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On Multipartite Superactivation of Quantum Channel Capacities

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Abstract—We consider a generalization of the notion of superactivation of quantum channel capacities to the case of $n > 2$ channels. An explicit example of such superactivation for the 1-shot quantum zero-error capacity is constructed for $n = 3$. An interpretation of this example in terms of quantum measurements is given.

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1. GENERAL OBSERVATIONS

Superactivation of quantum channel capacities is one of the most impressive quantum effects having no classical counterpart. It means that the particular capacity C of a tensor product of two quantum channels Φ_1 and Φ_2 can be positive despite the same capacity of each of these channels is zero; i.e.,

$$C(\Phi_1 \otimes \Phi_2) > 0 \quad \text{while} \quad C(\Phi_1) = C(\Phi_2) = 0. \tag{1}$$

This effect was originally observed by G. Smith and J. Yard for the case of quantum ε-error capacity [1]. Then the possibility of superactivation of other capacities, in particular classical and quantum zero-error capacities, was shown [2–5].

A natural generalization of the superactivation effect (1) to the case of n channels $\Phi_1, \ldots, \Phi_n$ consists in the validity of the following property:

$$C(\Phi_1 \otimes \ldots \otimes \Phi_n) > 0 \quad \text{while} \quad C(\Phi_{i_1} \otimes \ldots \otimes \Phi_{i_k}) = 0, \tag{2}$$

for any proper subset $\Phi_{i_1}, \ldots, \Phi_{i_k}$ $(k < n)$ of the set $\Phi_1, \ldots, \Phi_n$. This property will be called *n-partite superactivation* of the capacity C.

Property (2) means that all the channels $\Phi_1, \ldots, \Phi_n$ are required to transmit (classical or quantum) information by using a protocol corresponding to the capacity C; i.e., excluding any channel from the set $\Phi_1, \ldots, \Phi_n$ makes other channels useless for information transmission.

The obvious difficulty in finding channels $\Phi_1, \ldots, \Phi_n$ that demonstrate property (2) for a given capacity C consists in the necessity of proving the vanishing of $C(\Phi_{i_1} \otimes \ldots \otimes \Phi_{i_k})$ for any subset $\Phi_{i_1}, \ldots, \Phi_{i_k}$.

In this paper we construct an example of tripartite superactivation in the case where $C = \bar{Q}_0$ is the 1-shot quantum zero-error capacity (its definition is given in Section 2).

In [6] it is shown how a channel Ψ_n for a given n can be constructed such that

$$\bar{Q}_0(\Psi_n^{\otimes n}) = 0 \qquad \text{and} \qquad \bar{Q}_0(\Psi_n^{\otimes m}) > 0, \tag{3}$$

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where m is a natural number satisfying the inequality

$$n/m \le 2\ln(3/2)/\pi < 1.$$

Relations (3) imply the existence of $\widetilde{n} > n$ not greater than m such that (2) holds for $n = \widetilde{n}$, $C = \bar{Q}_0$, and $\Phi_1 = \ldots = \Phi_{\widetilde{n}} = \Psi_n$. Unfortunately, the approach used in [6] does not allow to determine this number $\widetilde{n}$.

In this paper we modify the example in [6] (by appropriately extending its noncommutative graph) to construct a family of channels $\{\Phi_\theta\}$ with $d_A = 4$ and $d_E = 3$ having the following property:

$$\bar{Q}_0(\Phi_{\theta_1} \otimes \Phi_{\theta_2} \otimes \Phi_{\theta_3}) > 0 \quad \text{while} \quad \bar{Q}_0(\Phi_{\theta_i} \otimes \Phi_{\theta_j}) = 0, \quad \forall i \neq j, \tag{4}$$

where θ_1, θ_2, and θ_3 are positive numbers such that $\theta_1 + \theta_2 + \theta_3 = \pi$. Thus, the channels Φ_{θ_1}, Φ_{θ_2}, and Φ_{θ_3} demonstrate the 3-partite superactivation of the 1-shot quantum zero-error capacity.

Property (4) means that all the channels Φ_{θ_i} and all the bipartite channels $\Phi_{\theta_i} \otimes \Phi_{\theta_j}$ have no ideal (perfectly reversible) subchannels, but the tripartite channel $\Phi_{\theta_1} \otimes \Phi_{\theta_2} \otimes \Phi_{\theta_3}$ has.

By using the observation in [7, Section 4], the superactivation property (4) can be reformulated in terms of quantum measurement theory as the existence of quantum observables $\mathcal{M}_{\theta_1}$, $\mathcal{M}_{\theta_2}$, and $\mathcal{M}_{\theta_3}$ such that all the observables $\mathcal{M}_{\theta_i}$ and all the bipartite observables $\mathcal{M}_{\theta_i} \otimes \mathcal{M}_{\theta_j}$ have no indistinguishable subspaces but the tripartite observable $\mathcal{M}_{\theta_1} \otimes \mathcal{M}_{\theta_2} \otimes \mathcal{M}_{\theta_3}$ has (see Corollary 2).

2. PRELIMINARIES

Let $\mathcal{H}$ be a finite dimensional Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all linear operators in $\mathcal{H}$, and $\mathfrak{S}(\mathcal{H})$ the closed convex subset of $\mathfrak{B}(\mathcal{H})$ consisting of positive operators with unit trace, called *states* [8,9]. The algebra $\mathfrak{B}(\mathcal{H})$ can be identified with the algebra $\mathfrak{M}_n$ of all $n \times n$ matrices, where $n = \dim \mathcal{H}$.

Let $\Phi\colon \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_B)$ be a quantum channel, i.e., a completely positive trace-preserving linear map [8,9]. This map has the Kraus representation

$$\Phi(A) = \sum_k V_k A V_k^*, \quad A \in \mathfrak{B}(\mathcal{H}_A), \tag{5}$$

where $\{V_k\}$ is a set of linear operators from $\mathcal{H}_A$ into $\mathcal{H}_B$ such that $\sum\limits_k V_k^* V_k = I_{\mathcal{H}_A}$ is the identity operator in $\mathcal{H}_A$. The minimal number of terms in such representation is called the *Choi rank* of Φ and is denoted by d_E (since d_E is the minimal dimension of an environment space $\mathcal{H}_E$ [8, Ch. 6]). We will also use the notation $d_A \doteq \dim \mathcal{H}_A$ and $d_B \doteq \dim \mathcal{H}_B$.

