

# On Approximation of Infinite-Dimensional Quantum Channels<sup>1</sup>

M. E. Shirokov and A. S. Holevo

*Steklov Mathematical Institute, RAS, Moscow*  
msh@mi.ras.ru    holevo@mi.ras.ru

Received December 11, 2007

**Abstract**—We develop an approximation approach to infinite-dimensional quantum channels based on a detailed investigation of continuity properties of entropic characteristics of quantum channels and operations (trace-nonincreasing completely positive maps) as functions of a pair “channel, input state.” Obtained results are then applied to the problems of continuity of the  $\chi$ -capacity as a function of a channel, strong additivity of the  $\chi$ -capacity for infinite-dimensional channels, and approximating representation for the convex closure of the output entropy of an arbitrary quantum channel.

**DOI:** 10.1134/S0032946008020014

## 1. INTRODUCTION

Though major attention in quantum information theory was so far paid to finite-dimensional systems and channels, there is an increasing interest in infinite-dimensional generalizations (see [1–7] and references therein). An essential feature of infinite-dimensional channels is discontinuity and unboundedness of main entropic characteristics, which makes a straightforward generalization of results obtained in finite dimensions impossible. A natural way to study infinite-dimensional quantum channels is to approximate them in an appropriate topology by channels with continuous characteristics (for example, channels with finite-dimensional output spaces). This approach was (implicitly) used in [5] to derive strong additivity of the Holevo capacity ( $\chi$ -capacity in what follows) for some classes of infinite-dimensional channels from the corresponding finite-dimensional results and to prove that validity of the additivity conjecture in finite dimensions implies strong additivity of the  $\chi$ -capacity for all infinite-dimensional channels.

In the present paper we develop an approximation approach to infinite-dimensional quantum channels based on a detailed investigation of continuity properties of entropic characteristics of quantum channels related to the classical capacity as functions of a pair “channel, input state”. It appears that often it is convenient to approximate a channel by *operations*, i.e., trace-nonincreasing completely positive maps, rather than by channels (from the point of view of noncommutative probability, an operation is a sub-Markov map, while a channel is a Markov map). Thus, we have to extend definitions of entropic characteristics to operations and study continuity properties of these characteristics on the extended domain.

The paper is organized as follows. Section 2 presents basic notions and some results of previous works used in this paper. In Section 3 we consider the topology of strong convergence on the set of all quantum operations, which appears to be a proper topology for approximation purposes. It is

---

<sup>1</sup> Supported in part by the program “Modern Problems of Theoretical Mathematics” of the Russian Academy of Sciences, the Russian Foundation for Basic Research, project no. 06-01-00164-a, and NSH, grant no. 4129.2006.1.

shown that it is this topology in which the set of all quantum operations is homeomorphic to a particular subset of states of a composite system (the generalized Choi–Jamiołkowski isomorphism). This homeomorphism implies a simple compactness criterion for subsets of quantum operations. In Section 4, continuity properties of the convex closure of the output entropy and of the constrained  $\chi$ -capacity are explored and several sufficient continuity conditions are obtained. In Section 5 these results are applied to the following problems:

1. Continuity of the  $\chi$ -capacity as a function of a channel;
2. Strong additivity of the  $\chi$ -capacity for infinite-dimensional channels;
3. Approximating representation for the convex closure of the output entropy of an arbitrary quantum channel.

Thus, approximation of infinite-dimensional quantum channels by operations in the topology of strong convergence appears to be a useful tool in studying characteristics related to the classical capacity. Further we plan to apply it to other characteristics of quantum channels, such as the entanglement-assisted capacity and quantum capacity.

## 2. PRELIMINARIES

Let  $\mathcal{H}$  be a separable Hilbert space,  $\mathfrak{B}(\mathcal{H})$  be the set of all bounded operators on  $\mathcal{H}$ , and  $\mathfrak{T}(\mathcal{H})$  be the Banach space of all trace-class operators with the trace norm  $\|\cdot\|_1$ . Let

$$\mathfrak{T}_1(\mathcal{H}) = \{A \in \mathfrak{T}(\mathcal{H}) \mid A \geq 0, \operatorname{Tr} A \leq 1\} \quad \text{and} \quad \mathfrak{S}(\mathcal{H}) = \{A \in \mathfrak{T}_1(\mathcal{H}) \mid \operatorname{Tr} A = 1\}$$

be closed convex subsets of  $\mathfrak{T}(\mathcal{H})$ , which are complete separable metric spaces with the metric defined by the trace norm. Operators in  $\mathfrak{S}(\mathcal{H})$  are called density operators. Each density operator uniquely defines a normal state on  $\mathfrak{B}(\mathcal{H})$  (see [8]), so in what follows we for brevity use the term *state*.

We denote by  $\operatorname{co} \mathcal{A}$  ( $\overline{\operatorname{co}} \mathcal{A}$ ) the convex hull (closure) of a set  $\mathcal{A}$  and denote by  $\operatorname{co} f$  ( $\overline{\operatorname{co}} f$ ) the convex hull (closure) of a function  $f$  [9]. We denote by  $\operatorname{extr} \mathcal{A}$  the set of all extreme points of a convex set  $\mathcal{A}$ .

Let  $\mathcal{P}(\mathcal{A})$  be the set of all Borel probability measures on a complete separable metric space  $\mathcal{A}$  endowed with the topology of weak convergence [10, 11]. This set can also be considered as a complete separable metric space [11]. The subset of  $\mathcal{P}(\mathcal{A})$  consisting of measures with finite support will be denoted by  $\mathcal{P}^f(\mathcal{A})$ . In what follows we will also use the abbreviations  $\mathcal{P} = \mathcal{P}(\mathfrak{S}(\mathcal{H}))$  and  $\widehat{\mathcal{P}} = \mathcal{P}(\operatorname{extr} \mathfrak{S}(\mathcal{H}))$ .

The *barycenter* of a measure  $\mu \in \mathcal{P}$  is the state defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \sigma \mu(d\sigma).$$

For an arbitrary subset  $\mathcal{A} \subset \mathfrak{S}(\mathcal{H})$ , let  $\mathcal{P}_{\mathcal{A}}$  (respectively,  $\widehat{\mathcal{P}}_{\mathcal{A}}$ ) be the subset of  $\mathcal{P}$  (respectively,  $\widehat{\mathcal{P}}$ ) consisting of all measures with barycenter in  $\mathcal{A}$ .

A collection of states  $\{\rho_i\}$  with a corresponding probability distribution  $\{\pi_i\}$  is conventionally called an *ensemble* and is denoted by  $\{\pi_i, \rho_i\}$ . In this paper we consider an ensemble of states as a particular case of a probability measure, so that the notation  $\{\pi_i, \rho_i\} \in \mathcal{P}_{\{\rho\}}$  means that  $\rho = \sum_i \pi_i \rho_i$ .

We will use the following two extensions of the von Neumann entropy  $S(\rho) = -\operatorname{Tr} \rho \log \rho$  of a state  $\rho$  to the set  $\mathfrak{T}_1(\mathcal{H})$  (cf. [12]):

$$S(A) = -\operatorname{Tr} A \log A \quad \text{and} \quad H(A) = S(A) - \eta(\operatorname{Tr} A), \quad \forall A \in \mathfrak{T}_1(\mathcal{H}),$$

where  $\eta(x) = -x \log x$ .

The nonnegativity, concavity, and lower semicontinuity of the von Neumann entropy  $S$  on the set  $\mathfrak{S}(\mathcal{H})$  imply the same properties of the functions  $S$  and  $H$  on the set  $\mathfrak{T}_1(\mathcal{H})$ . The definition and well-known properties of the von Neumann entropy (see [13]) also imply the following relations:

$$H(\lambda A) = \lambda H(A), \quad A \in \mathfrak{T}_1(\mathcal{H}), \quad \lambda \geq 0, \tag{1}$$

$$H(A) + H(B - A) \leq H(B) \leq H(A) + H(B - A) + \text{Tr } B h_2\left(\frac{\text{Tr } A}{\text{Tr } B}\right), \tag{2}$$

where  $A, B \in \mathfrak{T}_1(\mathcal{H})$ ,  $A \leq B$ , and  $h_2(x) = \eta(x) + \eta(1 - x)$ .

The subadditivity property of the quantum entropy implies the following inequality:

$$S(C) \leq S(\text{Tr}_{\mathcal{H}} C) + S(\text{Tr}_{\mathcal{K}} C) - \eta(\text{Tr } C), \quad \forall C \in \mathfrak{T}_1(\mathcal{H} \otimes \mathcal{K}). \tag{3}$$

The relative entropy for two operators  $A$  and  $B$  in  $\mathfrak{T}_1(\mathcal{H})$  is defined as (cf. [12])

$$H(A \| B) = \sum_i \langle i | (A \log A - A \log B + B - A) | i \rangle,$$

where  $\{|i\rangle\}$  is an orthonormal basis of eigenvectors of  $A$ .

Let  $\mathcal{H}$  and  $\mathcal{H}'$  be a pair of separable Hilbert spaces, which we call, respectively, the input and output space. A quantum operation  $\Phi$  is a linear positive trace-nonincreasing map from  $\mathfrak{T}(\mathcal{H})$  to  $\mathfrak{T}(\mathcal{H}')$  such that the dual map  $\Phi^*: \mathfrak{B}(\mathcal{H}') \mapsto \mathfrak{B}(\mathcal{H})$  is completely positive [8]. The convex set of all quantum operations from  $\mathfrak{T}(\mathcal{H})$  to  $\mathfrak{T}(\mathcal{H}')$  will be denoted by  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ . If  $\Phi$  is trace preserving, then it is called a quantum channel. The convex set of all channels from  $\mathfrak{T}(\mathcal{H})$  to  $\mathfrak{T}(\mathcal{H}')$  will be denoted by  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$ .

Since the functions  $\rho \mapsto H_{\Phi}(\rho) = H(\Phi(\rho))$ ,  $\rho \mapsto S_{\Phi}(\rho) = S(\Phi(\rho))$ , and  $\rho \mapsto H(\Phi(\rho) \| A)$ , where  $\Phi$  is a given quantum operation in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  and  $A$  is a given operator in  $\mathfrak{T}_1(\mathcal{H})$ , are nonnegative and lower semicontinuous on the set  $\mathfrak{S}(\mathcal{H})$ , the functionals

$$\widehat{H}_{\Phi}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} H_{\Phi}(\rho) \mu(d\rho), \quad \widehat{S}_{\Phi}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} S_{\Phi}(\rho) \mu(d\rho),$$

and

$$\chi_{\Phi}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho) \| \Phi(\bar{\rho}(\mu))) \mu(d\rho)$$

are well defined on the set  $\mathcal{P}$ .

