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INFORMATION THEORY

On Superactivation of One-Shot Quantum Zero-Error Capacity and the Related Property of Quantum Measurements

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Abstract—We give a detailed description of a low-dimensional quantum channel (input dimension 4, Choi rank 3) demonstrating the symmetric form of superactivation of one-shot quantum zero-error capacity. This property means appearance of a noiseless (perfectly reversible) subchannel in the tensor square of a channel having no noiseless subchannels. Then we describe a quantum channel with an arbitrary given level of symmetric superactivation (including the infinite value). We also show that superactivation of one-shot quantum zero-error capacity of a channel can be reformulated in terms of quantum measurement theory as appearance of an indistinguishable subspace for the tensor product of two observables having no indistinguishable subspaces.

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1. INTRODUCTION

Quantum channels, i.e., completely positive trace-preserving linear maps between spaces of trace-class operators (spaces of matrices in the finite-dimensional case), can be considered as noncommutative analogs of classical communication channels. There exist various protocols for transmission of classical and quantum information over quantum channels, determined by used resources, security requirements, etc. For each such protocol and for a given quantum channel there is an ultimate rate of errorless (or asymptotically errorless) information transmission. Similarly to the classical case, it is called the capacity of this channel (corresponding to this protocol) [1, 2].

The superactivation phenomenon means that a particular capacity of the tensor product of two quantum channels may be positive despite that the same capacity of each of these channels is zero. This phenomenon, which has no classical counterpart, has been discovered by G. Smith and J. Yard in 2008 for the case of quantum capacity [3].

Later it was shown that superactivation may hold for different quantum channel capacities, in particular, for (one-shot and asymptotic) classical and quantum zero-error capacities [4, 6]. The effect of (so-called) extreme superactivation of zero-error capacities was also discovered [5].

This paper is devoted to superactivation of one-shot quantum zero-error capacity. This capacity $\bar{Q}_0(\Phi)$ (strictly defined in Section 2) characterizes the possibility of zero-error transmission of

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quantum information over quantum channel Φ with one use of this channel. Superactivation of this capacity means that

$$\bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi_1 \otimes \Phi_2) > 0 \tag{1}$$

for some channels Φ_1 and Φ_2.

The superactivation property (1) can be reformulated without using the term *capacity* as appearance of a noiseless (i.e., perfectly reversible) subchannel in the tensor product of two channels each of which has no noiseless subchannels. Thus, analysis of property (1) seems to be interesting for the theory of completely positive maps between operator algebras.

Existence of quantum channels for which (1) holds follows from existence of quantum channels demonstrating the so-called extreme superactivation of asymptotic zero-error capacities. This result is shown in [5] in an implicit way, and hence it neither gives an explicit form of channels demonstrating property (1) nor says anything about their minimal dimensions.

In our recent paper [7] we explicitly describe low-dimensional channels $\Phi_1 \neq \Phi_2$ (input dimension 8, Choi rank 5) demonstrating the extreme superactivation of one-shot zero-error capacity, which means (1) with the condition $\bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0$ replaced by a stronger condition $\bar{C}_0(\Phi_1) = \bar{C}_0(\Phi_2) = 0$ (where $\bar{C}_0$ is the one-shot classical zero-error capacity). For these channels, property (1) obviously holds.

In this paper we use the same approach to construct a *simpler* example of superactivation (1). It turns out that the change of prerequisites

$$\bar{C}_0(\Phi_1) = \bar{C}_0(\Phi_2) = 0 \quad \longrightarrow \quad \bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0$$

makes it possible to essentially decrease the dimensions (input dimension 4, Choi rank 3) and construct a symmetrical example $\Phi_1 = \Phi_2$, i.e., a channel Φ such that

$$\bar{Q}_0(\Phi) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi \otimes \Phi) > 0.$$

Moreover, this channel Φ is defined via a very simple noncommutative graph, which makes it possible to write a minimal Kraus representation of Φ in an explicit form.

In Section 3 we explicitly describe a quantum channel Φ such that

$$\bar{Q}_0(\Phi) = 0, \qquad \bar{Q}_0(\Phi \otimes \Phi) \geq \log n,$$

where n is any natural number or $+\infty$ (in the last case, Φ is an infinite-dimensional channel).

In Section 4 we show that the superactivation property (1) has a counterpart in quantum measurement theory. Namely, it can be reformulated as appearance of an indistinguishable subspace for the tensor product of two quantum observables having no indistinguishable subspaces.

A general way to write the Kraus representation of a channel with a given noncommutative graph is considered in the Appendix.

2. SUPERACTIVATION OF ONE-SHOT QUANTUM ZERO-ERROR CAPACITY

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ the Banach spaces of all bounded operators in $\mathcal{H}$ and of all trace-class operators in $\mathcal{H}$, respectively, and $\mathfrak{S}(\mathcal{H})$ the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of positive operators with unit trace, called *quantum states* [1,2]. The support $\operatorname{supp}\rho$ of a state ρ is the orthogonal complement of its kernel $\ker\rho$; the dimension of the support is called the rank of a state: $\operatorname{rank}\rho = \dim\operatorname{supp}\rho$. If $\dim\mathcal{H} = n < +\infty$, then we may identify $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ with the space $\mathfrak{M}_n$ of all $n \times n$ matrices (endowed with an appropriate norm).

Let $\Phi\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ be a quantum channel, i.e., a completely positive trace-preserving linear map [1, 2]. Stinespring's theorem implies existence of a Hilbert space $\mathcal{H}_E$ and an isometry $V\colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \mathrm{Tr}_{\mathcal{H}_E} V\rho V^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A). \tag{2}$$

The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni \rho \mapsto \widehat{\Phi}(\rho) = \mathrm{Tr}_{\mathcal{H}_B} V\rho V^* \in \mathfrak{T}(\mathcal{H}_E) \tag{3}$$

is said to be *complementary* to the channel Φ [1,8]. The complementary channel is defined uniquely up to isometric equivalence [8, Appendix].

By using the Stinespring representation (2), one can obtain the Kraus representation

$$\Phi(\rho) = \sum_k V_k \rho V_k^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A), \tag{4}$$

in which $\{V_k\}$ is a set of bounded linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ such that $\sum\limits_k V_k^* V_k = I_{\mathcal{H}_A}$. These operators are defined by the relation

$$\langle \varphi | V_k \psi \rangle = \langle \varphi \otimes k | V\psi \rangle, \quad \varphi \in \mathcal{H}_B, \quad \psi \in \mathcal{H}_A,$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_E$ [1, ch. 6].

Representation (4) with a minimal number of nonzero summands is called the *minimal* Kraus representation of a channel Φ. This minimal number is a characteristic of the channel Φ called the *Choi rank* [1,2]. In what follows, the Choi rank of Φ will be denoted by $\dim \mathcal{H}_E$, since it coincides with the minimal dimension of $\mathcal{H}_E$.

The one-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel Φ is defined as $\sup\limits_{\mathcal{H} \in q_0(\Phi)} \log \dim \mathcal{H}$, where $q_0(\Phi)$ is the set of all subspaces $\mathcal{H}_0$ of $\mathcal{H}_A$ on which the channel Φ is perfectly reversible (in the sense that there is a channel Θ such that $\Theta(\Phi(\rho)) = \rho$ for all states ρ supported by $\mathcal{H}_0$). The (asymptotic) quantum zero-error capacity is defined by regularization: $Q_0(\Phi) = \sup\limits_n n^{-1} \bar{Q}_0(\Phi^{\otimes n})$ [4–6, 9, 10].

It is well known that a channel Φ is perfectly reversible on a subspace $\mathcal{H}_0$ if and only if the restriction of the complementary channel $\widehat{\Phi}$ to the subset $\mathfrak{S}(\mathcal{H}_0)$ is completely depolarizing, i.e., $\widehat{\Phi}(\rho_1) = \widehat{\Phi}(\rho_2)$ for all states ρ_1 and ρ_2 supported by $\mathcal{H}_0$ [1, ch 10]. It follows that the one-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel Φ is completely determined by the set $\mathcal{G}(\Phi) \doteq \widehat{\Phi}^*(\mathfrak{B}(\mathcal{H}_E))$, called the *noncommutative graph* of Φ [9].

