below shows that the Gaussian channel $\Phi_{K,\alpha}$ corresponding to case B) is characterized by the existence of completely depolarizing subchannels and that any such channel can be represented as a partial trace over some input modes followed by a Gaussian channel which either corresponds to case A) or satisfies the condition $\operatorname{Ran} K=Z_A$. Therefore, we will focus attention on case A) and will find all mixed states $\rho$ with finite entropy such that $\overline{C}(\Phi_{K,\alpha},\rho)=I(\Phi_{K,\alpha},\rho)$ by describing the set $\Pi(\Phi_{K,\alpha})$ in the Schrödinger representation.

By Lemma 8 in § 6.3, in case A) there exists a symplectic basis $\{\widetilde{e}_k,\widetilde{h}_k\}$ in $Z_A$ such that $\{\widetilde{e}_1,\dots,\widetilde{e}_{s_A},\widetilde{h}_{d+1},\dots,\widetilde{h}_{s_A}\}$ is a basis in $\operatorname{Ran} K$, $d\leqslant s_A$. Let $Z_B^0$ be

[8] $[\operatorname{Ran} K]^{\perp}$ is the skew-orthogonal complement to $\operatorname{Ran} K\subseteq Z_A$ (see § 6.3). We will always use this sense of the symbol $\perp$ when dealing with a subspace of a symplectic space.

[9] The definitions of isotropic and symplectic subspaces of a symplectic space can be found in § 6.3.

a subspace of $Z_B$ with the basis $\{z_1^e, \ldots, z_{s_A}^e, z_{d+1}^h, \ldots, z_{s_A}^h\}$ such that $\widetilde{e}_k = Kz_k^e$ for all $k = 1, \ldots, s_A$ and $\widetilde{h}_k = Kz_k^h$ for all $k = d+1, \ldots, s_A$. For any vector $z \in Z_B^0$ represented in the form

$$z = \sum_{k=1}^{s_A} x_k z_k^e + \sum_{k=d+1}^{s_A} y_k z_k^h, \qquad (x_1, \ldots, x_{s_A}) \in \mathbb{R}^{s_A}, \quad (y_{d+1}, \ldots, y_{s_A}) \in \mathbb{R}^{s_A - d},$$

it follows from (4.1) that

$$\begin{aligned} W_A(Kz) &= W_A\left(\sum_{k=1}^{s_A} x_k \widetilde{e}_k + \sum_{k=d+1}^{s_A} y_k \widetilde{h}_k\right) \\ &= \lambda W_A(x_1 \widetilde{e}_1) \cdots W_A(x_{s_A} \widetilde{e}_{s_A}) \cdot W_A(y_{d+1} \widetilde{h}_{d+1}) \cdots W_A(y_{s_A} \widetilde{h}_{s_A}), \end{aligned}$$

where $\lambda = e^{i[x_{d+1} y_{d+1} + \cdots + x_{s_A} y_{s_A}]} \neq 0$.

By identifying $\mathscr{H}_A$ with the space $L_2(\mathbb{R}^{s_A})$ of complex-valued functions of $s_A$ variables (denoted by $\xi_1, \ldots, \xi_{s_A}$) and the Weyl operators $W_A(\widetilde{e}_k)$ and $W_A(\widetilde{h}_k)$ with the operators

$$\psi(\xi_1, \ldots, \xi_{s_A}) \mapsto e^{i\xi_k} \psi(\xi_1, \ldots, \xi_{s_A}), \qquad \psi(\xi_1, \ldots, \xi_{s_A}) \mapsto \psi(\xi_1, \ldots, \xi_k + 1, \ldots, \xi_{s_A}),$$

the equality in (4.4) for the vector $z$ can be written as follows

$$\begin{aligned} \int \cdots \int & \overline{\varphi_i(\xi_1, \ldots, \xi_{s_A})} (S_{y_{d+1}, \ldots, y_{s_A}} \varphi_j)(\xi_1, \ldots, \xi_{s_A}) \\ & \times e^{i(x_1 \xi_1 + \cdots + x_{s_A} \xi_{s_A})} \, d\xi_1 \cdots d\xi_{s_A} = 0, \end{aligned}$$

where $(S_{y_{d+1}, \ldots, y_{s_A}} \varphi_j)(\xi_1, \ldots, \xi_{s_A}) = \varphi_j(\xi_1, \ldots, \xi_d, \xi_{d+1} + y_{d+1}, \ldots, \xi_{s_A} + y_{s_A})$.

This equality holds for all $(x_1, \ldots, x_{s_A}) \in \mathbb{R}^{s_A}$ and $(y_{d+1}, \ldots, y_{s_A}) \in \mathbb{R}^{s_A - d}$ (that is, for all $z \in Z_B^0$) if and only if

$$\overline{\varphi_i(\xi_1, \ldots, \xi_{s_A})} (S_{y_{d+1}, \ldots, y_{s_A}} \varphi_j)(\xi_1, \ldots, \xi_{s_A}) = 0$$

for almost all $(\xi_1, \ldots, \xi_{s_A}) \in \mathbb{R}^{s_A}$ and all $(y_{d+1}, \ldots, y_{s_A}) \in \mathbb{R}^{s_A - d}$. Since $\operatorname{Ran} K = K(Z_B^0)$, this means that condition (4.4) holds if and only if

$$\varphi_i \cdot S_{y_{d+1}, \ldots, y_{s_A}} \varphi_j = 0 \quad (\text{in } L_2(\mathbb{R}^{s_A})) \quad \forall\, (y_{d+1}, \ldots, y_{s_A}) \in \mathbb{R}^{s_A - d}, \quad \forall\, i \neq j, \quad (4.5)$$

where $S_{y_{d+1}, \ldots, y_{s_A}}$ is the shift operator in $L_2(\mathbb{R}^{s_A})$ along the last $s_A - d$ coordinates:

$$(S_{y_{d+1}, \ldots, y_{s_A}} \psi)(\xi_1, \ldots, \xi_{s_A}) = \psi(\xi_1, \ldots, \xi_d, \xi_{d+1} + y_{d+1}, \ldots, \xi_{s_A} + y_{s_A}).$$

Condition (4.5) means, roughly speaking, that all shifts in $\mathbb{R}^{s_A}$ of the supports of all functions of the family $\{\varphi_i\}$ along the last $s_A - d$ coordinates do not intersect each other.