**Proposition 1.** *The functionals  $\widehat{H}_{\Phi}(\mu)$ ,  $\widehat{S}_{\Phi}(\mu)$ , and  $\chi_{\Phi}(\mu)$  are lower semicontinuous on  $\mathcal{P}$ . If  $S_{\Phi}(\bar{\rho}(\mu)) < +\infty$ , then*

$$\chi_{\Phi}(\mu) = S_{\Phi}(\bar{\rho}(\mu)) - \widehat{S}_{\Phi}(\mu). \tag{4}$$

This proposition can be proved by an obvious modification of arguments used in the proof of Proposition 1 in [3].

**Corollary 1.** *Let  $\mathcal{P}_0$  be a subset of  $\mathcal{P}$  such that the function  $S_{\Phi}$  is continuous on the set  $\{\bar{\rho}(\mu)\}_{\mu \in \mathcal{P}_0}$ . Then the functionals  $\widehat{H}_{\Phi}(\mu)$ ,  $\widehat{S}_{\Phi}(\mu)$ , and  $\chi_{\Phi}(\mu)$  are continuous on  $\mathcal{P}_0$ .*

Corollary 1 implies in particular the continuity of the functionals  $\widehat{H}_{\Phi}(\mu)$ ,  $\widehat{S}_{\Phi}(\mu)$ , and  $\chi_{\Phi}(\mu)$  on the set  $\mathcal{P}_{\{ \rho \}}$  if  $S_{\Phi}(\rho) < +\infty$ .

An important characteristic of a quantum channel  $\Phi$  is the convex closure  $\overline{\text{co}} H_{\Phi}$  of the output entropy  $H_{\Phi}$  ( $= S_{\Phi}$ ) [7]. In this paper we consider the convex closures  $\overline{\text{co}} H_{\Phi}$  and  $\overline{\text{co}} S_{\Phi}$  of the functions  $H_{\Phi}$  and  $S_{\Phi}$ , respectively, for an arbitrary quantum operation  $\Phi$  in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ .

**Proposition 2.** *Let  $\Phi$  be an arbitrary quantum operation in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ , and let  $\rho$  be an arbitrary state in  $\mathfrak{S}(\mathcal{H})$ .*

(A) *We have the expressions*

$$\overline{\text{co}} H_{\Phi}(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \widehat{H}_{\Phi}(\mu) = \inf_{\mu \in \widehat{\mathcal{P}}_{\{\rho\}}} \widehat{H}_{\Phi}(\mu) \quad (5)$$

and

$$\overline{\text{co}} S_{\Phi}(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \widehat{S}_{\Phi}(\mu) = \inf_{\mu \in \widehat{\mathcal{P}}_{\{\rho\}}} \widehat{S}_{\Phi}(\mu). \quad (6)$$

*The infima in these expressions are attained at some measures in  $\widehat{\mathcal{P}}_{\{\rho\}}$ .*

(B) *We have the inequalities*

$$\overline{\text{co}} H_{\Phi}(\rho) \leq \overline{\text{co}} S_{\Phi}(\rho) \leq \overline{\text{co}} H_{\Phi}(\rho) + \eta(\text{Tr } \Phi(\rho)).$$

(C) *If  $\overline{\text{co}} S_{\Phi}(\rho) < +\infty$ , then*

$$\{S_{\Phi}(\rho) < +\infty\} \iff \{\overline{\text{co}} S_{\Phi}(\rho) = \text{co } S_{\Phi}(\rho)\},$$

*where  $\text{co } S_{\Phi}$  is the convex hull of the function  $S_{\Phi}$  defined by the expression*

$$\text{co } S_{\Phi}(\rho) = \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_{\{\rho\}}^f} \sum_i \pi_i S_{\Phi}(\rho_i).$$

**Proof.** All assertions in (A) follow from Theorem 1 in [14].

The inequalities in (B) are easily deduced from the representations in (A) and the concavity of the function  $\eta$ .

The implication  $\Rightarrow$  in (C) follows from Lemma 1 in [3] and Corollary 1. Since the set of all states  $\rho$  with finite  $S_{\Phi}(\rho)$  is convex,  $S_{\Phi}(\rho) = +\infty$  implies  $\text{co } S_{\Phi}(\rho) = +\infty$ . This observation proves the implication  $\Leftarrow$  in (C).  $\triangle$

The  $\chi$ -function of a channel  $\Phi$  is defined by the expression (cf. [3, 15])

$$\chi_{\Phi}(\rho) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}_{\{\rho\}}^f} \chi_{\Phi}(\{\pi_i, \rho_i\}) = \sup_{\mu \in \mathcal{P}_{\{\rho\}}} \chi_{\Phi}(\mu), \quad (7)$$

where the last equality follows from the lower semicontinuity of the functional  $\chi_{\Phi}$  and Lemma 1 in [3].

In this paper we will consider the  $\chi$ -function of an arbitrary quantum operation  $\Phi$  in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ . By using Propositions 1 and 2, it is easy to deduce from (7) that

$$\chi_{\Phi}(\rho) = S_{\Phi}(\rho) - \text{co } S_{\Phi}(\rho) = S_{\Phi}(\rho) - \overline{\text{co}} S_{\Phi}(\rho) \quad (8)$$

for an arbitrary state  $\rho \in \mathfrak{S}(\mathcal{H})$  such that  $S_{\Phi}(\rho) < +\infty$ .

### 3. THE TOPOLOGY OF STRONG CONVERGENCE

The set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  of all quantum operations from  $\mathfrak{T}(\mathcal{H})$  to  $\mathfrak{T}(\mathcal{H}')$  can be endowed with various topologies; in particular, with the topology of *uniform convergence*, defined by the metric

$$d(\Phi, \Psi) = \sup_{\rho \in \mathfrak{S}(\mathcal{H})} \|\Phi(\rho) - \Psi(\rho)\|_1,$$

or with the topology defined by the norm of complete boundedness [16].

But to approximate an arbitrary quantum channel by a sequence of quantum operations with “smooth characteristics,” it is convenient to use a weaker topology of *strong convergence* on the set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ , generated by the strong operator topology on the set of all linear bounded operators from the Banach space  $\mathfrak{T}(\mathcal{H})$  to the Banach space  $\mathfrak{T}(\mathcal{H}')$ . Strong convergence of a sequence  $\{\Phi_n\} \subset \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  to a quantum operation  $\Phi_0 \in \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  means that

$$\lim_{n \rightarrow +\infty} \Phi_n(\rho) = \Phi_0(\rho), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

In what follows we consider the set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  as a topological space with the topology of strong convergence. The separability of the set  $\mathfrak{S}(\mathcal{H})$  implies that the topology of strong convergence on the set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  is metrisable.

*Remark 1.* Since the operator norm of any quantum operation in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  is not greater than 1, it is easily seen that the topology of strong convergence on the set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  coincides with the topology of uniform convergence on compact subsets of  $\mathfrak{S}(\mathcal{H})$ .

The advantage of the topology of strong convergence consists in the possibility to approximate an arbitrary channel  $\Phi$  in  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  by a sequence of quantum operations with a finite-dimensional output space, for example, by the sequence  $\{\Phi_n(\cdot) = P_n \Phi(\cdot) P_n\}$ , where  $\{P_n\}$  is an arbitrary sequence of finite rank projectors in  $\mathfrak{B}(\mathcal{H}')$  increasing to the unit operator  $I_{\mathcal{H}'}$ .

The following proposition shows that it is the topology of strong convergence that makes the set of all operations topologically isomorphic to a special subset of states of a composite system (the generalized Choi–Jamiolkowski isomorphism [17]).

For a given full rank state  $\sigma = \sum_i \lambda_i |i\rangle\langle i|$  in  $\mathfrak{S}(\mathcal{K})$ , let  $\mathfrak{T}(\sigma)$  be the subset of  $\mathfrak{T}_1(\mathcal{K})$  consisting of all operators  $A$  such that  $\left\| \frac{\langle i|A|j\rangle}{\sqrt{\lambda_i \lambda_j}} \right\| \leq E$ , where  $E$  is the unit matrix (this means that  $\sum_{i,j} \frac{\langle i|A|j\rangle}{\sqrt{\lambda_i \lambda_j}} |i\rangle\langle j| \leq I_{\mathcal{K}}$ ).

**Proposition 3.** *Let  $\mathcal{H}, \mathcal{H}'$ , and  $\mathcal{K}$  be separable Hilbert spaces. Let  $|\Omega\rangle$  be a unit vector in  $\mathcal{H} \otimes \mathcal{K}$  such that  $\sigma = \text{Tr}_{\mathcal{H}} |\Omega\rangle\langle\Omega|$  is a full rank state in  $\mathcal{K}$ . Then the map*

$$\mathfrak{V}: \Phi \mapsto A_{\Phi} = \Phi \otimes \text{Id}(|\Omega\rangle\langle\Omega|)$$

is a homeomorphism from  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  onto the subset

$$\mathfrak{T}_1(\mathcal{H}') \otimes \mathfrak{T}(\sigma) = \{A \in \mathfrak{T}_1(\mathcal{H}' \otimes \mathcal{K}) \mid \text{Tr}_{\mathcal{H}'} A \in \mathfrak{T}(\sigma)\}.$$

The restriction of the map  $\mathfrak{V}$  to the set  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  of channels is a homeomorphism from  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  onto the subset

$$\mathfrak{S}(\mathcal{H}') \otimes \{\sigma\} = \{\omega \in \mathfrak{S}(\mathcal{H}' \otimes \mathcal{K}) \mid \text{Tr}_{\mathcal{H}'} \omega = \sigma\}.$$

**Proof.** The second assertion of the proposition obviously follows from the first one.

Let  $\sigma = \sum_i \lambda_i |i\rangle\langle i|$  and  $|\Omega\rangle = \sum_i \sqrt{\lambda_i} |i\rangle \otimes |i\rangle$ , where  $\{|i\rangle\}$  is an orthonormal basis in  $\mathcal{H} \cong \mathcal{H}' \cong \mathcal{K}$ .

Let  $\Phi(\cdot) = \sum_k V_k(\cdot)V_k^*$  be a quantum operation in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ , so that  $\sum_k V_k^* V_k \leq I_{\mathcal{H}}$ . We have

$$\langle i| \text{Tr}_{\mathcal{H}'} \Phi \otimes \text{Id}(|\Omega\rangle\langle\Omega|) |j\rangle = \sqrt{\lambda_i \lambda_j} \text{Tr} \Phi(|i\rangle\langle j|) = \sqrt{\lambda_i \lambda_j} \langle j| \sum_k V_k^* V_k |i\rangle.$$

This implies  $\text{Tr}_{\mathcal{H}'} \Phi \otimes \text{Id}(|\Omega\rangle\langle\Omega|) \in \mathfrak{T}(\sigma)$ .