Lemma 1. *A channel* $\Phi\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ *is perfectly reversible on the subspace* $\mathcal{H}_0 \subseteq \mathcal{H}_A$ *spanned by the family* $\{\varphi_i\}_{i=1}^n$, $n \le +\infty$, *of orthogonal unit vectors* (*which means that* $\bar{Q}_0(\Phi) \ge \log n$) *if and only if*

$$\langle \varphi_i | A\varphi_j \rangle = 0 \quad \textit{and} \quad \langle \varphi_i | A\varphi_i \rangle = \langle \varphi_j | A\varphi_j \rangle, \quad \forall i, j, \quad \forall A \in \mathfrak{L}, \tag{5}$$

where $\mathfrak{L} = \mathcal{G}(\Phi)$, *or, equivalently,* $\mathfrak{L}$ *is any subset of* $\mathfrak{B}(\mathcal{H}_A)$ *such that*

$$\text{w-o-cl}(\mathrm{lin}\, \mathfrak{L}) = \text{w-o-cl}(\mathcal{G}(\Phi)), \tag{6}$$

where w-o-cl$(\cdot)$ *is the weak operator closure and* $\mathrm{lin}\, \mathfrak{L}$ *is the linear span of* $\mathfrak{L}$.

Proof. It is easy to see that relations (5) with $\mathfrak{L} = \mathcal{G}(\Phi)$ mean that the complementary channel $\widehat{\Phi}$ has a completely depolarizing restriction to the subset $\mathfrak{S}(\mathcal{H}_0)$.

Validity of relations (5) with any $\mathfrak{L}$ satisfying condition (6) implies validity of these relations with $\mathfrak{L} = \mathcal{G}(\Phi)$. △

Remark 1. Since a subspace $\mathfrak{L}$ of the algebra $\mathfrak{M}_n$ of $n \times n$ matrices is a noncommutative graph of a particular channel if and only if

$$\mathfrak{L} \textit{ is symmetric } (\mathfrak{L} = \mathfrak{L}^*) \textit{ and contains the unit matrix} \tag{7}$$

(see [6, Lemma 2] or [7, Proposition 2]), Lemma 1 shows that one can construct a channel Φ with $\dim \mathcal{H}_A = n$ and positive (respectively, zero) one-shot quantum zero-error capacity by taking a subspace $\mathfrak{L} \subset \mathfrak{M}_n$ satisfying (7) for which the following condition is valid (respectively, not valid):

$$\exists\, \varphi, \psi \in [\mathbb{C}^n]_1 \quad \text{such that} \quad \langle \psi | A\varphi \rangle = 0 \quad \text{and} \quad \langle \varphi | A\varphi \rangle = \langle \psi | A\psi \rangle, \quad \forall A \in \mathfrak{L}, \tag{8}$$

where $[\mathbb{C}^n]_1$ is the unit sphere of $\mathbb{C}^n$.

If m is a natural number such that $\dim \mathfrak{L} \le m^2$, then Corollary 1 in [7] and Proposition 3 in the Appendix give explicit expressions for a channel Φ such that $\mathcal{G}(\Phi) = \mathfrak{L}$ and $\dim \mathcal{H}_E = m$.

Superactivation of one-shot quantum zero-error capacity means that

$$\bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0 \quad \text{but} \quad \bar{Q}_0(\Phi_1 \otimes \Phi_2) > 0 \tag{9}$$

for some channels Φ_1 and Φ_2. As is mentioned in Section 1, existence of channels Φ_1 and Φ_2 for which (9) holds follows from the results in [5], but explicit examples of such channels with minimal dimensions are not known.

Below we will construct a channel Φ with $\dim \mathcal{H}_A = 4$, $\dim \mathcal{H}_E = 3$, and $\dim \mathcal{H}_B = 12$ such that (9) holds with $\Phi_1 = \Phi_2 = \Phi$.

By Remark 1, the problem of finding channels for which (9) holds reduces to the problem of finding subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ satisfying (7) such that condition (8) is not valid for $\mathfrak{L} = \mathfrak{L}_1$ and for $\mathfrak{L} = \mathfrak{L}_2$ but is valid for $\mathfrak{L} = \mathfrak{L}_1 \otimes \mathfrak{L}_2$. Now we will consider a symmetric example ($\mathfrak{L}_1 = \mathfrak{L}_2$) of such subspaces in $\mathfrak{M}_4$.

Let U be the unitary operator in $\mathbb{C}^2$ determined (in the canonical basis) by the matrix

$$U = \begin{bmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{bmatrix},$$

where $\eta = \exp[i\pi/4]$. Consider the 5D subspace

$$\mathfrak{L}_0 = \left\{ M = \begin{bmatrix} A & \lambda U^* \\ \lambda U & A \end{bmatrix},\ A \in \mathfrak{M}_2,\ \lambda \in \mathbb{C} \right\}$$

of $\mathfrak{M}_4$. It obviously satisfies condition (7).

Theorem 1. *Condition* (8) *is not valid for* $\mathfrak{L} = \mathfrak{L}_0$ *but is valid for* $\mathfrak{L} = \mathfrak{L}_0 \otimes \mathfrak{L}_0$ *with the vectors*

$$|\varphi_t\rangle = \frac{1}{\sqrt{2}} \left[|1\rangle \otimes |1\rangle + e^{it} |2\rangle \otimes |2\rangle \right], \qquad |\psi_t\rangle = \frac{1}{\sqrt{2}} \left[|3\rangle \otimes |3\rangle + e^{it} |4\rangle \otimes |4\rangle \right], \tag{10}$$

where $\{|k\rangle\}_{k=1}^4$ *is the canonical basis in* $\mathbb{C}^4$ *and* t *is a fixed number in* $[0, 2\pi)$.

Proof. Throughout the proof, we will identify $\mathbb{C}^4$ with $\mathbb{C}^2 \oplus \mathbb{C}^2$.

Assume there exist unit vectors $\varphi = [x_1, x_2]$ and $\psi = [y_1, y_2]$, $x_i, y_i \in \mathbb{C}^2$, such that $\langle \psi | M\varphi \rangle = 0$ and $\langle \psi | M\psi \rangle = \langle \varphi | M\varphi \rangle$ for all $M \in \mathfrak{L}_0$. It follows that

$$\langle y_1 | A x_1 \rangle + \langle y_2 | A x_2 \rangle = 0, \quad \forall A \in \mathfrak{M}_2, \tag{11}$$

$$\langle y_1 | U^* x_2 \rangle + \langle y_2 | U x_1 \rangle = 0, \tag{12}$$

$$\langle y_1 | A y_1 \rangle + \langle y_2 | A y_2 \rangle = \langle x_1 | A x_1 \rangle + \langle x_2 | A x_2 \rangle, \quad \forall A \in \mathfrak{M}_2, \tag{13}$$

$$\langle y_1 | U^* y_2 \rangle + \langle y_2 | U y_1 \rangle = \langle x_1 | U^* x_2 \rangle + \langle x_2 | U x_1 \rangle. \tag{14}$$

If $x_1 \nparallel x_2$, then there is $A_0 \in \mathfrak{M}_2$ such that $y_1 = A_0 x_1$ and $y_2 = A_0 x_2$. Hence, it follows from (11) that $\langle y_1|y_1\rangle + \langle y_2|y_2\rangle = 0$, i.e., $y_1 = y_2 = 0$. Similarly, if $y_1 \nparallel y_2$, then (11) implies $x_1 = x_2 = 0$.