The 1-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel Φ is defined as $\sup\limits_{\mathcal{H} \in q_0(\Phi)} \log_2 \dim \mathcal{H}$, where $q_0(\Phi)$ is the set of all subspaces $\mathcal{H}_0$ of $\mathcal{H}_A$ on which the channel Φ is perfectly reversible (in the sense that there is a channel Θ such that $\Theta(\Phi(\rho)) = \rho$ for all states ρ supported by $\mathcal{H}_0$). The (asymptotic) quantum zero-error capacity is defined by regularization: $Q_0(\Phi) = \sup\limits_n n^{-1} \bar{Q}_0(\Phi^{\otimes n})$ [3,10,11].

The capacities $\bar{Q}_0(\Phi)$ and $Q_0(\Phi)$ are completely determined by the *noncommutative graph* $\mathcal{G}(\Phi)$ of the channel Φ, which can be defined as the subspace of $\mathfrak{B}(\mathcal{H}_A)$ spanned by the operators $V_k^* V_l$, where V_k are the operators from any Kraus representation (5) of Φ [11]. In particular, the Knill–Laflamme error-correcting condition (see [12]) implies the following lemma.

Lemma 1. *A channel $\Phi\colon \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_B)$ is perfectly reversible on a subspace $\mathcal{H}_0 \subseteq \mathcal{H}_A$ spanned by vectors $\{\varphi_i\}_{i=1}^n$ (which means that $\bar{Q}_0(\Phi) \ge \log n$) if and only if*

$$\langle \varphi_i | A | \varphi_j \rangle = 0 \quad \textit{and} \quad \langle \varphi_i | A | \varphi_i \rangle = \langle \varphi_j | A | \varphi_j \rangle, \quad \forall\, i \neq j, \quad \forall A \in \mathfrak{L}, \tag{6}$$

where $\mathfrak{L}$ is any subset of $\mathfrak{B}(\mathcal{H}_A)$ such that $\operatorname{lin} \mathfrak{L} = \mathcal{G}(\Phi)$.

Since a subspace $\mathfrak{L}$ of the algebra $\mathfrak{M}_n$ of $n\times n$ matrices is a noncommutative graph of a particular channel if and only if

$$\mathfrak{L} \text{ is symmetric } (\mathfrak{L}=\mathfrak{L}^*) \text{ and contains the unit matrix} \tag{7}$$

(see [4, Lemma 2] or [7, Appendix]), Lemma 1 shows that one can "construct" a channel Φ with $\dim\mathcal{H}_A=n$ having positive (correspondingly, zero) 1-shot quantum zero-error capacity by taking a subspace $\mathfrak{L}\subset\mathfrak{M}_n$ satisfying (7) for which the following condition is valid (correspondingly, not valid):

$$\exists\,\varphi,\psi\in[\mathbb{C}^n]_1 \quad \text{such that} \quad \langle\psi|A|\varphi\rangle=0 \quad \text{and} \quad \langle\varphi|A|\varphi\rangle=\langle\psi|A|\psi\rangle, \quad \forall A\in\mathfrak{L}, \tag{8}$$

where $[\mathbb{C}^n]_1$ is the unit sphere of $\mathbb{C}^n$.

3. EXAMPLE OF TRIPARTITE SUPERACTIVATION

For a given $\theta\in(-\pi,\pi]$, consider the 8-D subspace

$$\mathfrak{N}_\theta=\left\{M=\begin{bmatrix} a & b & e & f\\ c & d & f & \bar{\gamma}e\\ g & h & a & b\\ h & \gamma g & c & d\end{bmatrix},\ a,b,c,d,e,f,g,h\in\mathbb{C}\right\} \tag{9}$$

of $\mathfrak{M}_4$ satisfying condition (7), where $\gamma=\exp[i\theta]$.

Denote by $\widehat{\mathfrak{N}}_\theta$ the set of all channels whose noncommutative graph coincides with $\mathfrak{N}_\theta$. In [7, Appendix] it is shown how to explicitly construct pseudo-diagonal channels in $\widehat{\mathfrak{N}}_\theta$ with $d_A=4$ and $d_E\geq 3$ (since $\dim\mathfrak{N}_\theta=8\leq 3^2$).

Theorem. *Let Φ_θ be a channel in $\widehat{\mathfrak{N}}_\theta$ and $n\in\mathbb{N}$ be arbitrary.*

A. $\bar{Q}_0(\Phi_\theta)>0$ *if and only if* $\theta=\pi$ *and* $\bar{Q}_0(\Phi_\pi)=1$;

B. *If* $\theta_1+\ldots+\theta_n=\pi \pmod{2\pi}$, *then* $\bar{Q}_0(\Phi_{\theta_1}\otimes\ldots\otimes\Phi_{\theta_n})>0$ *and the channel* $\Phi_{\theta_1}\otimes\ldots\otimes\Phi_{\theta_n}$ *is perfectly reversible on the subspace spanned by the vectors*[2]

$$|\varphi\rangle=\frac{1}{\sqrt{2}}\left[|1\ldots1\rangle+i|2\ldots2\rangle\right], \qquad |\psi\rangle=\frac{1}{\sqrt{2}}\left[|3\ldots3\rangle+i|4\ldots4\rangle\right], \tag{10}$$

where $\{|1\rangle,\ldots,|4\rangle\}$ *is the canonical basic in* $\mathbb{C}^4$;

C. *If* $|\theta_1|+|\theta_2|<\pi$, *then* $\bar{Q}_0(\Phi_{\theta_1}\otimes\Phi_{\theta_2})=0$;

D. *If* $|\theta_1|+\ldots+|\theta_n|\leq 2\ln(3/2)$, *then* $\bar{Q}_0(\Phi_{\theta_1}\otimes\ldots\otimes\Phi_{\theta_n})=0$.

Assertion C is the main progress of this theorem as compared with Theorem 1 in [6]. It is the proof of this assertion that requires using the extended subspace $\mathfrak{N}_\theta$ (instead of the subspace $\mathfrak{L}_\theta$ used in [6]).