It is clear that the map  $\mathfrak{V}$  is continuous. It is injective since

$$\Phi \otimes \text{Id}(|\Omega\rangle\langle\Omega|) = \sum_{i,j} \sqrt{\lambda_i \lambda_j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j| \quad (9)$$

and hence the operator  $\Phi \otimes \text{Id}(|\Omega\rangle\langle\Omega|)$  determines the action of the quantum operation  $\Phi$  on the operators  $|i\rangle\langle j|$  for all  $i$  and  $j$ . By generalizing the arguments in [18] to the infinite-dimensional case, we will show that for each operator  $A$  in  $\mathfrak{T}_1(\mathcal{H}') \otimes \mathfrak{T}(\sigma)$  there exists a quantum operation  $\Phi_A$  in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  such that  $A = \mathfrak{V}(\Phi_A)$ .

Let  $A = \sum_k \pi_k |\psi_k\rangle\langle\psi_k|$ , where  $|\psi_k\rangle = \sum_{i,j} c_{ij}^k |i\rangle \otimes |j\rangle$  is a unit vector in  $\mathcal{H}' \otimes \mathcal{K}$  for each  $k$ . Let  $\text{Tr}_{\mathcal{H}'} A = \sum_{i,j} a_{ij} |i\rangle\langle j|$ . The equality

$$\sum_{i,j} a_{ij} |i\rangle\langle j| = \text{Tr}_{\mathcal{H}'} A = \text{Tr}_{\mathcal{H}'} \sum_{k,i,j,p,t} \pi_k c_{ij}^k \overline{c_{pt}^k} |i\rangle\langle p| \otimes |j\rangle\langle t| = \sum_{k,i,j,t} \pi_k c_{ij}^k \overline{c_{it}^k} |j\rangle\langle t|$$

implies that

$$\sum_{k,i} \pi_k c_{ij}^k \overline{c_{it}^k} = a_{jt}, \quad \forall j, t; \quad (10)$$

in particular,

$$\sum_{k,i} \pi_k |c_{ij}^k|^2 = a_{jj}, \quad \forall j. \quad (11)$$

By using the condition  $\text{Tr}_{\mathcal{H}'} A \in \mathfrak{T}(\sigma)$  and equality (11), it is easy to show that  $\pi_k \sum_t |c_{ti}^k|^2 \leq \lambda_i$  for any  $i$  and  $k$ . Hence for each  $k$  we can introduce a bounded operator  $V_k$  from  $\mathcal{H}$  to  $\mathcal{H}'$  by defining its action on vectors  $\{|i\rangle\}$  as follows:

$$V_k |i\rangle = \sqrt{\frac{\pi_k}{\lambda_i}} \sum_t c_{ti}^k |t\rangle.$$

Direct computation shows that

$$A = \sum_k V_k \otimes I_{\mathcal{K}} |\Omega\rangle\langle\Omega| V_k^* \otimes I_{\mathcal{K}} = \Phi_A \otimes \text{Id}(|\Omega\rangle\langle\Omega|),$$

where  $\Phi_A(\cdot) = \sum_k V_k(\cdot)V_k^*$  is a completely positive map from  $\mathfrak{T}(\mathcal{H})$  to  $\mathfrak{T}(\mathcal{H}')$ .

It follows from equality (10) that  $\langle j| \sum_k V_k^* V_k |i\rangle = \frac{a_{ij}}{\sqrt{\lambda_i \lambda_j}}$ . Hence, the condition  $\text{Tr}_{\mathcal{H}'} A \in \mathfrak{T}(\sigma)$  means  $\sum_k V_k^* V_k \leq I_{\mathcal{H}}$ , so that  $\Phi_A \in \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ .

To complete the proof, we have to prove that the map  $\mathfrak{V}$  is open. By using expression (9), it is easy to see that for any sequence  $\{A_n\}$  of operators in  $\mathfrak{T}_1(\mathcal{H}') \otimes \mathfrak{T}(\sigma)$  converging to an operator  $A_0$ , the sequence  $\{\Phi_{A_n}(|i\rangle\langle j|)\}$  of trace class operators converges to the operator  $\Phi_{A_0}(|i\rangle\langle j|)$  (in the trace norm topology) for each  $i$  and  $j$ . Since the operator norm of a quantum operation in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  is not greater than 1, this implies the strong convergence of the sequence  $\{\Phi_{A_n}\}$  to the quantum operation  $\Phi_{A_0}$ .  $\triangle$

*Remark 2.* It follows from the proof of Proposition 3 that in infinite dimensions, the set of all completely positive maps is not isomorphic to the set of states of a composite quantum system, in contrast to the finite-dimensional case (cf. [18]).

Proposition 3 makes it possible to study properties of subsets of quantum operations (respectively, channels) by identifying these subsets with subsets of trace class operators (respectively, states). For example, it implies that the set  $\mathfrak{F}_{\sigma \rightarrow \rho}$  of all channels transforming a given full rank

state  $\sigma$  into a given arbitrary state  $\rho$  is topologically isomorphic to the set  $\mathcal{C}(\rho, \sigma)$  of all states  $\omega$  in  $\mathfrak{S}(\mathcal{H} \otimes \mathcal{H}')$  such that  $\text{Tr}_{\mathcal{H}'} \omega = \sigma$  and  $\text{Tr}_{\mathcal{H}} \omega = \rho$ .

Proposition 3 provides a simple proof of the following compactness criterion for subsets of quantum operations in the topology of strong convergence.

**Corollary 2.** (1) A subset  $\mathfrak{F}_0 \subseteq \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  is compact if there exists a full rank state  $\sigma$  in  $\mathfrak{S}(\mathcal{H})$  such that  $\{\Phi(\sigma)\}_{\Phi \in \mathfrak{F}_0}$  is a compact subset of  $\mathfrak{T}_1(\mathcal{H}')$ .

(2) If a subset  $\mathfrak{F}_0 \subseteq \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  is compact, then  $\{\Phi(\sigma)\}_{\Phi \in \mathfrak{F}_0}$  is a compact subset of  $\mathfrak{T}_1(\mathcal{H}')$  for an arbitrary state  $\sigma$  in  $\mathfrak{S}(\mathcal{H})$ .

**Proof.** (1) For an arbitrary state  $\sigma = \sum_i \lambda_i |i\rangle\langle i|$  in  $\mathfrak{S}(\mathcal{K})$ , the set  $\mathfrak{T}(\sigma)$  is a compact subset of  $\mathfrak{T}_1(\mathcal{K})$ . This follows from the compactness criterion for subsets of  $\mathfrak{T}_1(\mathcal{K})$  (see Proposition 11 in the Appendix). Indeed, if  $P_n = \sum_{i=1}^n |i\rangle\langle i|$ , then

$$\text{Tr } A(I_{\mathcal{K}} - P_n) = \sum_{i>n} \langle i|A|i\rangle \leq \sum_{i>n} \lambda_i, \quad \forall A \in \mathfrak{T}(\sigma).$$

Hence, the compactness of the set  $\mathfrak{F}_0$  in the topology of strong convergence follows from Proposition 3 and Corollary 6 (see the Appendix).

(2) This assertion obviously follows from the definition of the topology of strong convergence.  $\triangle$

*Example 1.* Let  $\sigma$  be a full rank state in  $\mathfrak{S}(\mathcal{H})$ , and let  $A$  be an arbitrary operator in  $\mathfrak{T}_1(\mathcal{H}')$ . By Corollary 2, the set

$$\mathfrak{F}_{\sigma \rightarrow A} = \{\Phi \in \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}') \mid \Phi(\sigma) = A\}$$

is compact in the topology of strong convergence. Note that this set is not compact in the topology of uniform convergence. Note also that the set of *all* completely positive maps transforming the state  $\sigma$  into the operator  $A$  is not compact in the topology of strong convergence.

*Example 2.* Let  $\sigma$  be a full rank state in  $\mathfrak{S}(\mathcal{H})$ , and let  $H'$  be an  $\mathfrak{H}$ -operator (a positive operator with eigenvalues of finite multiplicity tending to infinity, which can be interpreted as a Hamiltonian of a quantum system [3]) in the space  $\mathcal{H}'$ . Corollary 2 and a lemma in [2] imply that the set of channels

$$\{\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}') \mid \text{Tr } H' \Phi(\sigma) \leq h\}$$

is compact in the topology of strong convergence for any  $h > 0$ .

Let  $H$  be an arbitrary  $\mathfrak{H}$ -operator in the space  $\mathcal{H}$ . For a given  $k > 0$ , consider the set of channels

$$\mathfrak{F}_{H, H', k} = \left\{ \Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}') \mid \sup_{\substack{\rho \in \mathfrak{S}(\mathcal{H}) \\ \text{Tr } H \rho < +\infty}} \frac{\text{Tr } H' \Phi(\rho)}{\text{Tr } H \rho} \leq k \right\}. \quad (12)$$

If we consider the  $\mathfrak{H}$ -operators  $H$  and  $H'$  as Hamiltonians of the input and output systems, respectively, the set  $\mathfrak{F}_{H, H', k}$  can be treated as the set of channels with energy amplification factor of at most  $k$ . By the above observation, the set  $\mathfrak{F}_{H, H', k}$  is compact in the topology of strong convergence for each  $k$ .

#### 4. CONTINUITY PROPERTIES OF ENTROPIC CHARACTERISTICS

To realize the approximation procedures described in Section 1, we have to obtain sufficient conditions of continuity of the characteristics in question as functions of a pair “channel, state.” In this section we consider analytical properties of the functions  $(\Phi, \rho) \mapsto \chi_{\Phi}(\rho)$  and  $(\Phi, \rho) \mapsto \overline{\text{co}} H_{\Phi}(\rho)$  defined on the Cartesian product of the set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  of quantum operations (with the topology of strong convergence) and the set  $\mathfrak{S}(\mathcal{H})$  (with the trace norm topology).



**Proposition 4.** *The functions  $(\Phi, \rho) \mapsto \chi_\Phi(\rho)$  and  $(\Phi, \rho) \mapsto \overline{\text{co}} H_\Phi(\rho)$  are lower semicontinuous on the set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}') \times \mathfrak{S}(\mathcal{H})$ .*

**Proof.** The lower semicontinuity of the function  $(\Phi, \rho) \mapsto \chi_\Phi(\rho)$  can be proved by a simple modification of the proof of the lower semicontinuity of the function  $\rho \mapsto \chi_\Phi(\rho)$  (see [5, Proposition 3]).

The proof of the lower semicontinuity of the function  $(\Phi, \rho) \mapsto \overline{\text{co}} H_\Phi(\rho)$  is based on Lemma 1, given below, and on the compactness criterion for subsets of  $\mathcal{P}$ .