Thus, we have $x_1 \parallel x_2$ and $y_1 \parallel y_2$. Now we will obtain a contradiction to (11)–(14) by considering the following cases:

1. $x_2 = 0$, $x_1 \neq 0$. In this case (11) implies $\langle y_1|Ax_1\rangle = 0$ for all $A \in \mathfrak{M}_2$, which gives $y_1 = 0$. Then (13) implies $\langle x_1|Ax_1\rangle = \langle y_2|Ay_2\rangle$ for all $A \in \mathfrak{M}_2$, which can be valid only if $x_1 \parallel y_2$. By Lemma 2 below, this and (12) show that $y_2 = 0$. Thus, we obtain $y_1 = y_2 = 0$;

2. $y_2 = 0$, $y_1 \neq 0$. Similarly to Case 1 we obtain $x_1 = x_2 = 0$;

3. $x_2 \neq 0$, $y_2 \neq 0$. In this case $x_1 = \mu x_2$, $y_1 = \nu y_2$, and (13) implies

$$(1 + |\mu|^2)\langle x_2|Ax_2\rangle = (1 + |\nu|^2)\langle y_2|Ay_2\rangle, \quad \forall A \in \mathfrak{M}_2,$$

which can only be valid if $x_2 \parallel y_2$. Hence, we have $x_1 = \alpha y_2$ and $x_2 = \beta y_2$ (in addition to $y_1 = \nu y_2$). We may assume that $x_1 \neq 0$ and $y_1 \neq 0$, since otherwise (11) implies $\langle y_2|Ax_2\rangle = 0$ for all $A \in \mathfrak{M}_2$, which can only be valid if either $x_2 = 0$ or $y_2 = 0$.

It follows from (11) that $(\bar{\nu}\alpha + \beta)\langle y_2|y_2\rangle = 0$, and hence

$$\beta = -\bar{\nu}\alpha. \tag{15}$$

By Lemma 2 below, $z_0 = \langle y_2|Uy_2\rangle$ is a nonzero complex number. Thus, (14) and (15) imply $\operatorname{Re}(\nu z_0) = \operatorname{Re}(\alpha\bar{\beta}z_0) = -|\alpha|^2 \operatorname{Re}(\nu z_0)$, and hence

$$\operatorname{Re}(\nu z_0) = 0. \tag{16}$$

It follows from (12) and (15) that

$$\bar{\nu}\beta\bar{z}_0 + \alpha z_0 = \alpha(-\bar{\nu}^2\bar{z}_0 + z_0) = 0.$$

Since $\alpha \neq 0$ ($x_1 \neq 0$), we have $\nu^2 z_0 = \bar{z}_0$. This equality implies that νz_0 is a real number. Thus, (16) shows that $\nu = 0$, contradicting $y_1 \neq 0$.

Hence, condition (8) is not valid for $\mathfrak{L} = \mathfrak{L}_0$.

Now we will show that

$$\langle\psi_t|M_1 \otimes M_2\varphi_t\rangle = 0, \quad \forall M_1, M_2 \in \mathfrak{L}_0, \tag{17}$$

and

$$\langle\psi_t|M_1 \otimes M_2\psi_t\rangle = \langle\varphi_t|M_1 \otimes M_2\varphi_t\rangle, \quad \forall M_1, M_2 \in \mathfrak{L}_0, \tag{18}$$

where φ_t and ψ_t are vectors defined in (10). Since we identify $\mathbb{C}^4$ with $\mathbb{C}^2 \oplus \mathbb{C}^2$, these vectors are represented as follows:

$$|\varphi_t\rangle = \frac{1}{\sqrt{2}}\Big[|e_1, 0\rangle \otimes |e_1, 0\rangle + e^{it}|e_2, 0\rangle \otimes |e_2, 0\rangle\Big],$$
$$|\psi_t\rangle = \frac{1}{\sqrt{2}}\Big[|0, e_1\rangle \otimes |0, e_1\rangle + e^{it}|0, e_2\rangle \otimes |0, e_2\rangle\Big],$$

where $\{|e_i\rangle\}$ is the canonical basis in $\mathbb{C}^2$.

By setting $\alpha_1 = 1$ and $\alpha_2 = e^{it}$, we have

$$M_1 \otimes M_2|\varphi_t\rangle = \frac{1}{\sqrt{2}}\sum_{i=1}^{2} \alpha_i|A_1e_i, \lambda_1 Ue_i\rangle \otimes |A_2e_i, \lambda_2 Ue_i\rangle, \tag{19}$$

and hence

$$\begin{aligned}\langle\psi_t|M_1\otimes M_2\varphi_t\rangle &= \frac{1}{2}\sum_{i,j=1}^{2}\bar{\alpha}_i\alpha_j\langle 0,e_i|\otimes\langle 0,e_i|\cdot|A_1e_j,\lambda_1Ue_j\rangle\otimes|A_2e_j,\lambda_2Ue_j\rangle\\ &= \frac{1}{2}\lambda_1\lambda_2\sum_{i,j=1}^{2}\bar{\alpha}_i\alpha_j\langle e_i|Ue_j\rangle\langle e_i|Ue_j\rangle = \frac{1}{2}\lambda_1\lambda_2\left[\eta^2|\alpha_1|^2+\bar{\eta}^2|\alpha_2|^2\right]=0,\end{aligned}$$

Thus, (17) is valid. It follows from (19) that

$$\begin{aligned}\langle\varphi_t|M_1\otimes M_2\varphi_t\rangle &= \frac{1}{2}\sum_{i,j=1}^{2}\bar{\alpha}_i\alpha_j\langle e_i,0|\otimes\langle e_i,0|\cdot|A_1e_j,\lambda_1Ue_j\rangle\otimes|A_2e_j,\lambda_2Ue_j\rangle\\ &= \frac{1}{2}\sum_{i,j=1}^{2}\bar{\alpha}_i\alpha_j\langle e_i|A_1e_j\rangle\langle e_i|A_2e_j\rangle. \end{aligned}\tag{20}$$

Since

$$M_1\otimes M_2|\psi_t\rangle = \frac{1}{\sqrt{2}}\sum_{i=1}^{2}\alpha_i|\lambda_1U^*e_i,A_1e_i\rangle\otimes|\lambda_2U^*e_i,A_2e_i\rangle,$$

we have

$$\begin{aligned}\langle\psi_t|M_1\otimes M_2\psi_t\rangle &= \frac{1}{2}\sum_{i,j=1}^{2}\bar{\alpha}_i\alpha_j\langle 0,e_i|\otimes\langle 0,e_i|\cdot|\lambda_1U^*e_j,A_1e_j\rangle\otimes|\lambda_2U^*e_j,A_2e_j\rangle\\ &= \frac{1}{2}\sum_{i,j=1}^{2}\bar{\alpha}_i\alpha_j\langle e_i|A_1e_j\rangle\langle e_i|A_2e_j\rangle.\end{aligned}$$

This equality and (20) imply (18). △

Lemma 2. *If y is a nonzero vector in $\mathbb{C}^2$, then $\langle y|Uy\rangle\neq 0$.*

Proof. Let $y=[y_1,y_2]$; then $Uy=[\eta y_1,\bar{\eta}y_2]$ and $\langle y|Uy\rangle = |y_1|^2\eta+|y_2|^2\bar{\eta}\neq 0$ (since $\eta=\exp[i\pi/4]$). △

Theorem 1 (with Lemma 1) and Proposition 2 in [7] imply the following assertion.

Corollary 1. *There is a pseudo-diagonal*[3] *channel Φ with $\dim\mathcal{H}_A=4$, $\dim\mathcal{H}_E=3$, and $\dim\mathcal{H}_B=12$, such that $\mathcal{G}(\Phi)=\mathfrak{L}_0$, and hence*

$$\bar{Q}_0(\Phi)=0\quad but\quad \bar{Q}_0(\Phi\otimes\Phi)>0.$$

For each given $t\in[0,2\pi)$, the channel $\Phi\otimes\Phi$ is perfectly reversible on the 2D subspace $\mathcal{H}_t=\mathrm{lin}\{|\varphi_t\rangle,|\psi_t\rangle\}$, where φ_t,ψ_t are vectors defined in (10).