*Remark* 6. Condition (4.5) is completely determined by the subspace $\operatorname{Ran} K$. We will say that a family of functions $\{\varphi_i\}$ satisfies condition (4.5) for a certain subspace $Z_0 \subseteq Z_A$ (such that the subspace $[Z_0]^\perp$ is isotropic), if it satisfies the analogue of condition (4.5) constructed by using $Z_0$ instead of $\operatorname{Ran} K$.

Thus, in the Schrödinger representation, $\Pi(\Phi_{K,\alpha})$ consists of all orthogonal families $\{\varphi_i\}$ of functions in $L_2(\mathbb{R}^{s_A})$ with unit norm satisfying (4.5). So, by Theorem 3, the equality $\overline{C}(\Phi_{K,\alpha},\rho)=I(\Phi_{K,\alpha},\rho)$ holds for a mixed state $\rho$ with finite entropy if and only if $\rho$ has a basis of eigenvectors $\{|\varphi_i\rangle\}$ which (in the Schrödinger representation) satisfies condition (4.5). Theorem 3 also shows that this equality holds for any mixed state $\rho$ with infinite entropy having such a basis.

In the analysis of Bosonic systems, a special role is played by *Gaussian* states, states $\rho$ whose characteristic function $\phi_\rho(z)=\operatorname{Tr}W(z)\rho$ has Gaussian form:

$$\phi_\rho(z)=\exp\biggl[\mathrm{i}az-\frac{1}{2}z^\top\sigma_\rho z\biggr],$$

where $a$ is a $2s$-dimensional real row ($2s=\dim Z$) and $\sigma_\rho$ is a $(2s\times 2s)$-dimensional real symmetric matrix satisfying the inequality $\sigma_\rho\geqslant\frac{\mathrm{i}}{2}\Delta$. The row $a$ consists of the mean values of the canonical observables in $\rho$, while $\sigma_\rho$ is the *covariance* matrix of this state (see [1], [16], [17]). The spectral decomposition of a mixed Gaussian state in the Schrödinger representation (described in [1], Ch. 11) shows that its basis of eigenvectors cannot satisfy condition (4.5). Note also that any Gaussian state has finite entropy.

By these remarks, the above arguments imply parts A)–C) of the following proposition.

**Proposition 1.** *Let $\mathfrak{S}_{\mathrm{f}}$ be a subset of $\mathfrak{S}(\mathscr{H}_A)$ consisting of all states with finite entropy.*

A) *If $K\neq 0$ (that is, the channel $\Phi_{K,\alpha}$ is not completely depolarizing), then*

$$\overline{C}(\Phi_{K,\alpha},\rho)<I(\Phi_{K,\alpha},\rho) \tag{4.6}$$

*for all nondegenerate states $\rho\in\mathfrak{S}_{\mathrm{f}}$, in particular, for all nondegenerate Gaussian states $\rho$.*

B) *If $\operatorname{Ran}K=Z_A$, then (4.6) holds for all mixed states $\rho\in\mathfrak{S}_{\mathrm{f}}$, in particular, for all mixed Gaussian states $\rho$.*

C) *If $[\operatorname{Ran}K]^\perp$ is a nontrivial isotropic subspace of $Z_A$, then:*

- *inequality (4.6) holds for all mixed states $\rho\in\mathfrak{S}_{\mathrm{f}}$ which have no basis of eigenvectors satisfying condition (4.5), in particular, for all mixed Gaussian states $\rho$;*
- *$\overline{C}(\Phi_{K,\alpha},\rho)=I(\Phi_{K,\alpha},\rho)$ for all mixed states $\rho\in\mathfrak{S}(\mathscr{H}_A)$ which have a basis of eigenvectors satisfying condition (4.5).*

D) *If $[\operatorname{Ran}K]^\perp$ contains a nontrivial symplectic subspace, then there exist degenerate mixed Gaussian states $\rho$ such that $\overline{C}(\Phi_{K,\alpha},\rho)=I(\Phi_{K,\alpha},\rho)=0$.*

*Proof.* It remains to prove statement D).

If there exists a nontrivial symplectic subspace $Z_{A_0}$ of $[\operatorname{Ran}K]^\perp$, then $Z_A=Z_{A_0}\oplus Z_{A_*}$, where $Z_{A_*}=[Z_{A_0}]^\perp$ is a symplectic subspace of $Z_A$ (by Lemma 6 in § 6.3) and hence $\mathscr{H}_A=\mathscr{H}_{A_0}\otimes\mathscr{H}_{A_*}$. By using the concatenation rules for Gaussian channels (see [1], Ch. 11), it is easy to show that $\Phi_{K,\alpha}=\Phi_{K',\alpha}\circ\Psi$, where $\Psi$ is the partial trace in $\mathfrak{T}(\mathscr{H}_A)$ over $\mathscr{H}_{A_0}$ and $\Phi_{K',\alpha}$ is a Gaussian channel from $\mathfrak{T}(\mathscr{H}_{A_*})$ into $\mathfrak{T}(\mathscr{H}_B)$ determined by the 'output restriction' $K'$ of $K$ and by the same matrix $\alpha$.