Assume that the function  $(\Phi, \rho) \mapsto \overline{\text{co}} H_\Phi(\rho)$  is not lower semicontinuous. This means the existence of sequences  $\{\Phi_n\} \subset \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  and  $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H})$  converging to an operation  $\Phi_0$  and a state  $\rho_0$ , respectively, such that

$$\lim_{n \rightarrow +\infty} \overline{\text{co}} H_{\Phi_n}(\rho_n) < \overline{\text{co}} H_{\Phi_0}(\rho_0). \quad (13)$$

For each  $n > 0$ , Proposition 2 guarantees the existence of a measure  $\mu_n \in \mathcal{P}_{\{\rho_n\}}$  such that

$$\overline{\text{co}} H_{\Phi_n}(\rho_n) = \widehat{H}_{\Phi_n}(\mu_n).$$

By the compactness criterion for subsets of  $\mathcal{P}$  (Proposition 2 in [3]), the sequence  $\{\mu_n\}_{n>0}$  is relatively compact, and hence there exists a subsequence  $\{\mu_{n_k}\}_k$  converging to some measure  $\mu_0$ . The continuity of the map  $\mu \mapsto \bar{\rho}(\mu)$  implies that  $\mu_0 \in \mathcal{P}_{\{\rho_0\}}$ . By using Lemma 1, we obtain

$$\liminf_{k \rightarrow +\infty} \overline{\text{co}} H_{\Phi_{n_k}}(\rho_{n_k}) = \liminf_{k \rightarrow +\infty} \widehat{H}_{\Phi_{n_k}}(\mu_{n_k}) \geq \widehat{H}_{\Phi_0}(\mu_0) \geq \overline{\text{co}} H_{\Phi_0}(\rho_0),$$

which contradicts (13).  $\triangle$

**Lemma 1.** *The functional  $(\Phi, \mu) \mapsto \widehat{H}_\Phi(\mu)$  is lower semicontinuous on the set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}') \times \mathcal{P}$ .*

**Proof.** Assume that there exist sequences  $\{\Phi_n\} \subset \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  and  $\{\mu_n\} \subset \mathcal{P}$  converging to an operation  $\Phi_0$  and a measure  $\mu_0$ , respectively, such that

$$\lim_{n \rightarrow +\infty} \widehat{H}_{\Phi_n}(\mu_n) < \widehat{H}_{\Phi_0}(\mu_0). \quad (14)$$

Let  $\nu_n = \mu_n \circ \Phi_n^{-1}$  be the image of the measure  $\mu_n$  under the map  $\Phi_n$  for each  $n$ . By Theorem 6.1 in [11], to prove that the sequence  $\{\nu_n\}$  of measures in  $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}'))$  weakly converges to the measure  $\nu_0 = \mu_0 \circ \Phi_0^{-1}$ , it is sufficient to show that

$$\lim_{n \rightarrow +\infty} \int_{\mathfrak{T}_1(\mathcal{H}')} f(A) \nu_n(dA) = \int_{\mathfrak{T}_1(\mathcal{H}')} f(A) \nu_0(dA) \quad (15)$$

for any bounded uniformly continuous function  $f$  on the set  $\mathfrak{T}_1(\mathcal{H}')$ . By the construction of the sequence  $\{\nu_n\}$ , relation (15) is equivalent to

$$\lim_{n \rightarrow +\infty} \int_{\mathfrak{S}(\mathcal{H})} f(\Phi_n(\rho)) \mu_n(d\rho) = \int_{\mathfrak{S}(\mathcal{H})} f(\Phi_0(\rho)) \mu_0(d\rho). \quad (16)$$

By Prohorov's theorem (cf. [10, 11]), compactness of the sequence  $\{\mu_n\}_{n \geq 0}$  (taking into account the separability and completeness of the space  $\mathfrak{S}(\mathcal{H})$ ) implies that this sequence is *tight*, which means that for each  $\varepsilon > 0$  there exists a compact set  $\mathcal{C}_\varepsilon \subset \mathfrak{S}(\mathcal{H})$  such that  $\mu_n(\mathcal{C}_\varepsilon) > 1 - \varepsilon$  for



all  $n \geq 0$ . For each  $n$ , we have

$$\begin{aligned} & \left| \int_{\mathfrak{S}(\mathcal{H})} f(\Phi_n(\rho)) \mu_n(d\rho) - \int_{\mathfrak{S}(\mathcal{H})} f(\Phi_0(\rho)) \mu_0(d\rho) \right| \\ & \leq \left| \int_{\mathcal{C}_\varepsilon} f(\Phi_n(\rho)) \mu_n(d\rho) - \int_{\mathcal{C}_\varepsilon} f(\Phi_0(\rho)) \mu_0(d\rho) \right| + 2\varepsilon \sup_{A \in \mathfrak{T}_1(\mathcal{H})} |f(A)| \\ & \leq \sup_{\rho \in \mathcal{C}_\varepsilon} |f(\Phi_n(\rho)) - f(\Phi_0(\rho))| + \left| \int_{\mathcal{C}_\varepsilon} f(\Phi_0(\rho)) \mu_n(d\rho) - \int_{\mathcal{C}_\varepsilon} f(\Phi_0(\rho)) \mu_0(d\rho) \right| + 2\varepsilon \sup_{A \in \mathfrak{T}_1(\mathcal{H})} |f(A)|. \end{aligned}$$

The first term on the right-hand side of this inequality tends to zero as  $n \rightarrow +\infty$  due to the uniform continuity of the function  $f$  and uniform convergence of the sequence  $\{\Phi_n\}$  to the quantum operation  $\Phi_0$  on the compact set  $\mathcal{C}_\varepsilon$ , which follows from the strong convergence (see Remark 1). The second term tends to zero as  $n \rightarrow +\infty$  due to the weak convergence of the sequence  $\{\mu_n\}$  to the measure  $\mu_0$ . Since  $\varepsilon$  is arbitrary, this observation proves (16) and hence (15). The weak convergence of the sequence  $\{\nu_n = \mu_n \circ \Phi_n^{-1}\}$  to the measure  $\nu_0 = \mu_0 \circ \Phi_0^{-1}$  and the lower semicontinuity of the functional  $\widehat{H}(\nu) = \int_{\mathfrak{T}_1(\mathcal{H}')} H(A) \nu(dA)$  on the set  $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}'))$  (which follows from the nonnegativity and lower semicontinuity of the function  $H(A)$  on the set  $\mathfrak{T}_1(\mathcal{H}')$ ) imply

$$\liminf_{n \rightarrow +\infty} \widehat{H}_{\Phi_n}(\mu_n) = \liminf_{n \rightarrow +\infty} \widehat{H}(\nu_n) \geq \widehat{H}(\nu_0) = \widehat{H}_{\Phi_0}(\mu_0),$$

which contradicts (14).  $\triangle$

By the concavity of the entropy and convexity of the relative entropy, Proposition 4 implies the following observation.

**Corollary 3.** *For an arbitrary state  $\sigma$  in  $\mathfrak{S}(\mathcal{H})$ , the functions*

$$\Phi \mapsto \chi_\Phi(\sigma) \quad \text{and} \quad \Phi \mapsto \overline{\text{co}} H_\Phi(\sigma)$$

are lower semicontinuous convex and concave functions on the set  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ , respectively.

By Corollary 3, the function  $\Phi \mapsto \overline{\text{co}} H_\Phi(\sigma)$  attains its infimum on any convex compact subset of  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  at some extreme point of this subset. Hence, the set  $\mathfrak{F}_{\sigma \mapsto \rho}$  of all channels mapping a given full rank state  $\sigma$  in  $\mathfrak{S}(\mathcal{H})$  to a given state  $\rho$  in  $\mathfrak{S}(\mathcal{H}')$  (see Example 1 in Section 3) contains a channel  $\Phi_{\sigma, \rho}$  such that

$$\overline{\text{co}} H_{\Phi_{\sigma, \rho}}(\sigma) \leq \overline{\text{co}} H_\Phi(\sigma), \quad \forall \Phi \in \mathfrak{F}_{\sigma \mapsto \rho}.$$

If  $\rho \cong \sigma$ , then  $\Phi_{\sigma, \rho}(\cdot) = U(\cdot)U^*$  and  $\overline{\text{co}} H_{\Phi_{\sigma, \rho}}(\sigma) = 0$ , where  $U$  is any unitary map from  $\mathcal{H}$  onto  $\mathcal{H}'$  such that  $U\sigma U^* = \rho$ . In the general case, the channel  $\Phi_{\sigma, \rho}$  is the image of some extreme point of the compact convex set  $\mathcal{C}(\sigma, \rho)$  (defined before Corollary 2) under the map  $\mathfrak{Y}^{-1}$  and, in a sense, can be considered as a channel with minimal noise transforming the state  $\sigma$  into the state  $\rho$ .

Propositions 2(B) and 4 and relation (8) imply the following sufficient condition<sup>2</sup> of continuity of the functions  $(\Phi, \rho) \mapsto \chi_\Phi(\rho)$  and  $(\Phi, \rho) \mapsto \overline{\text{co}} H_\Phi(\rho)$ .

**Proposition 5.** *Let  $\{\Phi_n\}$  be a sequence of operations in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$  strongly converging to a channel  $\Phi_0$ , and let  $\{\rho_n\}$  be a sequence of states in  $\mathfrak{S}(\mathcal{H})$  converging to a state  $\rho_0$ . If*

$$\lim_{n \rightarrow +\infty} H_{\Phi_n}(\rho_n) = H_{\Phi_0}(\rho_0) < +\infty,$$

<sup>2</sup> Proposition 5 is a generalization of Theorem 1 in [6].

then

$$\lim_{n \rightarrow +\infty} \overline{\text{co}} H_{\Phi_n}(\rho_n) = \lim_{n \rightarrow +\infty} \overline{\text{co}} S_{\Phi_n}(\rho_n) = \overline{\text{co}} H_{\Phi_0}(\rho_0) \quad \text{and} \quad \lim_{n \rightarrow +\infty} \chi_{\Phi_n}(\rho_n) = \chi_{\Phi_0}(\rho_0).$$

As an application of this condition, consider the compact set  $\mathfrak{F}_{H,H',k} \times \mathcal{K}_{H,h}$ , where  $\mathfrak{F}_{H,H',k}$  is the compact subset of  $\mathfrak{F}(\mathcal{H}, \mathcal{H}')$  consisting of channels with a bounded energy amplification factor (defined in Example 2) and  $\mathcal{K}_{H,h}$  is the compact subset of  $\mathfrak{S}(\mathcal{H})$  consisting of states with bounded mean energy (defined by the inequality  $\text{Tr } H\rho \leq h$ ). Assume that  $\text{Tr } \exp(-\lambda H') < +\infty$  for all  $\lambda > 0$ . By using an observation made in [19] (presented in [13, Proposition 6.6]), it is easy to see that the function  $(\Phi, \rho) \mapsto H_{\Phi}(\rho)$  is continuous on the set  $\mathfrak{F}_{H,H',k} \times \mathcal{K}_{H,h}$  for each  $k$  and  $h$ . Proposition 5 implies that the functions  $(\Phi, \rho) \mapsto \overline{\text{co}} H_{\Phi}(\rho)$  and  $(\Phi, \rho) \mapsto \chi_{\Phi}(\rho)$  are continuous on the set  $\mathfrak{F}_{H,H',k} \times \mathcal{K}_{H,h}$ .