Remark 2. It is easy to see that the above subspace $\mathfrak{L}_0$ is not transitive. Thus, by Lemma 2 in [7], the corresponding channel Φ has positive one-shot classical zero-error capacity, and hence this channel does not demonstrate the extreme superactivation of one-shot zero-error capacity (in contrast to the channels considered in [7]).

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[3] A channel $\Phi\colon\mathfrak{T}(\mathcal{H}_A)\to\mathfrak{T}(\mathcal{H}_B)$ is said to be pseudo-diagonal if it has the representation

$$\Phi(\rho)=\sum_{i,j}c_{ij}\langle\psi_i|\rho|\psi_j\rangle|i\rangle\langle j|,\quad \rho\in\mathfrak{T}(\mathcal{H}_A),$$

where $\{c_{ij}\}$ is a Gram matrix of a collection of unit vectors, $\{|\psi_i\rangle\}$ is a collection of vectors in $\mathcal{H}_A$ such that $\sum_i|\psi_i\rangle\langle\psi_i|=I_{\mathcal{H}_A}$, and $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H}_B$ [11].

To obtain a minimal Kraus representation for one of the channels having the properties stated in Corollary 1, we have to find a basis $\{A_i\}_{i=1}^5$ of $\mathfrak{L}_0$ such that $A_i \geq 0$ for all i and $\sum_{i=1}^5 A_i = I_4$. Such a basis can easily be found; for instance,

$$A_1 = \frac{1}{6}\begin{bmatrix} 1 & 0 & \bar{\eta} & 0 \\ 0 & 2 & 0 & \eta \\ \eta & 0 & 1 & 0 \\ 0 & \bar{\eta} & 0 & 2 \end{bmatrix}, \qquad A_2 = \frac{1}{6}\begin{bmatrix} 1 & 0 & -\bar{\eta} & 0 \\ 0 & 2 & 0 & -\eta \\ -\eta & 0 & 1 & 0 \\ 0 & -\bar{\eta} & 0 & 2 \end{bmatrix}, \qquad A_3 = \frac{5}{9}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_4 = \frac{1}{18}\begin{bmatrix} 1 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 3 \end{bmatrix}, \qquad A_5 = \frac{1}{18}\begin{bmatrix} 1 & -\sqrt{3} & 0 & 0 \\ -\sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 \end{bmatrix}.$$

We also have to chose a collection $\{|\psi_i\rangle\}_{i=1}^5$ of unit vectors in $\mathbb{C}^3$ such that $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^5$ is a linearly independent subset of $\mathfrak{M}_3$. Let

$$|\psi_1\rangle = |1\rangle, \qquad |\psi_2\rangle = |2\rangle, \qquad |\psi_3\rangle = |3\rangle, \qquad |\psi_4\rangle = \frac{1}{\sqrt{2}}|1+3\rangle, \qquad |\psi_5\rangle = \frac{1}{\sqrt{2}}|2+3\rangle,$$

where $\{|1\rangle, |2\rangle, |3\rangle\}$ is the canonical basis in $\mathbb{C}^3$.

Now, by noting that $r_i = \operatorname{rank} A_i = 3$ for $i = 1, 2$ and $r_i = \operatorname{rank} A_i = 2$ for $i = 3, 4, 5$, we can apply Proposition 3 in the Appendix to obtain a minimal Kraus representation for the pseudo-diagonal channel Φ having the properties stated in Corollary 1. Direct computation gives the following Kraus operators:

$$V_1 = \frac{1}{6}\begin{bmatrix} \sqrt{6} & 0 & \sqrt{6}\bar{\eta} & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & \bar{\beta} & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad V_2 = \frac{1}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{6} & 0 & -\sqrt{6}\bar{\eta} & 0 \\ 0 & \alpha & 0 & -\beta \\ 0 & -\bar{\beta} & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} \end{bmatrix},$$

$$V_3 = \frac{1}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\sqrt{5} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{5} & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} \\ 1 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} \end{bmatrix},$$

where $\alpha = \dfrac{3+\sqrt{3}}{\sqrt{2}}$ and $\beta = \eta\dfrac{3-\sqrt{3}}{\sqrt{2}}$ ($\eta = e^{i\pi/4}$). Thus, $\Phi(\rho) = \sum_{k=1}^{3} V_k \rho V_k^*$ is a minimal Kraus representation of Φ.

3. SUPERACTIVATION WITH $\bar{Q}_0(\Phi\otimes\Phi)\geq\log n$

By generalizing the above construction, one can obtain the following result.

Theorem 2. *Let* $\dim\mathcal{H}_A=2n\leq+\infty$, $\{|k\rangle\}_{k=1}^{2n}$ *be an orthonormal basis in* $\mathcal{H}_A$, *and* m *the minimal natural number such that* $n^2-n+4\leq m^2$ *if* $n<+\infty$ *and* $m=+\infty$ *otherwise.*

There exists a pseudo-diagonal channel $\Phi\colon\mathfrak{T}(\mathcal{H}_A)\to\mathfrak{T}(\mathcal{H}_B)$ *with* $\dim\mathcal{H}_E=m$ *such that* $\bar{Q}_0(\Phi)=0$ *while the channel* $\Phi\otimes\Phi$ *is perfectly reversible on the subspace of* $\mathcal{H}_A\otimes\mathcal{H}_A$ *spanned by the vectors*

$$|\varphi_k^t\rangle=\frac{1}{\sqrt{2}}\Big[|2k-1\rangle\otimes|2k-1\rangle+e^{it}|2k\rangle\otimes|2k\rangle\Big],\quad k=1,2,\ldots,n,\tag{21}$$

where t *is a fixed number in* $[0,2\pi)$, *and hence* $\bar{Q}_0(\Phi\otimes\Phi)\geq\log n$.

Proof. Assume first that $n<+\infty$. Consider the subspace

$$\mathfrak{L}_n=\left\{M=\begin{bmatrix}A & \lambda_{12}U^* & \ldots & \lambda_{1n}U^*\\ \lambda_{21}U & A & \ldots & \lambda_{2n}U^*\\ \ldots & \ldots & \ldots & \ldots\\ \lambda_{n1}U & \lambda_{n2}U & \ldots & A\end{bmatrix},\quad A\in\mathfrak{M}_2,\quad \lambda_{ij}\in\mathbb{C}\right\}\tag{22}$$

of $\mathfrak{M}_{2n}$, where U is the unitary operator in $\mathbb{C}^2$ defined in Section 2 (it has the matrix $\operatorname{diag}\{\eta,\bar{\eta}\}$ in the canonical basis of $\mathbb{C}^2$, $\eta=\exp[i\pi/4]$).

The subspace $\mathfrak{L}_n$ satisfies condition (7) and $\dim\mathfrak{L}_n=n^2-n+4$. Thus, by Proposition 2 in [7], there is a pseudo-diagonal channel Φ with $\dim\mathcal{H}_A=2n$ and $\dim\mathcal{H}_E=m$ such that $\mathcal{G}(\Phi)=\mathfrak{L}_n$.

We will prove that $\bar{Q}_0(\Phi)=0$ by showing that condition (8) is not valid for $\mathfrak{L}=\mathfrak{L}_n$.