Remark. Since assertion D is proved by using quite coarse estimates, the other assertions of Theorem 1 make it reasonable to conjecture that assertion D can be strengthened as follows:

D′. *If* $|\theta_1|+\ldots+|\theta_n|<\pi$, *then* $\bar{Q}_0(\Phi_{\theta_1}\otimes\ldots\otimes\Phi_{\theta_n})=0$.

The proof of assertion C (i.e., D′ with $n=2$) given below cannot be generalized to the case of an arbitrary n. Thus, the question of the validity of conjecture D′ remains open.

The above theorem implies the following example of tripartite superactivation of the 1-shot quantum zero-error capacity.

[2] Here and in what follows $|1\ldots1\rangle$ denotes the vector $|1\otimes\ldots\otimes1\rangle$, etc.

Corollary 1. *Let θ_1, θ_2, and θ_3 be positive numbers such that $\theta_1+\theta_2+\theta_3=\pi$. Then*

$$\bar{Q}_0(\Phi_{\theta_1}\otimes\Phi_{\theta_2}\otimes\Phi_{\theta_3})>0 \quad \text{while} \quad \bar{Q}_0(\Phi_{\theta_i}\otimes\Phi_{\theta_j})=0, \quad \forall i\neq j.$$

The channel $\Phi_{\theta_1}\otimes\Phi_{\theta_2}\otimes\Phi_{\theta_3}$ is perfectly reversible on the subspace spanned by the vectors

$$|\varphi\rangle=\frac{1}{\sqrt{2}}\left[|111\rangle+i|222\rangle\right], \qquad |\psi\rangle=\frac{1}{\sqrt{2}}\left[|333\rangle+i|444\rangle\right]. \tag{11}$$

If conjecture D′ were valid for some $n>2$, then a similar assertion would be true for $n+1$ channels $\Phi_{\theta_1},\ldots,\Phi_{\theta_{n+1}}$. This would give an example of $(n+1)$-partite superactivation of the 1-shot quantum zero-error capacity.

For each θ one can (nonuniquely) choose a basis $\{M_k^\theta\}_{k=1}^8$ of the subspace $\mathfrak{N}_\theta$ consisting of positive operators such that $\sum_{k=1}^{8} M_k^\theta = I_{\mathcal{H}_A}$ (since the subspace $\mathfrak{N}_\theta$ satisfies condition (7); see [7]). This basis can be considered as a quantum observable $\mathcal{M}_\theta$. By using Proposition 1 in [7] and Lemma 1, one can reformulate Corollary 1 in terms of the theory of quantum measurements.

Corollary 2. *Let θ_1, θ_2, and θ_3 be positive numbers such that $\theta_1+\theta_2+\theta_3=\pi$. Then all the observables $\mathcal{M}_{\theta_i}$ and all the bipartite observables $\mathcal{M}_{\theta_i}\otimes\mathcal{M}_{\theta_j}$ have no indistinguishable subspaces, but the tripartite observable $\mathcal{M}_{\theta_1}\otimes\mathcal{M}_{\theta_2}\otimes\mathcal{M}_{\theta_3}$ has an indistinguishable subspace spanned by the vectors* (11).[3]

Note also that Theorem 1 gives an example of superactivation of the 2-shot quantum zero-error capacity (i.e., the quantity $\frac{1}{2}\bar{Q}_0(\Phi^{\otimes 2})$ determining the ultimate rate of zero-error transmission of quantum information by simultaneous use of two copies of a channel).

Corollary 3. *Let θ_1 and θ_2 be positive numbers such that $\theta_1+\theta_2=\pi/2$. Then*

$$\bar{Q}_0([\Phi_{\theta_1}\otimes\Phi_{\theta_2}]^{\otimes 2})>0 \quad \text{while} \quad \bar{Q}_0(\Phi_{\theta_1}^{\otimes 2})=\bar{Q}_0(\Phi_{\theta_2}^{\otimes 2})=\bar{Q}_0(\Phi_{\theta_1}\otimes\Phi_{\theta_2})=0.$$

Proof of the theorem. The subspace $\mathfrak{N}_\theta$ is an extension of the subspace $\mathfrak{L}_\theta$ used in [6], i.e., $\mathfrak{L}_\theta\subset\mathfrak{N}_\theta$ for each θ, and hence $\bar{Q}_0(\Phi_{\theta_1}\otimes\ldots\otimes\Phi_{\theta_n})\le\bar{Q}_0(\Psi_{\theta_1}\otimes\ldots\otimes\Psi_{\theta_n})$ for any channels $\Psi_{\theta_1}\in\widehat{\mathfrak{L}}_{\theta_1},\ldots,\Psi_{\theta_n}\in\widehat{\mathfrak{L}}_{\theta_n}$ (this follows from Lemma 1).

Thus, the equality $\bar{Q}_0(\Phi_\theta)=0$ for $\theta\neq\pi$, inequality $\bar{Q}_0(\Phi_\pi)\le 1$, and assertion D follow from the corresponding assertions of Theorem 1 in [6].

By using Lemma 1 it is easy to verify that the channel Φ_π is perfectly reversible on the subspace spanned by the vectors $|\varphi\rangle=[1,i,0,0]^\top$ and $|\psi\rangle=[0,0,1,i]^\top$. This implies $\bar{Q}_0(\Phi_\pi)=1$.

To prove assertion B, it suffices, by Lemma 1, to show that for any $M_1\in\mathfrak{N}_{\theta_1},\ldots,M_n\in\mathfrak{N}_{\theta_n}$ the equalities

$$\langle\psi|X|\varphi\rangle=0 \qquad \text{and} \qquad \langle\psi|X|\psi\rangle=\langle\varphi|X|\varphi\rangle \tag{12}$$

hold, where $X=M_1\otimes\ldots\otimes M_n$ and where φ and ψ are the vectors defined in (10).