A special choice of approximating sequences ensures the convergence of the functions  $\overline{\text{co}} H_{\Phi}$ ,  $\overline{\text{co}} S_{\Phi}$ , and  $\chi_{\Phi}$  without extra conditions on the output entropy.

**Proposition 6.** *Let  $\{\Phi_n\}$  be a sequence of operations strongly converging to a channel  $\Phi_0$ . The relations*

$$\lim_{n \rightarrow +\infty} \overline{\text{co}} H_{\Phi_n}(\rho) = \lim_{n \rightarrow +\infty} \overline{\text{co}} S_{\Phi_n}(\rho) = \overline{\text{co}} H_{\Phi_0}(\rho) \quad \text{and} \quad \lim_{n \rightarrow +\infty} \chi_{\Phi_n}(\rho) = \chi_{\Phi_0}(\rho)$$

hold for any state  $\rho$  in  $\mathfrak{S}(\mathcal{H})$  in the following cases:

(A)  $\Phi_n(\cdot) = P_n \Phi_0(\cdot) P_n$  for some sequence  $\{P_n\}$  of projectors in  $\mathfrak{B}(\mathcal{H}')$  increasing to the unit operator  $I_{\mathcal{H}'}$ ;

(B)  $\Phi_n(\rho) \leq \Phi_0(\rho)$  for all  $\rho$  in  $\mathfrak{S}(\mathcal{H})$ .

**Proof.** (A) For an arbitrary state  $\rho$  in  $\mathfrak{S}(\mathcal{H})$ , Lemma 3 in [12] and the monotonicity of the relative entropy imply that  $\overline{\text{co}} H_{\Phi_n}(\rho) \leq \overline{\text{co}} H_{\Phi_0}(\rho)$  and  $\chi_{\Phi_n}(\rho) \leq \chi_{\Phi_0}(\rho)$ , respectively. Hence, the limit relations in the proposition follow from Proposition 4.

(B) For an arbitrary state  $\rho$  in  $\mathfrak{S}(\mathcal{H})$ , inequality (2) and Lemma 2 (given below) imply that  $\overline{\text{co}} H_{\Phi_n}(\rho) \leq \overline{\text{co}} H_{\Phi_0}(\rho)$  and  $\chi_{\Phi_n}(\rho) \leq \chi_{\Phi_0}(\rho) + \eta(\text{Tr } \Phi_n(\rho)) + h_2(\text{Tr } \Phi_n(\rho))$ , respectively. Hence, the limit relations in the proposition follow from Proposition 4.  $\triangle$

**Lemma 2.** *Let  $\{\pi_i, A_i\}$  and  $\{\pi_i, B_i\}$  be two (finite) ensembles of operators in  $\mathfrak{T}_1(\mathcal{H})$  such that  $A_i \leq B_i, \forall i$ . Then*

$$\sum_i \pi_i H(A_i \| A) \leq \sum_i \pi_i H(B_i \| B) + \eta(\text{Tr } A) + \text{Tr } B h_2\left(\frac{\text{Tr } A}{\text{Tr } B}\right),$$

where  $A = \sum_i \pi_i A_i$  and  $B = \sum_i \pi_i B_i$ .

**Proof.** First assume that  $H(B) < +\infty$ . Then, by using inequality (2) and the concavity of the functions  $H$ ,  $h_2$ , and  $\eta$ , we obtain

$$\begin{aligned} \sum_i \pi_i H(B_i \| B) &= S(B) - \sum_i \pi_i S(B_i) = \left[ H(B) - \sum_i \pi_i H(B_i) \right] + \left[ \eta(\text{Tr } B) - \sum_i \pi_i \eta(\text{Tr } B_i) \right] \\ &\geq \left[ H(A) - \sum_i \pi_i H(A_i) \right] - \sum_i \pi_i \text{Tr } B_i h_2\left(\frac{\text{Tr } A_i}{\text{Tr } B_i}\right) \\ &\quad + \left[ \eta(\text{Tr } B) - \sum_i \pi_i \eta(\text{Tr } B_i) \right] + \left[ H(B - A) - \sum_i \pi_i H(B_i - A_i) \right] \end{aligned}$$

$$\begin{aligned} &\geq \left[ S(A) - \sum_i \pi_i S(A_i) \right] - \left[ \eta(\text{Tr } A) - \sum_i \pi_i \eta(\text{Tr } A_i) \right] - \sum_i \pi_i \text{Tr } B_i h_2 \left( \frac{\text{Tr } A_i}{\text{Tr } B_i} \right) \\ &\geq \sum_i \pi_i H(A_i \| A) - \text{Tr } B h_2 \left( \frac{\text{Tr } A}{\text{Tr } B} \right) - \eta(\text{Tr } A). \end{aligned}$$

In the case  $H(B) = +\infty$ , the above observation applied to the ensembles  $\{\pi_i, P_n A_i P_n\}$  and  $\{\pi_i, P_n B_i P_n\}$  for each  $n$ , where  $\{P_n\}$  is an arbitrary sequence of finite rank projectors increasing to the unit operator  $I_{\mathcal{H}}$ , implies the inequality

$$\sum_i \pi_i H(P_n A_i P_n \| P_n A P_n) \leq \sum_i \pi_i H(P_n B_i P_n \| P_n B P_n) + \eta(\text{Tr } P_n A) + \text{Tr } P_n B h_2 \left( \frac{\text{Tr } P_n A}{\text{Tr } P_n B} \right).$$

By using Lemma 4 in [12], we can pass to the limit in this inequality and obtain the assertion of the lemma.  $\triangle$

*Remark 3.* Theorem 1 in [6] and Proposition 6(A) imply that the  $\chi$ -function (respectively, the convex closure of the output entropy) of an arbitrary quantum channel can be represented as the least upper bound of an increasing sequence of concave (respectively, convex) *continuous* bounded functions.

## 5. SOME APPLICATIONS OF THE APPROXIMATION APPROACH

### 5.1. On Continuity of the $\chi$ -Capacity as a Function of a Channel

The  $\chi$ -capacity of a quantum channel  $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  constrained by an arbitrary subset  $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$  can be defined as (cf. [2, 3])

$$\bar{C}(\Phi, \mathcal{A}) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}_{\mathcal{A}}^f} \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})) = \sup_{\rho \in \mathcal{A}} \chi_{\Phi}(\rho). \tag{17}$$

By using the lower semicontinuity of the relative entropy, it is easy to show that the function  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}') \ni \Phi \mapsto \bar{C}(\Phi, \mathcal{A})$  is lower semicontinuous, i.e.,

$$\liminf_{n \rightarrow +\infty} \bar{C}(\Phi_n, \mathcal{A}) \geq \bar{C}(\Phi_0, \mathcal{A}) \tag{18}$$

for an arbitrary sequence  $\{\Phi_n\}$  of channels in  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  strongly converging to a channel  $\Phi_0$ . There are examples showing that the strict inequality in (18) can take place even in the case of uniform convergence of a sequence  $\{\Phi_n\}$  to a channel  $\Phi_0$  and that the difference between the left- and right-hand sides can be arbitrarily large [5].

If a sequence  $\{\Phi_n\}$  is such that the inequality  $\bar{C}(\Phi_n, \mathcal{A}) \leq \bar{C}(\Phi_0, \mathcal{A})$  can be proved for each  $n$ , then (18) implies that

$$\lim_{n \rightarrow +\infty} \bar{C}(\Phi_n, \mathcal{A}) = \bar{C}(\Phi_0, \mathcal{A}). \tag{19}$$

For example, by the monotonicity property of the relative entropy, this holds if  $\Phi_n = \Pi_n \circ \Phi_0$  for each  $n$ , where  $\{\Pi_n\}$  is a sequence of channels in  $\mathfrak{F}_{=1}(\mathcal{H}', \mathcal{H}')$  strongly converging to a noiseless channel.

The results of Section 4 make it possible to prove the following continuity condition for the  $\chi$ -capacity.

**Proposition 7.** *Let  $\{\Phi_n\}$  be a sequence of channels in  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  strongly converging to a channel  $\Phi_0$ , and let  $\mathcal{A}$  be a compact subset of  $\mathfrak{S}(\mathcal{H})$ .*

*If  $\lim_{n \rightarrow +\infty} H_{\Phi_n}(\rho_n) = H_{\Phi_0}(\rho_0) < +\infty$  for an arbitrary sequence  $\{\rho_n\}$  of states in  $\mathcal{A}$  converging to a state  $\rho_0$ , then (19) holds.*

**Proof.** To prove (19), it suffices to show that the assumption

$$\lim_{n \rightarrow +\infty} \bar{C}(\Phi_n, \mathcal{A}) > \bar{C}(\Phi_0, \mathcal{A})$$

leads to a contradiction. For each  $n$ , let  $\rho_n$  be a state in  $\mathcal{A}$  such that

$$\chi_{\Phi_n}(\rho_n) > \bar{C}(\Phi_n, \mathcal{A}) - 1/n. \quad (20)$$

The compactness of the set  $\mathcal{A}$  implies the existence of a subsequence  $\{\rho_{n_k}\}$  converging to some state  $\rho_0 \in \mathcal{A}$ . By the condition, we have  $\lim_{k \rightarrow +\infty} H_{\Phi_{n_k}}(\rho_{n_k}) = H_{\Phi_0}(\rho_0) < +\infty$ , and Proposition 5 implies that

$$\lim_{k \rightarrow +\infty} \chi_{\Phi_{n_k}}(\rho_{n_k}) = \chi_{\Phi_0}(\rho_0) \leq \bar{C}(\Phi_0, \mathcal{A}).$$

This, together with (20), leads to a contradiction.  $\triangle$

By using Proposition 7, it is possible to show that the  $\chi$ -capacity of a channel with an energy constraint is continuous on the set of channels with a bounded energy amplification factor considered in Example 2.