Assume that there exist unit vectors $\varphi=[x_1,x_2,\ldots,x_n]$ and $\psi=[y_1,y_2,\ldots,y_n]$, $x_i,y_i\in\mathbb{C}^2$, such that $\langle\psi|M\varphi\rangle=0$ and $\langle\psi|M\psi\rangle=\langle\varphi|M\varphi\rangle$ for all $M\in\mathfrak{L}_n$. It follows that

$$\sum_{i=1}^n\langle y_i|Ax_i\rangle=0,\quad\forall A\in\mathfrak{M}_2,\tag{23}$$

$$\langle y_i|U^*x_k\rangle=0,\quad\forall k>1,\quad i<k,\tag{24}$$

$$\langle y_i|Ux_k\rangle=0,\quad\forall k<n,\quad i>k,\tag{25}$$

$$\sum_{i=1}^n\langle y_i|Ay_i\rangle=\sum_{i=1}^n\langle x_i|Ax_i\rangle,\quad\forall A\in\mathfrak{M}_2.\tag{26}$$

Note that (26) means that

$$\sum_{i=1}^n|y_i\rangle\langle y_i|=\sum_{i=1}^n|x_i\rangle\langle x_i|.\tag{27}$$

It suffices to show that

$$\text{either}\quad x_1\parallel x_2\parallel x_3\parallel\ldots\parallel x_n\quad\text{or}\quad y_1\parallel y_2\parallel y_3\parallel\ldots\parallel y_n,\tag{28}$$

since this and (27) imply $x_i\parallel y_j$ for all i,j, which, by Lemma 2 in Section 2, contradicts (24) and (25) (if $x_i=y_i=0$ for all $i\neq k$, then $\langle y_k|x_k\rangle=\langle\psi|\varphi\rangle=0$).

We will assume that both vectors φ and ψ have at least two nonzero components (since (28) obviously holds otherwise).

Let k be the minimal number such that $x_i=y_i=0$ for all $i<k$ and either $x_k\neq0$ or $y_k\neq0$.

By symmetry, we may assume that $x_k\neq0$. Then (25) implies

$$y_{k+1}\parallel y_{k+2}\parallel\ldots\parallel y_n.\tag{29}$$

If $y_k = 0$, then (29) means (28). If $y_k \neq 0$, then we have the following three cases:

1. $x_i \neq 0$ and $y_j \neq 0$, where $i > j > k$. In this case (24) with $k = i$ implies

$$y_k \parallel y_{k+1} \parallel \ldots \parallel y_{i-1}.$$

This and (29) imply (28) (since $y_j \neq 0$ and $i \geq k+2$);

2. $x_i \neq 0$ $y_j \neq 0$, where $j > i > k$. Since $x_k \neq 0$ and $y_k \neq 0$, this case is reduced to the previous one by permuting φ and ψ;

3. $x_i = y_i = 0$ for all $i > k$ excluding $i = \ell > k$. In this case (23) implies

$$\langle y_k | A x_k \rangle + \langle y_\ell | A x_\ell \rangle = 0, \quad \forall A \in \mathfrak{M}_2.$$

If $x_k \nparallel x_\ell$, then there is $A_0 \in \mathfrak{M}_2$ such that $y_k = A_0 x_k$ and $y_\ell = A_0 x_\ell$. Thus, the above equality implies $\langle y_k | y_k \rangle + \langle y_\ell | y_\ell \rangle = 0$, which contradicts the assumption $y_k \neq 0$. Thus, $x_k \parallel x_\ell$, and (28) holds.

Hence, condition (8) is not valid for $\mathfrak{L} = \mathfrak{L}_n$.

Now we will show that

$$\langle \varphi_k^t | M_1 \otimes M_2 \varphi_\ell^t \rangle = 0, \quad \forall M_1, M_2 \in \mathfrak{L}_n, \quad k \neq \ell, \tag{30}$$

and

$$\langle \varphi_k^t | M_1 \otimes M_2 \varphi_k^t \rangle = \langle \varphi_\ell^t | M_1 \otimes M_2 \varphi_\ell^t \rangle, \quad \forall M_1, M_2 \in \mathfrak{L}_n, \quad k \neq \ell, \tag{31}$$

for the family $\{\varphi_k^t\}_{k=1}^n$ of vectors defined in (21). By Lemma 1 these relations mean perfect reversibility of the channel $\Phi \otimes \Phi$ on the subspace spanned by this family.

Let $|\xi_i^k\rangle = |0, \ldots, 0, e_i, 0, \ldots, 0\rangle$ be a vector in $\mathbb{C}^{2n} = [\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2]$, where e_i is in the kth position ($\{e_1, e_2\}$ is the canonical basis in $\mathbb{C}^2$). Then

$$|\varphi_k^t\rangle = \frac{1}{\sqrt{2}} \left[|\xi_1^k\rangle \otimes |\xi_1^k\rangle + e^{it} |\xi_2^k\rangle \otimes |\xi_2^k\rangle \right], \quad k = 1, 2, \ldots, n.$$

By setting $\alpha_1 = 1$ and $\alpha_2 = e^{it}$, we have

$$M_1 \otimes M_2 |\varphi_k^t\rangle = \frac{1}{\sqrt{2}} \sum_{j=1}^{2} \alpha_j |\psi(1,k,j)\rangle \otimes |\psi(2,k,j)\rangle, \tag{32}$$

where

$$|\psi(r,k,j)\rangle = |\lambda_{1k}^r U^* e_j, \lambda_{2k}^r U^* e_j, \ldots, \lambda_{[k-1]k} U^* e_j, A^r e_j, \lambda_{[k+1]k}^r U e_j, \ldots, \lambda_{nk}^r U e_j\rangle,$$

$r = 1, 2$ (A^r and λ_{ij}^r correspond to the matrix M_r). If $\ell > k$, then

$$\begin{aligned}
\langle \varphi_\ell^t | M_1 \otimes M_2 \varphi_k^t \rangle &= \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle \xi_i^\ell | \otimes \langle \xi_i^\ell | \cdot |\psi(1,k,j)\rangle \otimes |\psi(2,k,j)\rangle \\
&= \frac{1}{2} \lambda_{\ell k}^1 \lambda_{\ell k}^2 \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle e_i | U e_j \rangle \langle e_i | U e_j \rangle = \frac{1}{2} \lambda_{\ell k}^1 \lambda_{\ell k}^2 \left[\eta^2 |\alpha_1|^2 + \bar{\eta}^2 |\alpha_2|^2 \right] = 0.
\end{aligned}$$

Thus, (30) holds for all $\ell > k$ and hence for all $\ell \neq k$. It follows from (32) that

$$\begin{aligned}
\langle \varphi_k^t | M_1 \otimes M_2 \varphi_k^t \rangle &= \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle \xi_i^k | \otimes \langle \xi_i^k | \cdot |\psi(1,k,j)\rangle \otimes |\psi(2,k,j)\rangle \\
&= \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle e_i | A^1 e_j \rangle \langle e_i | A^2 e_j \rangle
\end{aligned} \tag{33}$$

and that

$$\langle\varphi^t_\ell|M_1\otimes M_2\varphi^t_\ell\rangle=\frac{1}{2}\sum_{i,j=1}^{2}\bar{\alpha}_i\alpha_j\langle\xi^\ell_i|\otimes\langle\xi^\ell_i|\cdot|\psi(1,\ell,j)\rangle\otimes|\psi(2,\ell,j)\rangle$$
$$=\frac{1}{2}\sum_{i,j=1}^{2}\bar{\alpha}_i\alpha_j\langle e_i|A^1e_j\rangle\langle e_i|A^2e_j\rangle.$$

This equality and (33) imply (31).

Consider the case $n=+\infty$. Let $\mathcal{H}_A$ be a separable Hilbert space represented as a countable direct sum of 2D Hilbert spaces $\mathbb{C}^2$. Each operator in $\mathfrak{B}(\mathcal{H}_A)$ can be identified with an infinite block matrix satisfying a particular "boundedness" condition.