Let $a_k,b_k,\ldots,h_k$ be elements of the matrix M_k (see (9)). We have

$$\begin{aligned}2\langle\psi|X|\varphi\rangle&=\langle 3\ldots 3|X|1\ldots 1\rangle+i\langle 3\ldots 3|X|2\ldots 2\rangle-i\langle 4\ldots 4|X|1\ldots 1\rangle+\langle 4\ldots 4|X|2\ldots 2\rangle\\&=g_1\ldots g_n(1+\gamma_1\ldots\gamma_n)+h_1\ldots h_n(i-i)=0,\end{aligned}$$

since $\gamma_1\ldots\gamma_n=-1$ by the condition $\theta_1+\ldots+\theta_n=\pi\ (\mathrm{mod}\ 2\pi)$,

$$\begin{aligned}2\langle\varphi|X|\varphi\rangle&=\langle 1\ldots 1|X|1\ldots 1\rangle+i\langle 1\ldots 1|X|2\ldots 2\rangle-i\langle 2\ldots 2|X|1\ldots 1\rangle+\langle 2\ldots 2|X|2\ldots 2\rangle\\&=a_1\ldots a_n+i(b_1\ldots b_n-c_1\ldots c_n)+d_1\ldots d_n,\end{aligned}$$

[3] We call a subspace $\mathcal{H}_0$ *indistinguishable* for an observable $\mathcal{M}$ if applying $\mathcal{M}$ to all states supported by $\mathcal{H}_0$ leads to the same outcomes probability distribution [7].

and

$$2\langle\psi|X|\psi\rangle = \langle 3\dots 3|X|3\dots 3\rangle + i\langle 3\dots 3|X|4\dots 4\rangle - i\langle 4\dots 4|X|3\dots 3\rangle + \langle 4\dots 4|X|4\dots 4\rangle$$
$$= a_1\dots a_n + i(b_1\dots b_n - c_1\dots c_n) + d_1\dots d_n.$$

Thus, both equalities in (12) are valid.

To prove assertion C, we have to show that the subspace $\mathfrak{N}_{\theta_1}\otimes\mathfrak{N}_{\theta_2}$ does not satisfy condition (8) if $|\theta_1|+|\theta_2|<\pi$. In the case $\theta_1=\theta_2=0$, this follows from assertion D. Thus, we may assume, by symmetry, that $\theta_2\neq 0$.

Throughout the proof we will use the isomorphism

$$\mathbb{C}^n\otimes\mathbb{C}^m \ni x\otimes y \;\longleftrightarrow\; [x_1y,\dots,x_ny]^\top \in \underbrace{\mathbb{C}^m\oplus\dots\oplus\mathbb{C}^m}_{n} \tag{13}$$

and the corresponding isomorphism

$$\mathfrak{M}_n\otimes\mathfrak{M}_m \ni A\otimes B \;\longleftrightarrow\; [a_{ij}B]\in\mathfrak{M}_{nm}. \tag{14}$$

Let U_1, U_2, V_1, and V_2 be unitary operators in $\mathbb{C}^2$ determined (in the canonical basis) by the matrices

$$U_1=\begin{bmatrix}1&0\\0&\gamma_1\end{bmatrix},\qquad V_1=\begin{bmatrix}1&0\\0&\gamma_2\end{bmatrix},\qquad U_2=V_2=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

We will identify $\mathbb{C}^4$ with $\mathbb{C}^2\oplus\mathbb{C}^2$. Thus, arbitrary matrices $M_1\in\mathfrak{N}_{\theta_1}$ and $M_2\in\mathfrak{N}_{\theta_2}$ can be represented as

$$M_1=\begin{bmatrix}A_1 & e_1U_1^*+f_1U_2^*\\ g_1U_1+h_1U_2 & A_1\end{bmatrix},\qquad M_2=\begin{bmatrix}A_2 & e_2V_1^*+f_2V_2^*\\ g_2V_1+h_2V_2 & A_2\end{bmatrix},$$

or, according to (14), as

$$M_1 = I_2\otimes A_1 + |2\rangle\langle 1|\otimes[g_1U_1+h_1U_2] + |1\rangle\langle 2|\otimes[e_1U_1^*+f_1U_2^*]$$

and

$$M_2 = I_2\otimes A_2 + |2\rangle\langle 1|\otimes[g_2V_1+h_2V_2] + |1\rangle\langle 2|\otimes[e_2V_1^*+f_2V_2^*],$$

where A_1 and A_2 are arbitrary matrices and I_2 is the unit matrix in $\mathfrak{M}_2$.

Assume the existence of orthogonal unit vectors φ and ψ in $\mathbb{C}^4\otimes\mathbb{C}^4$ such that

$$\langle\psi|M_1\otimes M_2|\varphi\rangle=0 \qquad\text{and}\qquad \langle\psi|M_1\otimes M_2|\psi\rangle=\langle\varphi|M_1\otimes M_2|\varphi\rangle, \tag{15}$$

for all $M_1\in\mathfrak{N}_{\theta_1}$ and $M_2\in\mathfrak{N}_{\theta_2}$.

By using the above representations of M_1 and M_2 we have

$$M_1\otimes M_2 = [I_2\otimes I_2]\otimes[A_1\otimes A_2] + [I_2\otimes|2\rangle\langle 1|]\otimes[A_1\otimes[g_2V_1+h_2V_2]]$$
$$+ [I_2\otimes|1\rangle\langle 2|]\otimes[A_1\otimes[e_2V_1^*+f_2V_2^*]] + [|2\rangle\langle 1|\otimes I_2]\otimes[[g_1U_1+h_1U_2]\otimes A_2] + \dots.$$

Since $\mathfrak{M}_2\otimes\mathfrak{M}_2=\mathfrak{M}_4$, by choosing $e_i=f_i=g_i=h_i=0$, $i=1,2$, we obtain from (15) that

$$\langle\psi|I_4\otimes A|\varphi\rangle=0 \quad\text{and}\quad \langle\psi|I_4\otimes A|\psi\rangle=\langle\varphi|I_4\otimes A|\varphi\rangle,\quad \forall A\in\mathfrak{M}_4.$$

According to (13) and (14), we have

$$I_4\otimes A=\begin{bmatrix}A&0&0&0\\0&A&0&0\\0&0&A&0\\0&0&0&A\end{bmatrix},\qquad |\varphi\rangle=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix},\qquad |\psi\rangle=\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix},$$

where x_i and y_i are vectors in $\mathbb{C}^4$. Thus, the above relations can be rewritten as

$$\sum_{i=1}^{4}\langle y_i|A|x_i\rangle = 0 \quad \text{and} \quad \sum_{i=1}^{4}\langle y_i|A|y_i\rangle = \sum_{i=1}^{4}\langle x_i|A|x_i\rangle, \quad \forall A \in \mathfrak{M}_4,$$

which are equivalent to the operator equalities

$$\sum_{i=1}^{4}|y_i\rangle\langle x_i| = 0 \tag{16}$$

and

$$\sum_{i=1}^{4}|y_i\rangle\langle y_i| = \sum_{i=1}^{4}|x_i\rangle\langle x_i|. \tag{17}$$