So, for any pure Gaussian state $\rho_* = |\psi_*\rangle\langle\psi_*|$ in $\mathfrak{S}(\mathscr{H}_{A_*})$, the subchannel of $\Phi_{K,\alpha}$ corresponding to the subspace $\mathscr{H}_{A_0} \otimes \{\lambda|\psi_*\rangle\}$ is completely depolarizing. Hence

$$\overline{C}(\Phi_{K,\alpha}, \rho_0 \otimes \rho_*) = I(\Phi_{K,\alpha}, \rho_0 \otimes \rho_*) = 0$$

for all Gaussian states $\rho_0$ in $\mathfrak{S}(\mathscr{H}_{A_0})$.

*Example* 1. The simplest Gaussian channels are one-mode channels ($s_A = s_B = 1$). In accordance with Holevo's classification there exist (up to natural isomorphism) the following six types of such channels:

$$A_1[N], \quad A_2[N], \quad B_1, \quad B_2[N], \quad C[k,N]\ (k>0,\ k\neq 1), \quad D[k,N]\ (k>0)$$

(the parameter $N$ determines the level of noise; for details, see [1], Ch. 11).

For all one-mode Gaussian channels, apart from those of types $A_1$ and $A_2$, the matrix $K$ is nondegenerate. Hence, by Proposition 1, for all these channels, a strict inequality holds in (4.6) for all mixed states with finite entropy.

Channels of type $A_1$ are completely depolarizing channels. So, channels of type $A_2$ are the only nontrivial one-mode Gaussian channels for which an equality in (4.6) can be valid for mixed states.

One-mode Gaussian channels of type $A_2$ are nondiscrete classical-quantum channels. The canonical representative $\Phi_{K,\alpha}$ of this type is determined by the parameters

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \alpha = \begin{bmatrix} N+\dfrac{1}{2} & 0 \\ 0 & N+\dfrac{1}{2} \end{bmatrix}, \quad N \geqslant 0.$$

This channel satisfies the condition of part C) of Proposition 1. In this case, the basis $\{\widetilde{e}_k, \widetilde{h}_k\}$ introduced in deriving (4.5) consists of the vectors $\widetilde{e}_1 = [1,0]^\top$, $\widetilde{h}_1 = [0,1]^\top$ and $\operatorname{Ran} K = \{\lambda \widetilde{e}_1\}$. Thus, in the corresponding Schrödinger representation (in which $\mathscr{H}_A = \mathscr{H}_B = L_2(\mathbb{R})$), condition (4.5) is written as follows:

$$\varphi_i(\xi)\varphi_j(\xi) = 0 \quad \text{almost everywhere in } \mathbb{R} \text{ for all } i \neq j. \tag{4.7}$$

By part C) of Proposition 1, $\overline{C}(\Phi_{K,\alpha}, \rho) = I(\Phi_{K,\alpha}, \rho)$ for all states in

$$\bigcup_{\{\varphi_i\}\in\Pi(\Phi_{K,\alpha})} \left\{ \rho = \sum_i \pi_i |\varphi_i\rangle\langle\varphi_i| \;\middle|\; \{\pi_i\} \text{ is a probability distribution} \right\}, \tag{4.8}$$

where $\Pi(\Phi_{K,\alpha})$ is the set of all families $\{\varphi_i(\xi)\}$ of functions in $L_2(\mathbb{R})$ with unit norm satisfying (4.7). An example of such a family can be constructed by taking a decomposition $\{D_i\}$ of $\mathbb{R}$ into disjoint measurable subsets and by choosing for each $i$ a function $\varphi_i(\xi)$ with unit norm vanishing in $\mathbb{R} \setminus D_i$.

Proposition 1 also shows that $\overline{C}(\Phi_{K,\alpha}, \rho) < I(\Phi_{K,\alpha}, \rho)$ for all states $\rho$ with finite entropy not lying in the set (4.8), in particular, for all mixed Gaussian states $\rho$.

## § 5. On equality in the complementary inequality

In § 3 and § 4 we focused attention on equality in (2.13), which is more important for applications, and used its connection with equality in (2.9) in the proofs of the main results (since the last equality is related to the reversibility property of a channel). In this section we reformulate these results as conditions for equality in (2.9) by using relation (2.15).

Theorem 3 can be reformulated as follows.

**Theorem 4.** *Let* $\Phi\colon \mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ *be a quantum channel and* $\widehat{\Pi}(\Phi)$ *be the set of all orthogonal families* $\{|\varphi_i\rangle\}$ *of unit vectors in* $\mathscr{H}_A$ *such that* $\operatorname{supp}\Phi(|\varphi_i\rangle\langle\varphi_i|)\perp\operatorname{supp}\Phi(|\varphi_j\rangle\langle\varphi_j|)$ *for all* $i\neq j$.

A) *Let* $\rho$ *be a mixed state in* $\mathfrak{S}(\mathscr{H}_A)$ *such that* $H(\rho)<+\infty$. *The following statements are equivalent*:

(i) $\overline{C}(\Phi,\rho)=H(\rho)$;

(ii) $\widehat{\Pi}(\Phi)$ *contains at least one basis of eigenvectors of* $\rho$;

(iii) *the subchannel* $\Phi|_{\mathfrak{T}(\mathscr{H}_\rho)}$ *is isometrically equivalent* (*see Definition* 2) *to the channel*

$$\varrho\mapsto\sum_{i,j=1}^{\dim\mathscr{H}_\rho}\langle\varphi_i|\varrho\varphi_j\rangle|\varphi_i\rangle\langle\varphi_j|\otimes\sum_{k,l=1}^{\dim\mathscr{H}_B}\langle\psi_{jl}|\psi_{ik}\rangle|k\rangle\langle l|$$

*from* $\mathfrak{T}(\mathscr{H}_\rho)$ *into* $\mathfrak{T}(\mathscr{H}_\rho\otimes\mathscr{H}_B)$, *where* $\mathscr{H}_\rho=\operatorname{supp}\rho$, $\{|\varphi_i\rangle\}$ *is a particular orthonormal basis of eigenvectors of* $\rho$, $\{|\psi_{ik}\rangle\}$ *is a collection of vectors in a Hilbert space such that* $\sum_{k=1}^{\dim\mathscr{H}_B}\|\psi_{ik}\|^2=1$ *for all* $i$, *and* $\{|k\rangle\}$ *is an orthonormal basis in* $\mathscr{H}_B$.