Let $\mathcal{L}_*$ be the set of all infinite block matrices M defined in (22) with $n=+\infty$ satisfying the condition

$$\Lambda^2=\sum_{i=1}^{+\infty}\sum_{j\neq i}|\lambda_{ij}|^2<+\infty. \tag{34}$$

This condition guarantees boundedness of the corresponding operator due to the following easily-derived inequality:

$$\|M\|^2_{\mathfrak{B}(\mathcal{H}_A)}\leq 2\big[\|A\|^2_{\mathfrak{B}(\mathbb{C}^2)}+\Lambda^2\big]. \tag{35}$$

Let $\overline{\mathcal{L}}_*$ be the operator norm closure of $\mathcal{L}_*$. It is clear that $\overline{\mathcal{L}}_*$ is a symmetric subspace of $\mathfrak{B}(\mathcal{H}_A)$ containing the unit operator $I_{\mathcal{H}_A}$. By using inequality (35) it is easy to show separability of the subspace $\overline{\mathcal{L}}_*$ in the operator norm topology (as a countable dense subset of $\overline{\mathcal{L}}_*$, one can take the set of all matrices M in which A and all λ_{ij} have rational components).

Symmetricity and separability of $\overline{\mathcal{L}}_*$ imply (by the proof of Proposition 2 in [7]) existence of a countable subset $\{\widetilde{M}_i\}_{i=2}^{+\infty}\subset\overline{\mathcal{L}}_*$ of positive operators generating $\overline{\mathcal{L}}_*$ (i.e., such that the operator norm closure of all linear combinations of the operators $\widetilde{M}_i$ coincides with $\overline{\mathcal{L}}_*$). Let $M_i=2^{-i}\|\widetilde{M}_i\|^{-1}\widetilde{M}_i$, $i=2,3,\ldots$ Since $I_{\mathcal{H}_A}\in\overline{\mathcal{L}}_*$ and the series $\sum_{i=2}^{+\infty}M_i$ converges in the operator norm topology, the positive operator $M_1=I_{\mathcal{H}_A}-\sum_{i=2}^{+\infty}M_i$ lies in $\overline{\mathcal{L}}_*$. Thus, $\{M_i\}_{i=1}^{+\infty}$ is a countable subset of positive operators generating the subspace $\overline{\mathcal{L}}_*$ such that

$$\sum_{i=1}^{+\infty}M_i=I_{\mathcal{H}_A}, \tag{36}$$

where the series converges in the operator norm topology.

Let $\{|e_i\rangle\}_{i=1}^{+\infty}$ be an orthonormal basis in a separable Hilbert space $\mathcal{H}_B$. Consider the unital completely positive map

$$\mathfrak{B}(\mathcal{H}_B)\ni X\mapsto\Psi^*(X)=\sum_{i=1}^{+\infty}\langle e_i|Xe_i\rangle M_i\in\mathfrak{B}(\mathcal{H}_A).$$

Apparently, all M_i lie in $\operatorname{Ran}\Psi^*\doteq\Psi^*(\mathfrak{B}(\mathcal{H}_B))$. Since the series in (36) converges in the operator norm topology, $\operatorname{Ran}\Psi^*\subseteq\overline{\mathcal{L}}_*$. Hence, $\operatorname{Ran}\Psi^*$ is a dense subset of $\overline{\mathcal{L}}_*$.

The predual map

$$\mathfrak{T}(\mathcal{H}_A)\ni\rho\mapsto\Psi(\rho)=\sum_{i=1}^{+\infty}[\operatorname{Tr}M_i\rho]|e_i\rangle\langle e_i|\in\mathfrak{T}(\mathcal{H}_B)$$

is an entanglement-breaking quantum channel [1, ch. 6]. Let Φ be the complementary channel to Ψ, so that Φ is a pseudo-diagonal channel (see [11]) and $\mathcal{G}(\Phi)=\operatorname{Ran}\Psi^*$.

To prove that $\bar{Q}_0(\Phi) = 0$, it suffices to show, by Lemma 1, that condition (8) is not valid for $\mathfrak{L} = \mathfrak{L}_*$ (since $\mathfrak{L}_*$ and $\mathrm{Ran}\,\Psi^*$ are dense in $\overline{\mathfrak{L}}_*$). This can be done by repeating the arguments from the proof of the same assertion in the case $n < +\infty$.

The vectors defined in (21) with $n = +\infty$ are represented as follows:

$$|\varphi_k^t\rangle = \frac{1}{\sqrt{2}}\left[|\xi_1^k\rangle \otimes |\xi_1^k\rangle + e^{it}|\xi_2^k\rangle \otimes |\xi_2^k\rangle\right], \quad k = 1, 2, 3, \ldots,$$

where $|\xi_i^k\rangle = |0, \ldots, 0, e_i, 0, 0, \ldots\rangle$ is a vector in $\mathcal{H}_A = [\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2 \oplus \ldots]$ containing e_i in the kth position ($\{e_1, e_2\}$ is the canonical basis in $\mathbb{C}^2$).

Since $\mathrm{Ran}\,\Psi^*$ is a dense subset of $\overline{\mathfrak{L}}_*$, $\mathrm{Ran}\,[\Psi^* \otimes \Psi^*]$ is a dense subset of $\overline{\mathfrak{L}}_* \bar{\otimes} \overline{\mathfrak{L}}_*$ (where $\bar{\otimes}$ denotes the spacial tensor product). Thus, to prove that the channel $\Phi \otimes \Phi$ is perfectly reversible on the subspace spanned by the family $\{|\varphi_k^t\rangle\}_{k=1}^{+\infty}$, it suffices to show, by Lemma 1, that relations (5) hold for any pair $|\varphi_k^t\rangle, |\varphi_\ell^t\rangle$ and $\mathfrak{L} = \{M_1 \otimes M_2 \mid M_1, M_2 \in \mathfrak{L}_*\}$. This can be done by the same way as in the proof of the similar relations in the case of $n < +\infty$. △

4. ONE PROPERTY OF QUANTUM MEASUREMENTS

In this section we will show that the effect of superactivation of one-shot quantum zero-error capacity has a counterpart in quantum measurement theory.

In accordance with the basic postulates of quantum mechanics, any measurement of a quantum system associated with a Hilbert space $\mathcal{H}$ corresponds to a positive operator-valued measure (POVM), also called a (generalized) *quantum observable* [1, 2]. A quantum observable with a finite or countable set of outcomes is a discrete resolution of identity in $\mathfrak{B}(\mathcal{H})$, i.e., a set $\{M_i\}_{i=1}^m$, $m \leq +\infty$, of positive operators in $\mathcal{H}$ such that $\sum_{i=1}^m M_i = I_{\mathcal{H}}$. An observable is said to be *sharp* if it corresponds to an orthogonal resolution of identity (in this case, $\{M_i\}_{i=1}^m$ consists of mutually orthogonal projectors).

If an observable $\mathcal{M} = \{M_i\}_{i=1}^m$ is applied to a quantum system in a given state ρ, then the probability of the ith outcome is $\mathrm{Tr}\, M_i\rho$. Thus, we may consider the observable $\mathcal{M}$ as the quantum-classical channel

$$\mathfrak{S}(\mathcal{H}) \ni \rho \mapsto \pi_{\mathcal{M}}(\rho) = \{\mathrm{Tr}\, M_i\rho\}_{i=1}^m \in \mathfrak{P}_m,$$

where $\mathfrak{P}_m$ is the set of all probability distributions with m outcomes.

In quantum measurement theory, the notion of *informational completeness* of an observable and its modifications are widely used [12–14]. An observable $\mathcal{M}$ is said to be informational complete if for any two different states ρ_1 and ρ_2 the probability distributions $\pi_{\mathcal{M}}(\rho_1)$ and $\pi_{\mathcal{M}}(\rho_2)$ are different.

Informational noncompleteness of an observable can be characterized by the following notion.

Definition. A subspace $\mathcal{H}_0 \subset \mathcal{H}$ is said to be *indistinguishable* for an observable $\mathcal{M}$ if $\pi_{\mathcal{M}}(\rho_1) = \pi_{\mathcal{M}}(\rho_2)$ for any states ρ_1 and ρ_2 supported by $\mathcal{H}_0$.