**Corollary 4.** *Let  $H$  and  $H'$  be  $\mathfrak{H}$ -operators in the spaces  $\mathcal{H}$  and  $\mathcal{H}'$ , respectively, such that  $\text{Tr exp}(-\lambda H') < +\infty$  for all  $\lambda > 0$ . The function  $\Phi \mapsto \bar{C}(\Phi, \mathcal{K}_{H,h})$  is continuous on the set  $\mathfrak{F}_{H,H',k}$  (defined in (12)).*

**Proof.** By a lemma in [2], the set  $\mathcal{K}_{H,h}$  is compact. Let  $h$  and  $k$  be fixed positive numbers. For arbitrary sequences  $\{\Phi_n\} \subset \mathfrak{F}_{H,H',k}$  and  $\{\rho_n\} \subset \mathcal{K}_{H,h}$ , the sequence  $\{\Phi_n(\rho_n)\}$  belongs to the set  $\mathcal{K}_{H',kh}$ , on which the entropy is continuous by an observation in [19] (see also [13, Proposition 6.6]).  $\triangle$

For an arbitrary quantum channel  $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  and an arbitrary convex subset  $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$  such that  $\bar{C}(\Phi, \mathcal{A}) < +\infty$ , there exists a unique state  $\Omega(\Phi, \mathcal{A})$  in  $\mathfrak{S}(\mathcal{H}')$ , called the output optimal average for the  $\mathcal{A}$ -constrained channel  $\Phi$  (see [6, Proposition 1]). This state inherits main properties of the image of the average state of an optimal ensemble for a finite-dimensional  $\mathcal{A}$ -constrained channel  $\Phi$  [15, 20]. If there exists an optimal measure  $\mu$  for an  $\mathcal{A}$ -constrained channel  $\Phi$  (see [3] for a definition), then  $\Omega(\Phi, \mathcal{A}) = \Phi(\bar{\rho}(\mu))$ . It is interesting to note that continuity of the function  $\Phi \mapsto \bar{C}(\Phi, \mathcal{A})$  on some set of channels implies continuity of the function  $\Phi \mapsto \Omega(\Phi, \mathcal{A})$  on this set.

**Proposition 8.** *Let  $\{\Phi_n\}$  be a sequence of channels in  $\mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  strongly converging to a channel  $\Phi_0$ , and let  $\mathcal{A}$  be a convex subset of  $\mathfrak{S}(\mathcal{H})$ .*

*If  $\lim_{n \rightarrow +\infty} \bar{C}(\Phi_n, \mathcal{A}) = \bar{C}(\Phi_0, \mathcal{A}) < +\infty$ , then  $\lim_{n \rightarrow +\infty} \Omega(\Phi_n, \mathcal{A}) = \Omega(\Phi_0, \mathcal{A})$ .*

**Proof.** By Proposition 1 in [6], for an arbitrary  $\varepsilon > 0$  there exists an ensemble  $\{\pi_i, \rho_i\}$  with average state in  $\mathcal{A}$  such that

$$\chi_{\Phi_0}(\{\pi_i, \rho_i\}) \geq \bar{C}(\Phi_0, \mathcal{A}) - \varepsilon \quad \text{and} \quad \left\| \sum_i \pi_i \Phi_0(\rho_i) - \Omega(\Phi_0, \mathcal{A}) \right\|_1 < \varepsilon. \quad (21)$$

The lower semicontinuity of the relative entropy implies that

$$\chi_{\Phi_n}(\{\pi_i, \rho_i\}) \geq \chi_{\Phi_0}(\{\pi_i, \rho_i\}) - \varepsilon$$

for all sufficiently large  $n$ . By the assumption, we have

$$\bar{C}(\Phi_n, \mathcal{A}) \leq \bar{C}(\Phi_0, \mathcal{A}) + \varepsilon$$

for all sufficiently large  $n$ .

Thus, for all sufficiently large  $n$  we have

$$0 \leq \bar{C}(\Phi_n, \mathcal{A}) - \chi_{\Phi_n}(\{\pi_i, \rho_i\}) \leq \bar{C}(\Phi_0, \mathcal{A}) - \chi_{\Phi_0}(\{\pi_i, \rho_i\}) + 2\varepsilon \leq 3\varepsilon,$$

and by using Proposition 3 in [6], we obtain

$$\begin{aligned} \frac{1}{2} \left\| \sum_i \pi_i \Phi_n(\rho_i) - \Omega(\Phi_n, \mathcal{A}) \right\|_1^2 &\leq H \left( \sum_i \pi_i \Phi_n(\rho_i) \parallel \Omega(\Phi_n, \mathcal{A}) \right) \\ &\leq \bar{C}(\Phi_n, \mathcal{A}) - \chi_{\Phi_n}(\{\pi_i, \rho_i\}) \leq 3\varepsilon. \end{aligned} \tag{22}$$

By the strong convergence of the sequence  $\{\Phi_n\}$  to the channel  $\Phi_0$ , we have

$$\left\| \sum_i \pi_i \Phi_n(\rho_i) - \sum_i \pi_i \Phi_0(\rho_i) \right\|_1 \leq \varepsilon \tag{23}$$

for all sufficiently large  $n$ .

By using (21), (22), and (23), we obtain

$$\begin{aligned} \|\Omega(\Phi_n, \mathcal{A}) - \Omega(\Phi_0, \mathcal{A})\|_1 &\leq \left\| \Omega(\Phi_n, \mathcal{A}) - \sum_i \pi_i \Phi_n(\rho_i) \right\|_1 + \left\| \sum_i \pi_i \Phi_n(\rho_i) - \sum_i \pi_i \Phi_0(\rho_i) \right\|_1 \\ &\quad + \left\| \sum_i \pi_i \Phi_0(\rho_i) - \Omega(\Phi_0, \mathcal{A}) \right\|_1 \leq 2\varepsilon + \sqrt{6\varepsilon} \end{aligned}$$

for all sufficiently large  $n$ .  $\triangle$

### 5.2. On Additivity of the $\chi$ -Capacity

The approximation procedure is an essential part of the proof that the additivity conjecture in finite dimensions implies strong additivity of the  $\chi$ -capacity for all infinite-dimensional channels [5]. It also provides the possibility to derive strong additivity of the  $\chi$ -capacity for two infinite-dimensional channels, one of them being noiseless or entanglement-breaking, from the corresponding finite-dimensional results<sup>3</sup> [15, 21].

In [7], strong additivity of the  $\chi$ -capacity for two infinite-dimensional channels, one of them being complementary to an entanglement-breaking channel, is proved *under the condition* that output entropies of both channels are finite on the set of pure input states. This condition seems to be essential since it is the coincidence of the output entropies of two complementary channels on the set of pure states which provides the “transition” of the additivity properties between pairs of complementary channels (see [22, proof of Theorem 1]), and infinite values of these output entropies make this transition impossible. But the condition of finiteness of the output entropy on the set of pure states for a given channel is difficult to verify in general, which is a real obstacle in applying the above result. Moreover, this condition is not valid for a large class of infinite-dimensional channels. Below we show that the approximation approach makes it possible to overcome the problem of infinite output entropies and prove the strong additivity of the  $\chi$ -capacity for two infinite-dimensional channels, one of them being complementary to an entanglement-breaking channel, even in the case where the output entropies of these channels are everywhere infinite.

<sup>3</sup> Note that a direct generalization of the proofs of these results to the infinite-dimensional case seems to be nontrivial. For example, the proof of Theorem 2 in [21] is based on the finiteness of the output entropy and on the decomposition of an arbitrary separable state into a *discrete* convex combination of pure product states, which is not valid in the infinite-dimensional case [4].

**Proposition 9.** *Let  $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  be a channel such that its complementary channel is entanglement-breaking, and let  $\Psi \in \mathfrak{F}_{=1}(\mathcal{K}, \mathcal{K}')$  be an arbitrary channel. Then the strong additivity of the  $\chi$ -capacity holds for the channels  $\Phi$  and  $\Psi$ .*

**Proof.** By using Lemma 5 and Proposition 6 in [5], it is possible to reduce the proof to the case of  $\dim \mathcal{K} < +\infty$  and  $\dim \mathcal{K}' < +\infty$ . By Proposition 6 in [5], it is sufficient to prove the inequality

$$\chi_{\Phi \otimes \Psi}(\omega) \leq \chi_{\Phi}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) \quad (24)$$

for an arbitrary state  $\omega$  in  $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$  such that  $\text{rank } \omega^{\mathcal{H}} < +\infty$ . Let  $\omega$  be such a state, and let  $\mathcal{H}_{\omega} = \text{supp } \omega^{\mathcal{H}}$  be the corresponding finite-dimensional subspace.

Let  $\Phi(\rho) = \text{Tr}_{\mathcal{H}''} V \rho V^*$ , where  $V$  is the Stinespring isometry from  $\mathcal{H}$  to  $\mathcal{H}' \otimes \mathcal{H}''$  [8, 22]. By the condition, the complementary channel  $\widehat{\Phi}(\rho) = \text{Tr}_{\mathcal{H}'} V \rho V^*$  is entanglement-breaking.

Let  $\{P_n\}$  be an arbitrary sequence of finite rank projectors in  $\mathfrak{B}(\mathcal{H}'')$  increasing to the unit operator  $I_{\mathcal{H}''}$ . Consider the quantum operations

$$\Phi_n(\rho) = \text{Tr}_{\mathcal{H}''} I_{\mathcal{H}'} \otimes P_n \cdot V \rho V^* \cdot I_{\mathcal{H}'} \otimes P_n = \text{Tr}_{\mathcal{H}''} I_{\mathcal{H}'} \otimes P_n \cdot V \rho V^*, \quad \rho \in \mathfrak{S}(\mathcal{H}),$$

and

$$\widehat{\Phi}_n(\rho) = \text{Tr}_{\mathcal{H}'} I_{\mathcal{H}'} \otimes P_n \cdot V \rho V^* \cdot I_{\mathcal{H}'} \otimes P_n = P_n \widehat{\Phi}(\rho) P_n, \quad \rho \in \mathfrak{S}(\mathcal{H}).$$

Let  $\widehat{\Psi}$  be the channel complementary to  $\Psi$ . Note that the restriction of the quantum operation  $\widehat{\Phi}_n$  to the set  $\mathfrak{S}(\mathcal{H}_{\omega})$  is a finite-dimensional entanglement-breaking operation. By using Proposition 2(C) and repeating arguments from the proof of Theorem 2 in [21], it is possible to show that there exists a sequence  $\{\sigma_n\} \subset \mathfrak{S}(\mathcal{K})$  converging to the state  $\omega^{\mathcal{K}}$  such that for each  $n$  we have the inequality

$$\overline{\text{co}} S_{\widehat{\Phi}_n \otimes \widehat{\Psi}}(\omega) = \text{co } S_{\widehat{\Phi}_n \otimes \widehat{\Psi}}(\omega) \geq \overline{\text{co}} S_{\widehat{\Phi}_n}(\omega^{\mathcal{H}}) + \alpha_n \overline{\text{co}} S_{\widehat{\Psi}}(\sigma_n), \quad (25)$$

where  $\alpha_n = \inf_{\rho \in \mathfrak{S}(\mathcal{H}_{\omega})} \text{Tr } \widehat{\Phi}_n(\rho)$ .