*For a state* $\rho$ *with* $H(\rho)=+\infty$, *these statements are related as follows*: (iii) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (i).

B) *The set* $\widehat{\mathfrak{S}}_{\Phi}^{=}$ *of all mixed states* $\rho$ *in* $\mathfrak{S}(\mathscr{H}_A)$ *with finite entropy for which* (i) *holds can be represented as follows*

$$\widehat{\mathfrak{S}}_{\Phi}^{=}=\bigcup_{\{|\varphi_i\rangle\}\in\widehat{\Pi}(\Phi)}\biggl\{\rho=\sum_i\pi_i|\varphi_i\rangle\langle\varphi_i|\;\bigg|\;\{\pi_i\}\in\mathfrak{P}_{\mathrm{f}}\biggr\},$$

*where* $\mathfrak{P}_{\mathrm{f}}$ *is the set of all probability distributions with finite Shannon entropy*.

*Proof*. By using (2.2), (2.3) and the Schmidt representation of the vectors $V|\varphi\rangle$ and $V|\psi\rangle$ it is easy to show that

$$\operatorname{supp}\Phi(|\varphi\rangle\langle\varphi|)\perp\operatorname{supp}\Phi(|\psi\rangle\langle\psi|)\quad\Longleftrightarrow\quad\widehat{\Phi}(|\varphi\rangle\langle\psi|)=0$$

for any vectors $\varphi,\psi\in\mathscr{H}_A$, and hence $\widehat{\Pi}(\Phi)=\Pi(\widehat{\Phi})$. Thus, the standard representation of a complementary channel (formula (11) in [8]) shows that statements (ii) and (iii) in Theorem 4 are equivalent, respectively, to statements (ii) and (iii) in Theorem 3 for the channel $\widehat{\Phi}$.

Since a noiseless channel (see Definition 6) is complementary to a completely depolarizing channel and vice versa (see [8]), Corollary 3 can be reformulated as follows.

**Corollary 4.** *If* $\overline{C}(\Phi,\rho)=H(\rho)$ *for any state* $\rho$ *of rank* 2, *then* $\Phi$ *is a noiseless channel and hence* $\overline{C}(\Phi,\rho)=H(\rho)$ *for any state* $\rho$.

It is well known that the complementary channel to an arbitrary Gaussian channel is also Gaussian (see [1], [17]). So, one can assume that $\widehat{\Phi}_{K,\alpha}=\Phi_{L,\beta}$, where $L$ is a linear operator $Z_E\to Z_A$ and $\beta$ is a $(2s_E\times 2s_E)$-dimensional real symmetric matrix (satisfying an inequality similar to (4.3)).

It follows from Lemma 5 in § 6.2 and the remark before it that

$$[\operatorname{Ran}L]^{\perp}=K(\ker\alpha) \tag{5.1}$$

and that the restriction of $K$ to the subspace $\ker\alpha$ is nondegenerate and symplectic, that is, $\Delta_A(Kz_1,Kz_2)=\Delta_B(z_1,z_2)$ for all $z_1$, $z_2$ in $\ker\alpha$. Hence

$$\begin{aligned}\{L=0\}\quad&\Longleftrightarrow\quad\{\Phi_{K,\alpha}\text{ is a noiseless channel}\},\\ \{\operatorname{Ran}L=Z_A\}\quad&\Longleftrightarrow\quad\{\det\alpha\neq0\},\\ \{\text{the subspace }[\operatorname{Ran}L]^{\perp}\text{ is isotropic}\}\quad&\Longleftrightarrow\quad\{\text{the subspace }\ker\alpha\text{ is isotropic}\}.\end{aligned}$$

By using these relations and by noting that $\operatorname{Ran}L=[\operatorname{Ran}L]^{\perp\perp}=[K(\ker\alpha)]^{\perp}$, Proposition 1 can be reformulated as follows.

**Proposition 2.** *Let* $\mathfrak{S}_{\mathrm{f}}$ *be a subset of* $\mathfrak{S}(\mathscr{H}_A)$ *consisting of all states with finite von Neumann entropy.*

A) *If* $\Phi_{K,\alpha}$ *is not a noiseless channel* (*see Definition* 6), *then*

$$\overline{C}(\Phi_{K,\alpha},\rho)<H(\rho) \tag{5.2}$$

*for all nondegenerate states* $\rho\in\mathfrak{S}_{\mathrm{f}}$, *in particular, for all nondegenerate Gaussian states* $\rho$.

B) *If* $\det\alpha\neq0$, *then* (5.2) *holds for all mixed states* $\rho\in\mathfrak{S}_{\mathrm{f}}$, *in particular, for all mixed Gaussian states* $\rho$.