By choosing $e_i = f_i = g_1 = h_1 = 0$, $i = 1,2$, $A_2 = 0$, $(g_2,h_2) = (1,0)$, and $(g_2,h_2) = (0,1)$, we obtain from (15) that

$$\langle\psi|[I_2 \otimes |2\rangle\langle 1|] \otimes [A_1 \otimes V_k]|\varphi\rangle = 0$$

and

$$\langle\psi|[I_2 \otimes |2\rangle\langle 1|] \otimes [A_1 \otimes V_k]|\psi\rangle = \langle\varphi|[I_2 \otimes |2\rangle\langle 1|] \otimes [A_1 \otimes V_k]|\varphi\rangle$$

for all A_1 in $\mathfrak{M}_2$ and $k = 1,2$. According to (14), we have

$$[I_2 \otimes |2\rangle\langle 1|] \otimes [A_1 \otimes V_k] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_1 \otimes V_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 \otimes V_k & 0 \end{bmatrix},$$

and hence the above equalities imply

$$\langle y_2|A \otimes V_k|x_1\rangle + \langle y_4|A \otimes V_k|x_3\rangle = 0, \quad \forall A \in \mathfrak{M}_2, \quad k = 1,2, \tag{18}$$

and

$$\langle y_2|A \otimes V_k|y_1\rangle + \langle y_4|A \otimes V_k|y_3\rangle = \langle x_2|A \otimes V_k|x_1\rangle + \langle x_4|A \otimes V_k|x_3\rangle, \quad \forall A \in \mathfrak{M}_2, \quad k = 1,2. \tag{19}$$

Similarly, by choosing $e_i = f_i = g_2 = h_2 = 0$, $i = 1,2$, $A_1 = 0$, $(g_1,h_1) = (1,0)$, and $(g_1,h_1) = (0,1)$, we obtain from (15) the equalities

$$\langle y_3|U_k \otimes A|x_1\rangle + \langle y_4|U_k \otimes A|x_2\rangle = 0, \quad \forall A \in \mathfrak{M}_2, \quad k = 1,2, \tag{20}$$

and

$$\langle y_3|U_k \otimes A|y_1\rangle + \langle y_4|U_k \otimes A|y_2\rangle = \langle x_3|U_k \otimes A|x_1\rangle + \langle x_4|U_k \otimes A|x_2\rangle, \quad \forall A \in \mathfrak{M}_2, \quad k = 1,2. \tag{21}$$

By the symmetry of condition (15) with respect to φ and ψ, relations (18) and (20) imply, respectively,

$$\langle x_2|A \otimes V_k|y_1\rangle + \langle x_4|A \otimes V_k|y_3\rangle = 0, \quad \forall A \in \mathfrak{M}_2, \quad k = 1,2, \tag{22}$$

and

$$\langle x_3|U_k \otimes A|y_1\rangle + \langle x_4|U_k \otimes A|y_2\rangle = 0, \quad \forall A \in \mathfrak{M}_2, \quad k = 1,2. \tag{23}$$

Finally, by taking $A_1 = A_2 = 0$ and choosing appropriate values of e_i, f_i, g_i, and h_i, $i = 1, 2$, one can obtain from (15) the following equalities:

$$\langle y_4|U_k \otimes V_l|x_1\rangle = \langle x_4|U_k \otimes V_l|y_1\rangle = 0, \qquad k, l = 1, 2, \tag{24}$$
$$\langle y_4|U_k \otimes V_l|y_1\rangle = \langle x_4|U_k \otimes V_l|x_1\rangle, \qquad k, l = 1, 2, \tag{25}$$
$$\langle y_3|U_k \otimes V_l^*|x_2\rangle = \langle x_3|U_k \otimes V_l^*|y_2\rangle = 0, \qquad k, l = 1, 2, \tag{26}$$
$$\langle y_3|U_k \otimes V_l^*|y_2\rangle = \langle x_3|U_k \otimes V_l^*|x_2\rangle, \qquad k, l = 1, 2. \tag{27}$$

Below we prove that the system (16)–(27) has no nontrivial solutions.

We will use the following lemmas.

Lemma 2. A. *Equations* (16) *and* (17) *imply that all the vectors x_i and y_i, $i = \overline{1,4}$, lie in some* 2-D *subspace of* $\mathbb{C}^4$.

B. *If $x_{i_0} = y_{i_0} = 0$ for some i_0, then equations* (16) *and* (17) *imply that all the vectors x_i and y_i, $i = \overline{1,4}$, are collinear.*

Proof. A. Consider the 4×4 matrices

$$X = [\langle x_i|x_j\rangle], \qquad Y = [\langle y_i|y_j\rangle], \qquad Z = [\langle x_i|y_j\rangle].$$

It is easy to see that (16) implies $XY = 0$, while (17) shows that $X^2 = ZZ^*$ and $Y^2 = Z^*Z$. Hence, $\operatorname{rank} X = \operatorname{rank} Y \le 2$.

Since (17) implies that the sets $\{x_i\}_{i=1}^4$ and $\{y_i\}_{i=1}^4$ have the same linear hull, the above inequality shows that the dimension of this linear hull is not greater than 2.

B. This assertion is proved similarly, since the same argumentation with 3×3 matrices X, Y, and Z implies $\operatorname{rank} X = \operatorname{rank} Y \le 1$.