Since

$$S_{\widehat{\Phi}_n}(\rho) = S_{\Phi_n}(\rho), \quad \forall \rho \in \text{extr } \mathfrak{S}(\mathcal{H}), \quad S_{\widehat{\Psi}}(\sigma) = S_{\Psi}(\sigma), \quad \forall \sigma \in \text{extr } \mathfrak{S}(\mathcal{K}),$$

and

$$S_{\widehat{\Phi}_n \otimes \widehat{\Psi}}(\omega) = S_{\Phi_n \otimes \Psi}(\omega), \quad \forall \omega \in \text{extr } \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}),$$

Proposition 2(A) implies that inequality (25) is equivalent to

$$\overline{\text{co}} S_{\Phi_n \otimes \Psi}(\omega) \geq \overline{\text{co}} S_{\Phi_n}(\omega^{\mathcal{H}}) + \alpha_n \overline{\text{co}} S_{\Psi}(\sigma_n). \quad (26)$$

Note that inequality (3) implies that

$$S_{\Phi_n \otimes \Psi}(\omega) \leq S_{\Phi_n}(\omega^{\mathcal{H}}) + S(\text{Tr}_{\mathcal{H}'} \Phi_n \otimes \Psi(\omega)) - \varepsilon_n, \quad (27)$$

where  $\varepsilon_n = \eta(\text{Tr } \Phi_n(\omega^{\mathcal{H}}))$ .

By using (8), (26), and (27), we obtain

$$\begin{aligned} \chi_{\Phi_n \otimes \Psi}(\omega) &= S_{\Phi_n \otimes \Psi}(\omega) - \overline{\text{co}} S_{\Phi_n \otimes \Psi}(\omega) \\ &\leq S_{\Phi_n}(\omega^{\mathcal{H}}) - \overline{\text{co}} S_{\Phi_n}(\omega^{\mathcal{H}}) + S(\text{Tr}_{\mathcal{H}'} \Phi_n \otimes \Psi(\omega)) - \overline{\text{co}} S_{\Psi}(\sigma_n) + (1 - \alpha_n) \overline{\text{co}} S_{\Psi}(\sigma_n) \\ &\leq \chi_{\Phi_n}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) + [(1 - \alpha_n) S_{\Psi}(\sigma_n)] + [S(\text{Tr}_{\mathcal{H}'} \Phi_n \otimes \Psi(\omega)) - S_{\Psi}(\omega^{\mathcal{K}})] \\ &\quad + [\overline{\text{co}} S_{\Psi}(\omega^{\mathcal{K}}) - \overline{\text{co}} S_{\Psi}(\sigma_n)]. \end{aligned}$$

The sequence of quantum operations  $\{\Phi_n\}$  strongly converges to the channel  $\Phi$  and satisfies condition (B) of Proposition 6. Propositions 6(B) and 4 make it possible to prove inequality (24) by passing to the limit in the above inequality since the terms in square brackets tend to zero as  $n \rightarrow +\infty$  due to the assumed finite dimensionality of the spaces  $\mathcal{H}_\omega$  and  $\mathcal{K}'$ .  $\triangle$

*Example 3.* By Proposition 9, the strong additivity of the  $\chi$ -capacity holds for an arbitrary channel  $\Psi$  and the channel  $\Phi_p^a$  considered in an example in [7] with an arbitrary probability density function  $p(t)$  and  $a \leq +\infty$ . This implies in particular that the classical capacity of the channel  $\Phi_p^a$  with an arbitrary constraint coincides with the  $\chi$ -capacity.

### 5.3. Approximating Representation for the Convex Closure of the Output Entropy

The convex closure of the output entropy (CCoOE) of a quantum channel is an important characteristic related to the classical capacity of the channel [6]. This notion also plays an essential role in the theory of entanglement: an important entanglement measure of a state of a composite quantum system, the entanglement of formation (EoF), can be defined as the CCoOE of a partial trace [23].

By Proposition 2, the CCoOE of a quantum channel  $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  is given by the expression

$$\overline{\text{co}} H_\Phi(\rho) = \inf_{\mu \in \widehat{\mathcal{P}}_{\{\rho\}}^{\text{extr}} \mathfrak{S}(\mathcal{H})} \int H_\Phi(\sigma) \mu(d\sigma), \quad \rho \in \mathfrak{S}(\mathcal{H}). \tag{28}$$

In [6] it is shown that for an arbitrary state  $\rho$  with a finite output entropy  $H_\Phi(\rho)$ , the infimum in this expression can be taken over atomic measures only, which means that

$$\overline{\text{co}} H_\Phi(\rho) = \inf_{\{\pi_i, \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi(\rho_i) \tag{29}$$

(where the infimum is over all countable ensembles  $\{\pi_i, \rho_i\}$  of pure states with average state  $\rho$ ).

But validity of expression (29) for an arbitrary state  $\rho$  remains an open question. The second example in [14, Remark 2] shows that a positive answer to this question cannot be obtained by using only general analytical properties of the (output) entropy. For a given channel  $\Phi$ , the validity of expression (29) for an arbitrary state  $\rho$  is equivalent to the lower semicontinuity of the right-hand side of this expression as a function on the input state space  $\mathfrak{S}(\mathcal{H})$ .

Thus, in the case of the general quantum channel  $\Phi$ , we have to use representation (28), which involves optimization over *all* measures with a given barycenter  $\rho$ . This provides some technical problems in dealing with CCoOE. Moreover, this expression looks unnatural from the physical point of view since for a given state  $\rho$  with *finite* mean energy produced in a physical experiment, the above optimization involves measures supported by states with *infinite* mean energy.<sup>4</sup>

In this section we obtain a representation for the CCoOE of an arbitrary quantum channel  $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  as a limit of an increasing sequence of *continuous* bounded convex functions on  $\mathfrak{S}(\mathcal{H})$  defined via expressions similar to (29).

Let  $n > 1$  be a fixed natural number. Consider the function

$$H_\Phi^n(\rho) = - \sum_{i=1}^n \lambda_i \log \lambda_i + \left( \sum_{i=1}^n \lambda_i \right) \log \left( \sum_{i=1}^n \lambda_i \right),$$

where  $\{\lambda_i\}_{i=1}^n$  is the set of  $n$  maximal eigenvalues of the state  $\Phi(\rho)$ , which can be called the truncated output entropy. By Lemma 4 in [12], the sequence  $\{H_\Phi^n\}$  of continuous bounded functions on  $\mathfrak{S}(\mathcal{H})$  is nondecreasing and converges pointwise to the output entropy  $H_\Phi$ .

<sup>4</sup> Any countable ensemble having an average state with finite mean energy consists of states with finite mean energy.



Let

$$\check{H}_\Phi^n(\rho) = \inf_{\{\pi_i, \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi^n(\rho_i), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

By Proposition 5 in [14], the function  $\check{H}_\Phi^n (= (H_\Phi^n)_*)$  is the convex *continuous* extension of the function  $\text{extr } \mathfrak{S}(\mathcal{H}) \ni \rho \mapsto H_\Phi^n(\rho)$  to the set  $\mathfrak{S}(\mathcal{H})$ .<sup>5</sup>

The sequence  $\{\check{H}_\Phi^n\}_n$  of convex continuous bounded functions on  $\mathfrak{S}(\mathcal{H})$  is an increasing sequence and is majorized by the function  $\overline{\text{co}} H_\Phi$ . The results of Section 4 make it possible to prove the following observation.<sup>6</sup>

**Proposition 10.** *For an arbitrary channel  $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$ , the function  $\overline{\text{co}} H_\Phi$  coincides with the pointwise limit of the increasing sequence  $\{\check{H}_\Phi^n\}$  of convex continuous bounded functions on  $\mathfrak{S}(\mathcal{H})$ .*

*Remark 4.* This proposition does not imply the validity of expression (29). There exists an increasing sequence  $\{f_n\}$  of concave continuous bounded functions on  $\mathfrak{S}(\mathcal{H})$  converging to a (concave, lower semicontinuous) bounded function  $f$  such that

$$\lim_{n \rightarrow +\infty} \inf_{\{\pi_i, \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f_n(\rho_i) = 0 \quad \text{and} \quad \inf_{\{\pi_i, \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i) = 1$$

for some state  $\rho \in \mathfrak{S}(\mathcal{H})$  (see [14, the second example in Remark 2]).

**Proof.** By the above observation, it suffices to show that

$$\liminf_{n \rightarrow +\infty} \check{H}_\Phi^n(\rho) \geq \overline{\text{co}} H_\Phi(\rho) \tag{30}$$

for an arbitrary state  $\rho \in \mathfrak{S}(\mathcal{H})$ .

Let  $\{P_n\}$  be a sequence of projectors in  $\mathfrak{B}(\mathcal{H}')$  increasing to the unit operator  $I_{\mathcal{H}'}$  such that  $\text{rank } P_n = n$ . Consider the sequence  $\{\Phi_n(\cdot) = P_n \Phi(\cdot) P_n\}$  of operations in  $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ .

Let  $\rho$  be an arbitrary pure state in  $\mathfrak{S}(\mathcal{H})$ . If  $\{\lambda_i\}_{i=1}^n$  and  $\{\lambda_i^n\}_{i=1}^n$  are the sets of maximal eigenvalues (in descending order) of the operators  $\Phi(\rho)$  and  $\Phi_n(\rho)$ , then the Ritz principle implies that  $\lambda_i \geq \lambda_i^n$  for each  $i = \overline{1, n}$ . Hence, by using (2), we obtain

$$H_\Phi^n(\rho) = H(\{\lambda_i\}_{i=1}^n) \geq H(\{\lambda_i^n\}_{i=1}^n) = H_{\Phi_n}(\rho).$$

It follows that

$$\inf_{\{\pi_i, \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi^n(\rho_i) \geq \inf_{\{\pi_i, \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_{\Phi_n}(\rho_i), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

Since the function  $H_{\Phi_n}$  is concave, continuous, and bounded on  $\mathfrak{S}(\mathcal{H})$ , Corollary 10 in [14] implies that the right-hand side of the above inequality coincides with  $\overline{\text{co}} H_{\Phi_n}(\rho)$ .

The sequence  $\{\Phi_n\}$  satisfies condition (A) of Proposition 6. Hence, for an arbitrary state  $\rho$  in  $\mathfrak{S}(\mathcal{H})$ , we obtain

$$\liminf_{n \rightarrow +\infty} \inf_{\{\pi_i, \rho_i\} \in \widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i H_\Phi^n(\rho_i) \geq \lim_{n \rightarrow +\infty} \overline{\text{co}} H_{\Phi_n}(\rho) = \overline{\text{co}} H_\Phi(\rho),$$

which means (30).  $\triangle$

<sup>5</sup> Since in the general case the function  $H_\Phi^n$  is not concave on  $\mathfrak{S}(\mathcal{H})$ , we cannot claim that  $\check{H}_\Phi^n = \overline{\text{co}} H_\Phi^n$ .