C) *If* $\ker\alpha$ *is a nontrivial isotropic subspace of* $Z_B$, *then*:

- *inequality* (5.2) *holds for all mixed states* $\rho\in\mathfrak{S}_{\mathrm{f}}$ *which have no basis of eigenvectors satisfying condition* (4.5) *determined by the subspace* $[K(\ker\alpha)]^{\perp}$ (*see Remark* 6), *in particular, for all mixed Gaussian states* $\rho$;
- $\overline{C}(\Phi_{K,\alpha},\rho)=H(\rho)$ *for all mixed states* $\rho\in\mathfrak{S}(\mathscr{H}_A)$ *which have a basis of eigenvectors satisfying* (4.5) *determined by the subspace* $[K(\ker\alpha)]^{\perp}$.

D) *If* $\ker\alpha$ *contains a nontrivial symplectic subspace, then there exist mixed Gaussian states* $\rho$ *such that*

$$\overline{C}(\Phi_{K,\alpha},\rho)=H(\rho).$$

*Example* 2. Proposition 2 shows that for all one-mode Gaussian channels, apart from the noiseless channel $B_2[0]$ and the channel $B_1$ (see Example 1), we have strict inequality in (5.2) for all mixed states with finite entropy.

The canonical one-mode Gaussian channel $\Phi_{K,\alpha}$ of type $B_1$ is determined by the parameters

$$K=\begin{bmatrix}1&0\\0&1\end{bmatrix},\qquad \alpha=\begin{bmatrix}0&0\\0&\frac{1}{2}\end{bmatrix}.$$

In the Schrödinger representation (in which $\mathscr{H}_A = \mathscr{H}_B = L_2(\mathbb{R})$), condition (4.5) determined by the subspace $[K(\ker\alpha)]^\perp = \{[\lambda, 0]^\top\}$ coincides[10] with condition (4.7).

Thus, Proposition 2 shows that $\overline{C}(\Phi_{K,\alpha}, \rho) = H(\rho)$ for all states in the set (4.8) and that $\overline{C}(\Phi_{K,\alpha}, \rho) < H(\rho)$ for all states $\rho$ with finite entropy not lying in the set (4.8), in particular, for all mixed Gaussian states $\rho$.

## § 6. Appendix

**6.1. Proof of Theorem 3 in the infinite-dimensional case.** The proof of the implication (i) $\Longrightarrow$ (iii) in Theorem 3 presented in § 3 is not easily generalized to infinite dimensions, since it is based on the existence of an ensemble $\{\pi_i, \rho_i\}$ with an average state $\rho$ such that $\overline{C}(\widehat{\Phi}, \rho) = \chi_{\widehat{\Phi}}(\{\pi_i, \rho_i\})$, which follows from the compactness of the set of input states. To avoid this problem we will use the notion of a generalized (continuous) ensemble.

Following [11], we will consider an arbitrary Borel probability measure $\mu$ on the set $\mathfrak{S}(\mathscr{H}_A)$ as a generalized input ensemble for the channel $\Phi\colon \mathfrak{T}(\mathscr{H}_A) \to \mathfrak{T}(\mathscr{H}_B)$. The $\chi$-quantity and the output $\chi$-quantity of this ensemble are defined, respectively, by the expressions

$$\chi(\mu) = \int_{\mathfrak{S}(\mathscr{H}_A)} H(\rho\|\overline{\rho}(\mu))\mu(d\rho) = H(\overline{\rho}(\mu)) - \int_{\mathfrak{S}(\mathscr{H}_A)} H(\rho)\mu(d\rho) \tag{6.1}$$

and

$$\chi_\Phi(\mu) = \int_{\mathfrak{S}(\mathscr{H}_A)} H(\Phi(\rho)\|\Phi(\overline{\rho}(\mu)))\mu(d\rho) = H(\Phi(\overline{\rho}(\mu))) - \int_{\mathfrak{S}(\mathscr{H}_A)} H(\Phi(\rho))\mu(d\rho),$$

in which

$$\overline{\rho}(\mu) \doteq \int_{\mathfrak{S}(\mathscr{H}_A)} \rho\mu(d\rho)$$

is the barycenter of $\mu$, and the second formulae are valid under the conditions $H(\overline{\rho}(\mu)) < +\infty$ and $H(\Phi(\overline{\rho}(\mu))) < +\infty$, respectively.

Denote by $\mathscr{P}_p(\mathfrak{S}(\mathscr{H}_A))$ the set of all Borel probability measures on the set $\operatorname{extr}\mathfrak{S}(\mathscr{H}_A)$ of all pure states in $\mathfrak{S}(\mathscr{H}_A)$. By Corollary 1 in [11], we have

$$\overline{C}(\Phi, \rho) = \sup_{\overline{\rho}(\mu)=\rho} \chi_\Phi(\mu), \tag{6.2}$$

where the supremum is taken over all measures in $\mathscr{P}_p(\mathfrak{S}(\mathscr{H}_A))$ with barycenter $\rho$.

In contrast to the finite-dimensional case, the supremum in (6.2) is not attainable in general, but there exist sufficient conditions for its attainability, the simplest of which is the following: $H(\Phi(\rho)) < +\infty$ (see [11], Corollary 2).

Now we can prove the implication (i) $\Longrightarrow$ (iii) in Theorem 3.

Assume first that $H(\Phi(\rho)) < +\infty$. By the triangle inequality (see [4]), we have

$$|H(\Phi(\rho)) - H(\widehat{\Phi}(\rho))| \leqslant H(\rho)$$

[10] This is not surprising and follows from (5.1), since a channel of type $A_2$ with $N = 0$ is complementary to a channel of type $B_1$ (see [1], Ch. 11).

and hence the finiteness of $H(\rho)$ and $H(\Phi(\rho))$ implies the finiteness of $H(\widehat{\Phi}(\rho))$. By the above-mentioned Corollary 2 in [11] there exists a measure $\mu$ in $\mathscr{P}_p(\mathfrak{S}(\mathscr{H}_A))$ with barycenter $\rho$ such that

$$\overline{C}(\widehat{\Phi},\rho)=\chi_{\widehat{\Phi}}(\mu).$$

If (i) holds, then (2.15) implies $\overline{C}(\widehat{\Phi},\rho)=H(\rho)$. Since $H(\rho)=\chi(\mu)$ by the second formula in (6.1), this means that $\chi_{\widehat{\Phi}}(\mu)=\chi(\mu)$.