Lemma 3. A. *The condition*

$$\langle z_4|U_k \otimes V_l|z_1\rangle = 0, \quad k, l = 1, 2, \tag{28}$$

holds if and only if the pair (z_1, z_4) has one of the following forms:

1. $z_1 = \begin{bmatrix}\mu_1\\ s\end{bmatrix} \otimes \begin{bmatrix}a\\ b\end{bmatrix}, \quad z_4 = \begin{bmatrix}\bar\mu_1\\ -s\end{bmatrix} \otimes \begin{bmatrix}c\\ d\end{bmatrix};$
2. $z_1 = \begin{bmatrix}a\\ b\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ s\end{bmatrix}, \quad z_4 = \begin{bmatrix}c\\ d\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ -s\end{bmatrix};$
3. $z_1 = a\begin{bmatrix}\mu_1\\ 1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ s\end{bmatrix} + b\begin{bmatrix}\mu_1\\ -1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ -s\end{bmatrix}, \quad z_4 = c\begin{bmatrix}\bar\mu_1\\ 1\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ -s\end{bmatrix} + d\begin{bmatrix}\bar\mu_1\\ -1\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ s\end{bmatrix};$
4. $z_1 = h\begin{bmatrix}\mu_1\\ s\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ t\end{bmatrix}, \quad z_4 = \begin{bmatrix}\bar\mu_1\\ -s\end{bmatrix} \otimes \begin{bmatrix}a\\ b\end{bmatrix} + \begin{bmatrix}c\\ d\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ -t\end{bmatrix};$
5. $z_1 = \begin{bmatrix}\mu_1\\ -s\end{bmatrix} \otimes \begin{bmatrix}a\\ b\end{bmatrix} + \begin{bmatrix}c\\ d\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ -t\end{bmatrix}, \quad z_4 = h\begin{bmatrix}\bar\mu_1\\ s\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ t\end{bmatrix},$

where $\mu_k = \sqrt{\gamma_k}$, $k = 1, 2$, $a, b, c, d, h \in \mathbb{C}$, $s = \pm 1$, and $t = \pm 1$.

B. *Validity of* (24) *and* (25) *for vectors x_i and y_i, $i = 1, 4$, implies*

$$\langle y_4|U_k \otimes V_l|y_1\rangle = \langle x_4|U_k \otimes V_l|x_1\rangle = 0, \quad k, l = 1, 2.$$

Lemma 4. A. *The condition*

$$\langle z_3|U_k \otimes V_l^*|z_2\rangle = 0, \quad k,l = 1,2, \tag{29}$$

holds if and only if the pair (z_2, z_3) *has one of the following forms*:

1. $z_2 = \begin{bmatrix}\mu_1\\ s\end{bmatrix} \otimes \begin{bmatrix}a\\ b\end{bmatrix}, \quad z_3 = \begin{bmatrix}\bar\mu_1\\ -s\end{bmatrix} \otimes \begin{bmatrix}c\\ d\end{bmatrix};$
2. $z_2 = \begin{bmatrix}a\\ b\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ s\end{bmatrix}, \quad z_3 = \begin{bmatrix}c\\ d\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ -s\end{bmatrix};$
3. $z_2 = a\begin{bmatrix}\mu_1\\ 1\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ s\end{bmatrix} + b\begin{bmatrix}\mu_1\\ -1\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ -s\end{bmatrix}, \quad z_3 = c\begin{bmatrix}\bar\mu_1\\ -1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ s\end{bmatrix} + d\begin{bmatrix}\bar\mu_1\\ 1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ -s\end{bmatrix};$
4. $z_2 = h\begin{bmatrix}\mu_1\\ s\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ t\end{bmatrix}, \quad z_3 = \begin{bmatrix}\bar\mu_1\\ -s\end{bmatrix} \otimes \begin{bmatrix}a\\ b\end{bmatrix} + \begin{bmatrix}c\\ d\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ -t\end{bmatrix};$
5. $z_2 = \begin{bmatrix}\mu_1\\ -s\end{bmatrix} \otimes \begin{bmatrix}a\\ b\end{bmatrix} + \begin{bmatrix}c\\ d\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\ -t\end{bmatrix}, \quad z_3 = h\begin{bmatrix}\bar\mu_1\\ s\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ t\end{bmatrix},$

where $\mu_k = \sqrt{\gamma_k}$, $k = 1,2$, $a,b,c,d,h \in \mathbb{C}$, $s = \pm 1$, *and* $t = \pm 1$.

B. *Validity of* (26) *and* (27) *for vectors* x_i *and* y_i, $i = 2,3$, *implies*

$$\langle y_3|U_k \otimes V_l^*|y_2\rangle = \langle x_3|U_k \otimes V_l^*|x_2\rangle = 0, \quad k,l = 1,2.$$

Lemmas 3 and 4 are proved in the Appendix.

Lemma 5. *Let* $|\theta_1| + |\theta_2| < \pi$. *Then* $\langle x|U_1|x\rangle \neq 0$ *and* $\langle x|V_1|x\rangle \neq 0$ *for any nonzero vector* $x \in \mathbb{C}^2$.

Proof. Since $\theta_1, \theta_2 \neq \pi$, we have $\langle x|U_1|x\rangle = |x_1|^2 + \gamma_1|x_2|^2 \neq 0$ and $\langle x|V_1|x\rangle = |x_1|^2 + \gamma_2|x_2|^2 \neq 0$ for any nonzero vector $|x\rangle = [x_1, x_2]^\top \neq 0$. △

Lemma 6. *Let* $|\theta_1| + |\theta_2| < \pi$. *Then* $\langle y|U_1 \otimes V_1|y\rangle \neq 0$ *and* $\langle y|U_1 \otimes V_1^*|y\rangle \neq 0$ *for any nonzero vector* $y \in \mathbb{C}^2 \otimes \mathbb{C}^2$.

Proof. Since $U_1 \otimes V_1 = \mathrm{diag}\{1, \gamma_2, \gamma_1, \gamma_1\gamma_2\}$, the equality $\langle y|U_1 \otimes V_1|y\rangle = 0$ for a vector $|y\rangle = [y_1, y_2, y_3, y_4]^\top$ means that

$$|y_1|^2 + |y_2|^2\gamma_2 + |y_3|^2\gamma_1 + |y_4|^2\gamma_1\gamma_2 = 0.$$

By the condition $|\theta_1| + |\theta_2| < \pi$, the numbers $0, 1, \gamma_2, \gamma_1, \gamma_1\gamma_2$ are extreme points of a convex polygon in the complex plane, so the last equality can be valid only if $y_i = 0$ for all i.