<sup>6</sup> This observation is nontrivial since the set  $\mathfrak{S}(\mathcal{H})$  is not compact.

**Corollary 5.** *Let  $\Phi \in \mathfrak{F}_{=1}(\mathcal{H}, \mathcal{H}')$  be an arbitrary channel, and let  $\mathcal{A}$  be a compact subset of  $\mathfrak{S}(\mathcal{H})$  such that the output entropy  $H_\Phi$  is continuous on  $\mathcal{A}$ . Then the increasing sequence  $\{\check{H}_\Phi^n\}$  of continuous functions converges to the function  $\overline{\text{co}} H_\Phi$  uniformly on  $\mathcal{A}$ .*

**Proof.** Theorem 1 in [6] implies the continuity of the function  $\overline{\text{co}} H_\Phi$  on the set  $\mathcal{A}$ . Hence, the assertion of the corollary follows from Proposition 10 and Dini's lemma.  $\triangle$

Corollary 5 shows that for an arbitrary Gaussian channel  $\Phi$ , the sequence  $\{\check{H}_\Phi^n\}$  provides a uniform approximation of the function  $\overline{\text{co}} H_\Phi$  on the set of states with bounded mean energy (see [3, remark after Proposition 3]).

Let  $\mathcal{H}$  and  $\mathcal{K}$  be separable Hilbert spaces. Consider the channel  $\Theta: \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \ni \omega \mapsto \text{Tr}_{\mathcal{K}} \omega \in \mathfrak{S}(\mathcal{H})$ . The entanglement of formation of a state  $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$  can be defined (see [6]) by

$$E_F(\omega) = \overline{\text{co}} H_\Theta(\omega) = \inf_{\mu \in \widehat{\mathcal{P}}_{\{\omega\}}^{\text{extr}} \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})} \int H_\Theta(\sigma) \mu(d\sigma).$$

Proposition 10 implies that the function  $E_F$  coincides with the increasing sequence of convex continuous bounded functions

$$\check{H}_\Theta^n(\omega) = \inf_{\{\pi_i, \omega_i\} \in \widehat{\mathcal{P}}_{\{\omega\}}} \sum_i \pi_i H_\Theta^n(\omega_i).$$

This proves the conjecture that  $E_F$  is a function of the class  $\widehat{P}(\mathfrak{S}(\mathcal{H} \otimes \mathcal{K}))$  (cf. [14]).

By Corollary 5 (taking into account Proposition 3 in [1]), the sequence  $\{\check{H}_\Theta^n\}$  provides a uniform approximation of the function  $E_F$  on the set of states of a composite system with bounded mean energy.

## APPENDIX

The following compactness criterion for subsets of  $\mathfrak{T}_1(\mathcal{H})$  can be proved by a simple modification of arguments used in the proof of the compactness criterion for subsets of  $\mathfrak{S}(\mathcal{H})$  presented in [3, Appendix].

**Proposition 11.** *A closed subset  $\mathcal{A}$  of  $\mathfrak{T}_1(\mathcal{H})$  is compact if and only if for an arbitrary  $\varepsilon > 0$  there exists a finite rank projector  $P_\varepsilon$  such that  $\text{Tr}(I_{\mathcal{H}} - P_\varepsilon)A < \varepsilon$  for all  $A \in \mathcal{A}$ .*

**Corollary 6.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be subsets of  $\mathfrak{T}_1(\mathcal{H})$  and  $\mathfrak{T}_1(\mathcal{K})$ , respectively. The subset  $\mathcal{A} \otimes \mathcal{B}$  of  $\mathfrak{T}_1(\mathcal{H} \otimes \mathcal{K})$  consisting of all operators  $C$  such that  $\text{Tr}_{\mathcal{K}} C \in \mathcal{A}$  and  $\text{Tr}_{\mathcal{H}} C \in \mathcal{B}$  is compact if and only if the sets  $\mathcal{A}$  and  $\mathcal{B}$  are compact.*

**Proof.** The compactness of the set  $\mathcal{A} \otimes \mathcal{B}$  implies the compactness of the sets  $\mathcal{A}$  and  $\mathcal{B}$  due to continuity of the partial trace.

Let  $\mathcal{A}$  and  $\mathcal{B}$  be compact. By Proposition 11, for an arbitrary  $\varepsilon > 0$  there exist finite rank projectors  $P_\varepsilon$  and  $Q_\varepsilon$  such that

$$\text{Tr } P_\varepsilon A > \text{Tr } A - \varepsilon, \quad \forall A \in \mathcal{A}, \quad \text{and} \quad \text{Tr } Q_\varepsilon B > \text{Tr } B - \varepsilon, \quad \forall B \in \mathcal{B}.$$

Since  $C^{\mathcal{H}} = \text{Tr}_{\mathcal{K}} C \in \mathcal{A}$  and  $C^{\mathcal{K}} = \text{Tr}_{\mathcal{H}} C \in \mathcal{B}$  for an arbitrary  $C \in \mathcal{A} \otimes \mathcal{B}$ , we have

$$\begin{aligned} \text{Tr}((P_\varepsilon \otimes Q_\varepsilon) \cdot C) &= \text{Tr}((P_\varepsilon \otimes I_{\mathcal{K}}) \cdot C) - \text{Tr}(P_\varepsilon \otimes (I_{\mathcal{K}} - Q_\varepsilon) \cdot C) \\ &\geq \text{Tr } P_\varepsilon C^{\mathcal{H}} - \text{Tr}(I_{\mathcal{K}} - Q_\varepsilon) C^{\mathcal{K}} > \text{Tr } C - 2\varepsilon. \end{aligned}$$

Proposition 11 implies the compactness of the set  $\mathcal{A} \otimes \mathcal{B}$ .  $\triangle$

## REFERENCES

1. Eisert, J., Simon, C., and Plenio, M.B., On the Quantification of Entanglement in Infinite-Dimensional Quantum Systems, *J. Phys. A*, 2002, vol. 35, no. 17, pp. 3911–3923.
2. Holevo, A.S., Entanglement-assisted Capacities of Constrained Quantum Channels, *Teor. Veroyatnost. i Primenen.*, 2003, vol. 48, no. 2, pp. 359–374 [*Theory Probab. Appl.* (Engl. Transl.), 2004, vol. 48, no. 2, pp. 243–255].
3. Holevo, A.S. and Shirokov, M.E., Continuous Ensembles and the Capacity of Infinite-Dimensional Quantum Channels, *Teor. Veroyatnost. i Primenen.*, 2005, vol. 50, no. 1, pp. 98–114 [*Theory Probab. Appl.* (Engl. Transl.), 2006, vol. 50, no. 1, pp. 86–98].
4. Werner, R.F., Holevo, A.S., and Shirokov, M.E., On the Concept of Entanglement in Hilbert Spaces, *Uspekhi Mat. Nauk*, 2005, vol. 60, no. 2, pp. 153–154 [*Russian Math. Surveys* (Engl. Transl.), 2005, vol. 60, no. 2, pp. 359–360].
5. Shirokov, M.E., The Holevo Capacity of Infinite Dimensional Channels and the Additivity Problem, *Comm. Math. Phys.*, 2006, vol. 262, no. 1, pp. 137–159.
6. Shirokov, M.E., On Properties of Quantum Channels Related to Their Classical Capacity, *Teor. Veroyatnost. i Primenen.*, 2007, vol. 52, no. 2, pp. 301–335 [*Theory Probab. Appl.* (Engl. Transl.), 2007, vol. 52, no. 2, pp. 250–276].
7. Shirokov, M.E., The Convex Closure of the Output Entropy of Infinite Dimensional Channels and the Additivity Problem, 2006, arXiv e-print quant-ph/0608090.
8. Holevo, A.S., *Statistical Structure of Quantum Theory*, Berlin: Springer, 2001.
9. Ioffe, A.D. and Tikhomirov, V.M., *Teoriya ekstremal'nykh zadach*, Moscow: Nauka, 1974. Translated under the title *Theory of Extremal Problems*, Amsterdam: North-Holland, 1979.
10. Billingsley, P., *Convergence of Probability Measures*, New York: Wiley, 1968. Translated under the title *Skhodimost' veroyatnostnykh mer*, Moscow: Nauka, 1977.
11. Parthasarathy, K.R., *Probability Measures on Metric Spaces*, New York: Academic, 1967.
12. Lindblad, G., Expectations and Entropy Inequalities for Finite Quantum Systems, *Comm. Math. Phys.*, 1974, vol. 39, no. 2, pp. 111–119.
13. Ohya, M. and Petz, D., *Quantum Entropy and Its Use*, Berlin: Springer, 1993.
14. Shirokov, M.E., Properties of Probability Measures on the Set of Quantum States and Their Applications, 2006, arXiv e-print math-ph/0607019.
15. Holevo, A.S. and Shirokov, M.E., On Shor's Channel Extension and Constrained Channels, *Comm. Math. Phys.*, 2004, vol. 249, no. 2, pp. 417–430.
16. Paulsen, V.I., *Completely Bounded Maps and Operators Algebras*, Cambridge: Cambridge Univ. Press, 2002.
17. Choi, M.D., Completely Positive Linear Maps on Complex Matrices, *Linear Algebra Appl.*, 1975, vol. 10, pp. 285–290.
18. Horodecki, M., Horodecki, P., and Horodecki, R., General Teleportation Channel, Singlet Fraction, and Quasidistillation, *Phys. Rev. A*, 1999, vol. 60, no. 3, pp. 1888–1898.
19. Wehrl, A., General Properties of Entropy, *Rev. Modern Phys.*, 1978, vol. 50, no. 2, pp. 221–260.
20. Schumacher, B. and Westmoreland, M.D., Optimal Signal Ensembles, *Phys. Rev. A*, 2001, vol. 63, no. 2, 022308.
21. Shor, P.W., Additivity of the Classical Capacity of Entanglement-Breaking Quantum Channels, *J. Math. Phys.*, 2002, vol. 43, no. 9, pp. 4334–4340.
22. Holevo, A.S., Complementary Channels and the Additivity Problem, *Teor. Veroyatnost. i Primenen.*, 2006, vol. 51, no. 1, pp. 133–143 [*Theory Probab. Appl.* (Engl. Transl.), 2007, vol. 51, no. 1, pp. 92–100].
23. Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., and Wootters, W.K., Mixed-State Entanglement and Quantum Error Correction, *Phys. Rev. A*, 1996, vol. 54, no. 5, pp. 3824–3851.