By Proposition 3 in [9] (a generalization of Theorem 1 to continuous ensembles), this is equivalent to the reversibility of the channel $\widehat{\Phi}$ with respect to some measurable family $\mathfrak{S}$ of pure states such that $\mu(\mathfrak{S})=1$. Let $\mathfrak{S}=\bigcup_k\mathfrak{S}_k$ be the decomposition of $\mathfrak{S}$ into orthogonally-indecomposable subfamilies (which are measurable, in view of their mutual orthogonality and the measurability of $\mathfrak{S}$) and $\{|\varphi_k^i\rangle\}_i$ be an orthonormal basis of eigenvectors of the positive operator $\rho_k=\displaystyle\int_{\mathfrak{S}_k}\rho\mu(d\rho)$. Since $\rho=\sum_k\rho_k$ and $\operatorname{supp}\rho_k\perp\operatorname{supp}\rho_l$ for all $k\neq l$, $\{|\varphi_k^i\rangle\}_{ik}$ is an orthonormal basis of eigenvectors of $\rho$.

By part B) of Theorem 2, the reversibility of $\widehat{\Phi}$ with respect to $\mathfrak{S}$ implies the reversibility of this channel with respect to the orthogonal family $\{|\varphi_k^i\rangle\langle\varphi_k^i|\}_{ik}$ (contained in the family $\widehat{\mathfrak{S}}$). Since $\Phi$ and $\widehat{\widehat{\Phi}}$ are isometrically equivalent, part A) of Theorem 2 implies (iii).

If $H(\Phi(\rho))=+\infty$, then we choose an increasing sequence $\{P_n\}$ of finite rank projectors in $\mathscr{H}_B$, strongly converging to $I_{\mathscr{H}_B}$, and consider the sequence

$$\{\Phi_n\doteq\Pi_n\circ\Phi\}$$

of channels from $\mathfrak{T}(\mathscr{H}_A)$ into $\mathfrak{T}(\mathscr{H}_B)$, where $\Pi_n(\sigma)=P_n\sigma P_n+[\operatorname{Tr}(I_{\mathscr{H}_B}-P_n)\sigma]\tau$ is a channel from $\mathfrak{T}(\mathscr{H}_B)$ into itself and $\tau$ is a given pure state in $\mathfrak{S}(\mathscr{H}_B)$. By Lemma 2, it follows from (i) that

$$\overline{C}(\Phi_n,\rho)=I(\Phi_n,\rho)$$

for each $n$. Since $H(\Phi_n(\rho))<+\infty$, the previous part of the proof of this implication shows the validity of (iii), and hence of (ii), for the channels $\Phi_n$ for each $n$, that is,

$$\Phi_n(|\varphi_i^n\rangle\langle\varphi_j^n|)=0\quad\forall\, i\neq j,\quad\forall\, n,\tag{6.3}$$

where $\{|\varphi_i^n\rangle\}$ is a basis of eigenvectors of $\rho$ (depending on $n$).

If $\rho$ has no multiple eigenvalues, then it has a unique (up to a permutation and scalar multiplication) basis of eigenvectors $\{|\varphi_i\rangle\}$ and (6.3) implies

$$\Phi(|\varphi_i\rangle\langle\varphi_j|)=\lim_{n\to+\infty}\Phi_n(|\varphi_i\rangle\langle\varphi_j|)=0\quad\forall\, i\neq j,$$

that is, the validity of (ii) for $\Phi$.

If $\rho$ has multiple eigenvalues, then the required basis $\{|\varphi_i\rangle\}$ can be constructed as follows.

For any natural $m$, let $\mathscr{H}_m$ be the direct sum of the eigensubspaces of $\rho$ corresponding to its $m$ maximal eigenvalues. Let $d_m=\dim\mathscr{H}_m$. We can assume that the first $d_m$ vectors of the above basis $\{|\varphi_i^n\rangle\}$ form a basis of $\mathscr{H}_m$ for each $m$.

Let $n_k^1$ be a sequence of natural numbers such that there exists

$$\lim_{k\to+\infty}|\varphi_i^{n_k^1}\rangle=|\varphi_i^1\rangle,\qquad i=1,\dots,d_1$$

(the existence of this sequence and of all the subsequences below follows from the compactness of the unit ball of $\mathscr{H}_m$ for each $m$).

For given $m>1$, let $n_k^m$ be a subsequence of the sequence $n_k^{m-1}$ such that there exists

$$\lim_{k\to+\infty}|\varphi_i^{n_k^m}\rangle=|\varphi_i^m\rangle,\qquad i=1,\dots,d_m.$$

It follows from (6.3) that

$$\Phi(|\varphi_i^m\rangle\langle\varphi_j^m|)=\lim_{k\to+\infty}\Phi_{n_k^m}(|\varphi_i^{n_k^m}\rangle\langle\varphi_j^{n_k^m}|)=0 \tag{6.4}$$

for all $i\neq j$ not exceeding $d_m$.