Similarly one can show that $\langle y|U_1 \otimes V_1^*|y\rangle = 0$ implies $y = 0$. △

Lemma 7. *Let* p *and* q *be complex numbers such that* $|p|^2 + |q|^2 = 1$. *If* $\{|x_i\rangle\}_{i=1}^4$ *and* $\{|y_i\rangle\}_{i=1}^4$ *satisfy the system* (16)–(27), *then* $\{|px_i - qy_i\rangle\}_{i=1}^4$ *and* $\{|\bar q x_i + \bar p y_i\rangle\}_{i=1}^4$ *also satisfy* (16)–(27).

Proof. It suffices to note that the condition

$$\langle\varphi|A|\psi\rangle = \langle\psi|A|\varphi\rangle = \langle\psi|A|\psi\rangle - \langle\varphi|A|\varphi\rangle = 0$$

is invariant under the "rotation" $|\varphi\rangle \mapsto p|\varphi\rangle - q|\psi\rangle$, $|\psi\rangle \mapsto \bar q|\varphi\rangle + \bar p|\psi\rangle$. △

Lemma 8. *If* $|\theta_1| + |\theta_2| < \pi$, *then the system* (16)–(27) *has no nontrivial solution of the form* $|x_i\rangle = \alpha_i|z\rangle$ *and* $|y_i\rangle = \beta_i|z\rangle$, $i = \overline{1,4}$.

Proof. Assume that $|x_i\rangle = \alpha_i|z\rangle$ and $|y_i\rangle = \beta_i|z\rangle$, $i = \overline{1,4}$, form a nontrivial solution of the system (16)–(27). Then (16) implies that $|\alpha\rangle = [\alpha_1, \ldots, \alpha_4]^\top$ and $|\beta\rangle = [\beta_1, \ldots, \beta_4]^\top$ are orthogonal nonzero vectors of the same norm. By Lemma 6, it follows from (24)–(27) and the second parts of Lemmas 3 and 4 that

$$\alpha_1\alpha_4 = \alpha_1\beta_4 = \beta_1\alpha_4 = \beta_1\beta_4 = \alpha_2\alpha_3 = \alpha_2\beta_3 = \beta_2\alpha_3 = \beta_2\beta_3 = 0.$$

This is possible if and only if one of the pairs (α_1, β_1) and (α_4, β_4) and one of the pairs (α_2, β_2) and (α_3, β_3) are equal to $(0, 0)$.

Assume that $\alpha_1 = \beta_1 = 0$. Then $|\alpha_4| + |\beta_4| > 0$, since otherwise $\langle\beta|\alpha\rangle \neq 0$, and by Lemma 7 we may assume that $\alpha_4 \neq 0$. By Lemma 6 it follows from (22) with $A = U_1$ and (23) with $A = V_1$ that $\alpha_4\beta_2 = \alpha_4\beta_3 = 0$, which implies $\beta_2 = \beta_3 = 0$. Hence, the condition $\langle\beta|\alpha\rangle = 0$ can be valid only if $|\beta\rangle = 0$.

In a similar way one can show that the assumption $\alpha_4 = \beta_4 = 0$ leads to a contradiction. △

Assume that the collections $\{x_i\}_1^4$ and $\{y_i\}_1^4$ form a nontrivial solution of the system (16)–(27).

If $x_i \nparallel y_i$ for some i, then (24)–(27) and the second parts of Lemmas 3 and 4 imply

$$\langle y_{5-i}|W_i|x_i\rangle = \langle x_{5-i}|W_i|y_i\rangle = \langle x_{5-i}|W_i|x_i\rangle = \langle y_{5-i}|W_i|y_i\rangle = 0,$$

where $W_1 = U_1 \otimes V_1$, $W_2 = U_1 \otimes V_1^*$, $W_3 = U_1^* \otimes V_1$, and $W_4 = U_1^* \otimes V_1^*$. Since $x_{5-i}, y_{5-i} \in \mathrm{lin}\{x_i, y_i\}$ by claim A of Lemma 2, the above equalities show that $\langle x_{5-i}|W_i|x_{5-i}\rangle = \langle y_{5-i}|W_i|y_{5-i}\rangle = 0$. Lemma 6 implies $x_{5-i} = y_{5-i} = 0$. By claim B of Lemma 2, this contradicts the assumption $x_i \nparallel y_i$.

Thus, $x_i \parallel y_i$ for all $i = \overline{1,4}$. By Lemma 8 we may assume in what follows that

$$|x_i\rangle = \alpha_i|z_i\rangle \quad \text{and} \quad |y_i\rangle = \beta_i|z_i\rangle, \quad \text{where } |z_i\rangle \text{ are noncollinear vectors.}^4 \tag{30}$$

Claim B of Lemma 2 implies $|\alpha_i| + |\beta_i| > 0$, $i = \overline{1,4}$, and equations (16) and (17) can be rewritten as follows:

$$\sum_{i=1}^{4} \bar{\beta}_i\alpha_i|z_i\rangle\langle z_i| = 0, \tag{31}$$

$$\sum_{i=1}^{4} \Big[|\beta_i|^2 - |\alpha_i|^2\Big]\,|z_i\rangle\langle z_i| = 0. \tag{32}$$

By Lemma 7 we may assume that $\beta_1 = 0$ and hence $\alpha_1 \neq 0$. There are two cases:

1. If $\beta_i\alpha_i \neq 0$ for all $i > 1$, then (31) and Lemma 9 (in the Appendix) imply $z_2 \parallel z_3 \parallel z_4$. Then it follows from (32) that

$$|\alpha_1|^2|z_1\rangle\langle z_1| + [\ldots]|z_2\rangle\langle z_2| = 0,$$

and hence $z_1 \parallel z_2 \parallel z_3 \parallel z_4$, contradicting the assumption (30).

2. If there is $k > 1$ such that $\beta_k\alpha_k = 0$, then (31) implies that either $\beta_i\alpha_i \neq 0$ and $\beta_j\alpha_j \neq 0$ or $\beta_i\alpha_i = \beta_j\alpha_j = 0$, where i and $j > i$ are complementary indices to 1 and k.

If $\beta_i\alpha_i \neq 0$ and $\beta_j\alpha_j \neq 0$, then it follows from (31) that $z_i \parallel z_j$, and (32) implies

$$|\alpha_1|^2|z_1\rangle\langle z_1| + p|z_k\rangle\langle z_k| + [\ldots]|z_i\rangle\langle z_i| = 0,$$

where p is a nonzero number (equal to either $|\alpha_k|^2$ or $-|\beta_k|^2$). Hence, $z_1 \parallel z_k$ by Lemma 9.