If $\mathcal{M} = \{M_i\}$ is a sharp observable, then all its indistinguishable subspaces coincide with the ranges of the projectors M_i of rank ≥ 2. Thus, a sharp observable has no indistinguishable subspaces if and only if it consists of rank-one projectors. This is not true for unsharp observables (see the example after Corollary 2).

To describe indistinguishable subspaces of a given observable, one can use the following characterization of such subspaces.

Proposition 1. *Let $\mathcal{M} = \{M_i\}_{i=1}^m$, $m \leq +\infty$, be an observable in a Hilbert space $\mathcal{H}$, and let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$. The following statements are equivalent*:

(i) $\mathcal{H}_0$ *is an indistinguishable subspace for the observable* $\mathcal{M}$;

(ii) $\langle\psi|M_i\varphi\rangle = 0$ *for all* i *and any orthogonal vectors* φ *and* ψ *in* $\mathcal{H}_0$;

(iii) *There exists an orthonormal basis* $\{|\varphi_k\rangle\}$ *in* $\mathcal{H}_0$ *such that*

$$\langle\varphi_k|M_i\varphi_j\rangle = 0 \quad \textit{and} \quad \langle\varphi_k|M_i\varphi_k\rangle = \langle\varphi_j|M_i\varphi_j\rangle, \quad \forall i,j,k.$$

Proof. Note that the subspace $\mathcal{H}_0$ is indistinguishable for the observable $\mathcal{M}$ if and only if the quantum channel

$$\mathfrak{T}(\mathcal{H}) \ni \rho \mapsto \sum_{i=1}^{m}[\operatorname{Tr} M_i\rho]|i\rangle\langle i| \in \mathfrak{T}(\mathcal{H}_m), \tag{37}$$

where $\{|i\rangle\}$ is an orthonormal basis in the m-dimensional Hilbert space $\mathcal{H}_m$, has a completely depolarizing restriction to the subset $\mathfrak{S}(\mathcal{H}_0) \subset \mathfrak{S}(\mathcal{H})$. Thus, assertions of the proposition follow from the well-known characterizations of completely depolarizing channels [1]. △

Nonexistence of indistinguishable subspaces for a quantum observable can be treated as recognition quality of this observable. Thus, if we have two observables, $\mathcal{M}_1$ and $\mathcal{M}_2$, having no indistinguishable subspaces, it is natural to ask about existence of indistinguishable subspaces for their tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$.[4] It turns out that this question is closely related to the superactivation of one-shot quantum zero-error capacity.

Proposition 2. *Let* $\mathcal{H}_A^1$ *and* $\mathcal{H}_A^2$ *be finite-dimensional Hilbert spaces. The following statements are equivalent*:

(i) *There exist channels* $\Phi_1\colon \mathfrak{T}(\mathcal{H}_A^1) \to \mathfrak{T}(\mathcal{H}_B^1)$ *and* $\Phi_2\colon \mathfrak{T}(\mathcal{H}_A^2) \to \mathfrak{T}(\mathcal{H}_B^2)$ *with* $\dim\mathcal{G}(\Phi_1) = m_1$ *and* $\dim\mathcal{G}(\Phi_2) = m_2$ *such that*

$$\bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0 \quad \textit{and} \quad \bar{Q}_0(\Phi_1 \otimes \Phi_2) \geq \log n;$$

(ii) *There exist observables* $\mathcal{M}_1 = \{M_i^1\}_{i=1}^{m_1}$ *and* $\mathcal{M}_2 = \{M_i^2\}_{i=1}^{m_2}$ *in the spaces* $\mathcal{H}_A^1$ *and* $\mathcal{H}_A^2$ *having no indistinguishable subspaces such that the observable* $\mathcal{M}_1 \otimes \mathcal{M}_2$ *has an* n*-dimensional indistinguishable subspace.*

If $\Phi_1 = \Phi_2$ *in* (i), *then* $\mathcal{M}_1 = \mathcal{M}_2$ *in* (ii), *and vice versa.*

Proof. An observable $\mathcal{M} = \{M_i\}_{i=1}^m$ has an n-dimensional indistinguishable subspace if and only if the one-shot quantum zero-error capacity of the channel complementary to channel (37) is not less than $\log n$, and this observable $\mathcal{M}$ has no indistinguishable subspaces if and only if the above capacity is zero. This follows from Lemma 1 and Proposition 1, since the output set of the channel dual to channel (37) coincides with the subspace of $\mathfrak{B}(\mathcal{H}_A)$ generated by the family $\{M_i\}_{i=1}^m$.

This observation directly implies (ii) ⇒ (i).

(i) ⇒ (ii). By the proof of Proposition 2 in [7], there exist bases $\{A_i^1\}_{i=1}^{m_1}$ and $\{A_i^2\}_{i=1}^{m_2}$ of the subspaces $\mathcal{G}(\Phi_1)$ and $\mathcal{G}(\Phi_2)$ consisting of positive operators such that $\sum_{i=1}^{m_1} A_i^1 = I_{\mathcal{H}_A^1}$ and $\sum_{i=1}^{m_2} A_i^2 = I_{\mathcal{H}_A^2}$. If we consider these bases as observables $\mathcal{M}_1$ and $\mathcal{M}_2$, then validity of (ii) can be shown by using the above observation. △

Remark 3. By the above proof, the implication (ii) ⇒ (i) in Proposition 2 holds for infinite-dimensional Hilbert spaces $\mathcal{H}_A^1, \mathcal{H}_A^2$ and $n \leq \infty$. The implication (i) ⇒ (ii) can be generalized to this case if the noncommutative graphs $\mathcal{G}(\Phi_1), \mathcal{G}(\Phi_2)$ are separable (in the operator norm). This can be done by using the arguments at the end of the proof of Theorem 2 instead of Proposition 2 in [7].

[4] If $\mathcal{H}_1$ and $\mathcal{H}_2$ are indistinguishable subspaces for observables $\mathcal{M}_1$ and $\mathcal{M}_2$, then it is easy to see that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is an indistinguishable subspaces for the observable $\mathcal{M}_1 \otimes \mathcal{M}_2$, but there is a possibility of existence of *entangled* indistinguishable subspaces for the observable $\mathcal{M}_1 \otimes \mathcal{M}_2$.

Proposition 2 and Corollary 1 imply the following result.

Corollary 2. *There exists a quantum observable* $\mathcal{M} = \{M_i\}_{i=1}^5$ *in a 4D Hilbert space with no indistinguishable subspaces such that the observable* $\mathcal{M} \otimes \mathcal{M}$ *has a continuous family of 2D indistinguishable subspaces.*

As a concrete example of such an observable $\mathcal{M}$, one can take the resolution of identity $\{A_i\}_{i=1}^5$ described after Corollary 1. In this case each 2D subspace of $\mathbb{C}^4 \otimes \mathbb{C}^4$ spanned by the vectors (10) is indistinguishable for $\mathcal{M} \otimes \mathcal{M}$.

Proposition 2 (with Remark 3) and Theorem 2 imply the following observation.

Corollary 3. *Let* $n \in \mathbb{N}$ *or* $n = +\infty$. *There exists a quantum observable* $\mathcal{M} = \{M_i\}_{i=1}^{n^2-n+4}$ *in a* $2n$*-dimensional Hilbert space with no indistinguishable subspaces such that the observable* $\mathcal{M} \otimes \mathcal{M}$ *has a continuous family of* n*-dimensional indistinguishable subspaces.*[5]

Remark 4. The above effect of appearance of (entangled) indistinguishable subspace for the tensor product of two observables $\mathcal{M}_1$ and $\mathcal{M}_2$ having no indistinguishable subspaces *does not hold for sharp observables* $\mathcal{M}_1$ and $\mathcal{M}_2$ (since the tensor product of two observables consisting of mutually orthogonal rank-1 projectors is an observable consisting of mutually orthogonal rank-1 projectors as well).