By construction, $|\varphi_i^m\rangle=|\varphi_i^{m-1}\rangle$ for $i=1,\dots,d_{m-1}$. Thus we have the increasing sequence

$$\{|\varphi_i^1\rangle\}_{i=1}^{d_1}\subset\{|\varphi_i^2\rangle\}_{i=1}^{d_2}\subset\cdots\subset\{|\varphi_i^m\rangle\}_{i=1}^{d_m}\cdots$$

of orthonormal collections of eigenvectors of $\rho$ such that $\{|\varphi_i^m\rangle\}_{i=1}^{d_m}$ is a basis of $\mathscr{H}_m$, for which (6.4) holds.

It is clear that the union $\{|\varphi_i\rangle\}_{i=1}^{+\infty}$ of all these collections is a basis of eigenvectors of $\rho$ such that $\Phi(|\varphi_i\rangle\langle\varphi_j|)=0$ for all $i\neq j$.

The above arguments prove the following 'continuous' version of Corollary 2.

**Corollary 5.** *Let $\Phi\colon\mathfrak{T}(\mathscr{H}_A)\to\mathfrak{T}(\mathscr{H}_B)$ be a quantum channel and $\mu$ be an arbitrary measure in $\mathscr{P}_p(\mathfrak{S}(\mathscr{H}_A))$. If $\Phi$ is reversible with respect to $\mu$-almost all pure states in $\mathfrak{S}(\mathscr{H}_A)$, then $\Phi$ is reversible with respect to the family of orthogonal pure states corresponding to a particular basis of eigenvectors of the state $\overline{\rho}(\mu)\doteq\int_{\mathfrak{S}(\mathscr{H}_A)}\rho\mu(d\rho)$.*

**6.2. On the complementary channel to a Bosonic Gaussian channel.** It is known that the complementary channel to any Gaussian channel is also Gaussian. This fact is proved by constructing the Bosonic unitary dilation for an arbitrary centred channel $\Phi_{K,\alpha}$, that is, by finding Bosonic systems $D$ and $E$ such that $\Phi_{K,\alpha}$ is represented as a restriction of a particular unitary evolution of the composite Bosonic system $AD$ (described by the symplectic space $Z=Z_A\oplus Z_D=Z_B\oplus Z_E$) under the condition that $D$ is in some pure Gaussian state $\rho_D$. This means that

$$\Phi^*_{K,\alpha}(W_B(z))=\operatorname{Tr}_{\mathscr{H}_D}(I_{\mathscr{H}_A}\otimes\rho_D)U_T^*(W_B(z)\otimes I_{\mathscr{H}_E})U_T,\qquad z\in Z_B, \tag{6.5}$$

where $U_T$ is a unitary operator in the space $\mathscr{H}_A\otimes\mathscr{H}_D\cong\mathscr{H}_B\otimes\mathscr{H}_E$ corresponding to a symplectic transformation

$$T=\begin{bmatrix}K & L\\ K_D & L_D\end{bmatrix} \tag{6.6}$$

of $Z$ (here $L\colon Z_E\to Z_A$, $K_D\colon Z_B\to Z_D$, $L_D\colon Z_E\to Z_D$ are the corresponding linear operators); see [1], [17], [18].

If the above dilation of $\Phi_{K,\alpha}$ is constructed, then the complementary channel is defined by the expression

$$\begin{aligned}[\widehat{\Phi}_{K,\alpha}]^*(W_E(z)) &= \mathrm{Tr}_{\mathscr{H}_D}(I_{\mathscr{H}_A}\otimes\rho_D)U_T^*(I_{\mathscr{H}_B}\otimes W_E(z))U_T\\ &= \mathrm{Tr}_{\mathscr{H}_D}(I_{\mathscr{H}_A}\otimes\rho_D)(W_A(Lz)\otimes W_D(L_Dz))\\ &= W_A(Lz)\phi_{\rho_D}(L_Dz), \qquad z\in Z_E,\end{aligned} \tag{6.7}$$

where $\phi_{\rho_D}$ is the characteristic function of $\rho_D$. This expression shows that $\widehat{\Phi}_{K,\alpha}$ is a Bosonic Gaussian channel $\Phi_{L,\beta}$ with $\beta = L_D^\top\sigma_{\rho_D}L_D$, where $\sigma_{\rho_D}$ is the covariance matrix of $\rho_D$ (see [18]).

Note that by expanding (6.5) in a similar fashion to (6.7), it is easy to show that $\alpha = K_D^\top\sigma_{\rho_D}K_D$. Hence $\ker\alpha = \ker K_D$, since the covariance matrix $\sigma_{\rho_D}$ is nondegenerate. Thus, the following lemma shows that $\mathrm{Ran}\,L = [K(\ker\alpha)]^\perp$.

**Lemma 5.** *Let $T\colon Z_B\oplus Z_E\to Z_A\oplus Z_D$ be the symplectic transformation determined by the matrix (6.6). Then*

$$[\mathrm{Ran}\,L]^\perp = K(\ker K_D), \qquad \ker K_D = \Delta_B K^\top\Delta_A([\mathrm{Ran}\,L]^\perp),$$

*where $[\mathrm{Ran}\,L]^\perp$ is the skew-orthogonal complement to the subspace $\mathrm{Ran}\,L\subseteq Z_A$.*

*The restrictions of the operators $K$ and $\Delta_BK^\top\Delta_A$ to $\ker K_D$ and $[\mathrm{Ran}\,L]^\perp$ (respectively) are nondegenerate and symplectic, that is, they preserve the corresponding skew-symmetric forms $\Delta_X$, $X=A,B$.*

*Proof.* Note that $[\mathrm{Ran}\,L]^\perp = \ker[L^\top\Delta_A]$.

Since $T$ is symplectic, we have (see [18])

$$\begin{aligned}\Delta_B &= K^\top\Delta_AK + K_D^\top\Delta_DK_D,\\ 0 &= L^\top\Delta_AK + L_D^\top\Delta_DK_D,\\ \Delta_E &= L^\top\Delta_AL + L_D^\top\Delta_DL_D.\end{aligned} \tag{6.8}$$

Since $T^\top$ is also symplectic, we have

$$\begin{aligned}\Delta_A &= K\Delta_BK^\top + L\Delta_EL^\top,\\ 0 &= K_D\Delta_BK^\top + L_D\Delta_EL^\top,\\ \Delta_D &= K_D\Delta_BK_D^\top + L_D\Delta_EL_D^\top.\end{aligned} \tag{6.9}$$

The second equations in (6.8) and in (6.9) imply the inclusions

$$K(\ker K_D)\subseteq\ker[L^\top\Delta_A], \qquad \Delta_BK^\top\Delta_A(\ker[L^\top\Delta_A])\subseteq\ker K_D, \tag{6.10}$$

while the first equations in (6.8) and in (6.9) show that

$$\ker K\cap\ker K_D = \{0\}, \qquad \ker[\Delta_BK^\top\Delta_A]\cap\ker[L^\top\Delta_A] = \{0\}$$

since the matrices $\Delta_A$ and $\Delta_B$ are nondegenerate. Dimensional arguments imply equalities in both inclusions in (6.10).

The last assertion of the lemma follows from the first equations in (6.8) and in (6.9).

**6.3. Some results from the theory of symplectic spaces.** Let $Z$ be a $2s$-dimensional symplectic space with nondegenerate skew-symmetric form $\Delta$ (see [1], [19], [20]). A set of vectors $\{e_1, \dots, e_s, h_1, \dots, h_s\}$ is called a *symplectic basis* in $Z$ if

$$\Delta(e_k, e_l) = \Delta(h_k, h_l) = 0 \quad \text{for all } k, l, \quad \text{but } \Delta(e_k, h_l) = \delta_{kl}.$$

For any subspace $L \subset Z$ one can define its skew-orthogonal complement $L^{\perp} = \{z \in Z \mid \Delta(z, z') = 0 \ \forall\, z' \in L\}$. Despite the fact that $L \cap L^{\perp} \neq \{0\}$ in general, the usual relations hold:

$$[L^{\perp}]^{\perp} = L, \qquad \dim L + \dim L^{\perp} = \dim Z. \tag{6.11}$$

A linear transformation $T\colon Z \to Z$ is called *symplectic* if $\Delta(Tz_1, Tz_2) = \Delta(z_1, z_2)$ for all $z_1, z_2 \in Z$. A symplectic transformation transforms any symplectic basis into a symplectic basis and, contrariwise, any two symplectic bases are related by some symplectic transformation.

A subspace $L$ in $Z$ is called *symplectic* if $\Delta$ is nondegenerate on $L$; in this case, $L$ has even dimension and can be regarded as a symplectic space itself. We will use the following simple observation (see [19], [20]).

**Lemma 6.** *If $L$ is a symplectic subspace of $Z$, then $L^{\perp}$ is also a symplectic subspace of $Z$ and $Z = L + L^{\perp}$ (that is, $Z = \operatorname{lin}(L \cup L^{\perp})$ and $L \cap L^{\perp} = \{0\}$).*

A union of symplectic bases in $L$ and in $L^{\perp}$ is a symplectic basis in $Z$.

A subspace $L$ of $Z$ is called *isotropic* if $\Delta$ is equal to zero in $L$. In this case, $L$ has dimension $\leqslant s$. We will use the following known result.

**Lemma 7.** *If $L$ is an isotropic subspace of $Z$, then there exists a symplectic basis $\{\widetilde{e}_k, \widetilde{h}_k\}$ in $Z$ such that $\{\widetilde{e}_1, \dots, \widetilde{e}_d\}$ is a basis in $L$.*

*Proof*. Let $\{e_k, h_k\}$ be a symplectic basis in $Z$ and $L'$ be an isotropic subspace of $Z$ generated by the vectors $e_1, \dots, e_d$. Since $L$ and $L'$ have the same dimension, there exists a symplectic transformation $T$ such that $L = T(L')$ (see [20]). The basis $\{\widetilde{e}_k = Te_k, \widetilde{h}_k = Th_k\}$ has the required properties.

Now we can prove the lemma used in § 4.

**Lemma 8.** *For any subspace $L \subset Z$, there exists a symplectic basis in $Z$ such that* $\dim L$ *of its vectors lie in $L$.*

*Proof*. If $L$ is either symplectic or isotropic, then the assertion of the lemma follows, respectively, from Lemma 6 (with the remark after it) or Lemma 7.

If $L$ is neither symplectic nor isotropic, then $L_1 = L \cap L^{\perp} = \{z \in L \mid \Delta(z, z') = 0 \ \forall\, z' \in L\}$ is a nontrivial subspace of $L$. Let $L_2$ be a subspace such that $L = L_1 + L_2$, that is, $L = \operatorname{lin}(L_1 \cup L_2)$ and $L_1 \cap L_2 = \{0\}$. Then $L_2$ is symplectic. Indeed, if there exists a vector $z_0 \in L_2$ such that $\Delta(z_0, z) = 0$ for all $z \in L_2$, then $\Delta(z_0, z + z') = 0$ for all $z' \in L_1$, $z \in L_2$. This implies that $z_0 \in L_1$, and hence $z_0 = 0$.

By Lemma 6, $L_2^{\perp}$ is symplectic. It is easy to see that it contains the isotropic subspace $L_1$. By Lemma 7, there exists a symplectic basis $\{e_k, h_k\}$ in $L_2^{\perp}$ such that $\{e_1, \dots, e_d\} \subset L_1$, where $d = \dim L_1$. By joining this basis and any symplectic basis in $L_2$ we obtain a basis with the required properties.

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