APPENDIX

The Kraus representation of a channel with a given noncommutative graph. The following proposition is a modification of Corollary 1 in [7].

Proposition 3. *Let* $\mathfrak{L}$ *be a subspace of* $\mathfrak{M}_n$, $n \geq 2$, *satisfying condition* (7), *and* $\{A_i\}_{i=1}^d$ *a basis of* $\mathfrak{L}$ *such that* $A_i \geq 0$ *for all* i *and* $\sum\limits_{i=1}^d A_i = I_n$.[6] *Let* m *be a natural number such that* $d = \dim \mathfrak{L} \leq m^2$, *and* $\{|\psi_i\rangle\}_{i=1}^d$ *a collection of unit vectors in* $\mathbb{C}^m$ *such that* $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^d$ *is a linearly independent set of matrices.*

For each $k = \overline{1, m}$, *let* V_k *be a linear operator from* $\mathcal{H}_A \doteq \mathbb{C}^n$ *to* $\mathcal{H}_B \doteq \bigoplus\limits_{i=1}^d \mathbb{C}^{r_i}$, *where* $r_i = \operatorname{rank} A_i$, *defined as*

$$V_k = \sum_{i=1}^d \langle k|\psi_i\rangle W_i A_i^{1/2},$$

where $\{|k\rangle\}$ *is the canonical basis in* $\mathbb{C}^m$ *and* W_i *is a partial isometry from* $\mathcal{H}_A$ *to* $\mathcal{H}_B$ *with the initial subspace* $\operatorname{Ran} A_i$ *and final subspace* $\mathbb{C}^{r_i}$. *Then the channel*

$$\mathfrak{M}_n \ni \rho \mapsto \Phi(\rho) = \sum_{k=1}^m V_k \rho V_k^* \in \mathfrak{M}_{r_1+\ldots+r_d} \tag{38}$$

is pseudo-diagonal, and its noncommutative graph $\mathcal{G}(\Phi)$ *coincides with* $\mathfrak{L}$.

Proof. In the proof of Corollary 1 in [7] it is shown that the channel

$$\mathfrak{M}_n \ni \rho \mapsto \Psi(\rho) = \sum_{i=1}^d [\operatorname{Tr} A_i \rho] |\psi_i\rangle\langle\psi_i| \in \mathfrak{M}_m$$

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[5] If $n = +\infty$, then $n^2 - n + 4 = +\infty$, and the n-dimensional Hilbert space (subspace) means a separable Hilbert space (subspace).

[6] Existence of a basis $\{A_i\}_{i=1}^d$ with the stated properties for any subspace $\mathfrak{L}$ satisfying condition (7) is shown in the proof of Proposition 2 in [7].

has the Stinespring representation

$$\Psi(\rho) = \mathrm{Tr}_{\mathbb{C}^n\otimes\mathbb{C}^d} V\rho V^*,$$

where

$$V\colon |\varphi\rangle \mapsto \sum_{i=1}^{d} A_i^{1/2}|\varphi\rangle \otimes |i\rangle \otimes |\psi_i\rangle$$

is an isometry from $\mathbb{C}^n$ to $\mathbb{C}^n \otimes \mathbb{C}^d \otimes \mathbb{C}^m$ (here $\{|i\rangle\}$ is the canonical basis in $\mathbb{C}^d$).

Since the channel Ψ is entanglement-breaking and $\Psi^*(\mathfrak{M}_m) = \mathfrak{L}$, its complementary channel

$$\widehat{\Psi}(\rho) = \mathrm{Tr}_{\mathbb{C}^m} V\rho V^*$$

is pseudo-diagonal (see [11]), and $\mathcal{G}(\widehat{\Psi}) = \mathfrak{L}$. Its Kraus representation is $\widehat{\Psi}(\rho) = \sum_{k=1}^{m} \widetilde{V}_k\rho\widetilde{V}_k^*$, where the operators $\widetilde{V}_k$ are defined by the relation

$$\langle\phi|\widetilde{V}_k\varphi\rangle = \langle\phi\otimes k|V\varphi\rangle, \quad \varphi\in\mathbb{C}^n, \quad \phi\in\mathbb{C}^n\otimes\mathbb{C}^d,$$

so that

$$\widetilde{V}_k|\varphi\rangle = \sum_{i=1}^{d}\langle k|\psi_i\rangle A_i^{1/2}|\varphi\rangle\otimes|i\rangle.$$

By identifying $\mathbb{C}^n\otimes\mathbb{C}^d$ with $\bigoplus_{i=1}^{d}\mathbb{C}^n$, it is easy to show that the channel Φ defined in (38) is isometrically equivalent to the channel $\widehat{\Psi}$ (see [8, Appendix]), and hence $\mathcal{G}(\Phi) = \mathcal{G}(\widehat{\Psi}) = \mathfrak{L}$.

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REFERENCES

1. Holevo, A.S., *Kvantovye sistemy, kanaly, informatsiya*, Moscow: MCCME, 2010. Translated under the title*Quantum Systems, Channels, Information: A Mathematical Introduction*, Berlin: De Gruyter, 2012.

2. Nielsen, M.A. and Chuang, I.L., *Quantum Computation and Quantum Information*, Cambridge: Cambridge Univ. Press, 2000. Translated under the title *Kvantovye vychisleniya i kvantovaya informatsiya*, Moscow: Mir, 2006.

3. Smith, G. and Yard, J., Quantum Comminication with Zero-Capacity Channels, *Science*, 2008, vol. 321, no. 5897, pp. 1812–1815.

4. Cubitt, T.S., Chen, J., and Harrow, A.W., Superactivation of the Asymptotic Zero-Error Classical Capacity of a Quantum Channel, *IEEE Trans. Inform. Theory*, 2011, vol. 57, no. 2, pp. 8114–8126.

5. Cubitt, T.S. and Smith, G., An Extreme Form of Superactivation for Quantum Zero-Error Capacities, *IEEE Trans. Inform. Theory*, 2012, vol. 58, no. 3, pp. 1953–1961.

6. Duan, R., Superactivation of Zero-Error Capacity of Noisy Quantum Channels, `arXiv:0906.2527 [quant-ph]`, 2009.

7. Shirokov, M.E. and Shulman, T.V., On Superactivation of Zero-Error Capacities and Reversibility of a Quantum Channel, `arXiv:1309.2610v5 [quant-ph]`, 2013.

8. Holevo, A.S., Complementary Channels and the Additivity Problem, *Teor. Veroyatnost. i Primenen.*, 2006, vol. 51, no. 1, pp. 133–143 [*Theory Probab. Appl.* (Engl. Transl.), 2007, vol. 51, no. 1, pp. 92–100].

9. Duan, R., Severini, S., and Winter, A., Zero-Error Communication via Quantum Channels, Noncommutative Graphs, and a Quantum Lovász Number, *IEEE Trans. Inform. Theory*, 2013, vol. 59, no. 2, pp. 1164–1174.

10. Medeiros, R.A.C. and de Assis, F.M., Quantum Zero-Error Capacity, *Int. J. Quantum Inform.*, 2005, vol. 3, no. 1, pp. 135–139.

11. Cubitt, T., Ruskai, M.-B., and Smith, G., The Structure of Degradable Quantum Channels, *J. Math. Phys.*, 2008, vol. 49, no. 10, p. 102104 (27).

12. Busch, P., Informationally Complete Sets of Physical Quantities, *Int. J. Theor. Phys.*, 1991, vol. 30, no. 9, pp. 1217–1227.

13. Carmeli, C., Heinosaari, T., Schultz, J., and Toigo, A., Tasks and Premises in Quantum State Determination, *J. Phys. A*, 2014, vol. 47, no. 7, pp. 075302 (24).

14. Prugovečki, E., Information-Theoretical Aspects of Quantum Measurement, *Int. J. Theor. Phys.*, 1977, vol. 16, no. 5, pp. 321–331.