Thus, $z_1 \parallel z_k$ and $z_i \parallel z_j$. By Lemma 6 it follows from (24) and (26) that $k \neq 4$ and $(i, j) \neq (2, 3)$. Thus, we have only two possibilities:

(a) $k = 2$, $i = 3$, $j = 4$. In this case $z_3 \parallel z_4$ and (22) with $A = U_1$ implies

$$\bar{\alpha}_4\beta_3\langle z_4|U_1 \otimes V_1|z_3\rangle = -\bar{\alpha}_2\beta_1\langle z_2|U_1 \otimes V_1|z_1\rangle = 0 \quad (\text{since } \beta_1 = 0).$$

Hence Lemma 6 shows that $\alpha_4\beta_3 = 0$, contradicting the assumption $\alpha_3\beta_3 \neq 0$ and $\alpha_4\beta_4 \neq 0$.

[4] In the sense that among the vectors $|z_i\rangle$, $i = \overline{1,4}$, there are noncollinear pairs.

(b) $k=3$, $i=2$, $j=4$. In this case $z_2 \parallel z_4$, and (23) with $A=V_1$ implies

$$\bar{\alpha}_4\beta_2\langle z_4|U_1\otimes V_1|z_2\rangle = -\bar{\alpha}_3\beta_1\langle z_3|U_1\otimes V_1|z_1\rangle = 0 \quad (\text{since } \beta_1=0).$$

Hence, Lemma 6 shows that $\alpha_4\beta_2=0$, contradicting the assumption $\alpha_2\beta_2\neq 0$ and $\alpha_4\beta_4\neq 0$.

Thus, we have $\beta_i\alpha_i=0$ for all $i=\overline{1,4}$. Since the vectors $z_1,\ldots,z_4$ are not collinear by assumption (30), equality (32) and claim B of Lemma 2 imply that there are two nonzero α_i and two nonzero β_i. Thus, there are the following cases (up to permutation):

$$\text{(a) } |\varphi\rangle,|\psi\rangle = \begin{bmatrix}x_1\\x_2\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\y_3\\y_4\end{bmatrix}; \qquad \text{(b) } |\varphi\rangle,|\psi\rangle = \begin{bmatrix}x_1\\0\\x_3\\0\end{bmatrix},\begin{bmatrix}0\\y_2\\0\\y_4\end{bmatrix}; \qquad \text{(c) } |\varphi\rangle,|\psi\rangle = \begin{bmatrix}x_1\\0\\0\\x_4\end{bmatrix},\begin{bmatrix}0\\y_2\\y_3\\0\end{bmatrix},$$

where $x_1 \nparallel x_k$ and $y_i \nparallel y_j$ (if either $x_1\parallel x_k$ or $y_i\parallel y_j$, then (32) implies $x_1\parallel x_k\parallel y_i\parallel y_j$, contradicting assumption (30)).

First we show that case (c) is not possible. It follows from (18) with $A=U_1$ and (20) with $A=V_1$ that

$$\langle y_2|U_1\otimes V_1|x_1\rangle = \langle y_3|U_1\otimes V_1|x_1\rangle = 0.$$

Since $y_2\nparallel y_3$, claim A of Lemma 2 shows that $x_1\in \operatorname{lin}\{y_2,y_3\}$ and the above equalities imply $\langle x_1|U_1\otimes V_1|x_1\rangle=0$. By Lemma 6 this is possible only if $x_1=0$.

It is more difficult to show the incompatibility of the system (16)–(27) in cases (a) and (b). We will consider these cases simultaneously by denoting $z_2=x_2$ and $z_3=y_3$ in case (a), $z_2=y_2$ and $z_3=x_3$ in case (b), and $z_1=x_1$ and $z_4=y_4$ in both cases. The system (16)–(27) implies the following equations:

$$|x_1\rangle\langle x_1| + |x_i\rangle\langle x_i| = |y_j\rangle\langle y_j| + |y_4\rangle\langle y_4|, \tag{33}$$

where $(i,j)=(2,3)$ in case (a) and $(i,j)=(3,2)$ in case (b),

$$\langle z_3|U_k\otimes A|x_1\rangle = -\sigma_*\langle y_4|U_k\otimes A|z_2\rangle,\quad \forall A\in\mathfrak{M}_2,\quad k=1,2, \tag{34}$$

$$\langle z_2|A\otimes V_k|x_1\rangle = +\sigma_*\langle y_4|A\otimes V_k|z_3\rangle,\quad \forall A\in\mathfrak{M}_2,\quad k=1,2, \tag{35}$$

where $\sigma_*=1$ in case (a) and $\sigma_*=-1$ in case (b),

$$\langle y_4|U_k\otimes V_l|x_1\rangle = 0,\quad k,l=1,2, \tag{36}$$

$$\langle z_3|U_k\otimes V_l^*|z_2\rangle = 0,\quad k,l=1,2. \tag{37}$$

It follows from (36) and (37) that the pairs (z_1,z_4) and (z_2,z_3) must have one of the forms 1–5 given in claims A of Lemmas 3 and 4, respectively.

Assume first that both pairs (z_1,z_4) and (z_2,z_3) have forms 1 or 2. In this case z_1, z_2, z_3, and z_4 are tensor product vectors (vectors of the form $u\otimes v$). By Lemma 10 (see the Appendix), equality (33) can only be valid in the following cases 1–4:

1. $|z_i\rangle = |p\rangle\otimes|a_i\rangle$, $i=\overline{1,4}$. It follows from (34) that

$$\langle p|U_1|p\rangle\langle a_3|A|a_1\rangle = -\sigma_*\langle p|U_1|p\rangle\langle a_4|A|a_2\rangle,\quad \forall A\in\mathfrak{M}_2.$$

Since $\langle p|U_1|p\rangle\neq 0$ by Lemma 5, we have $a_1\parallel a_2$ and $a_3\parallel a_4$. In case (a) this and (33) implies $x_1\parallel x_2\parallel y_3\parallel y_4$, contradicting (30). In case (b) this means that $x_1\parallel y_2$ and $x_3\parallel y_4$. The assumption $x_1\nparallel x_3$ and (33) show that this case can only be valid if $|x_1\rangle\langle x_1| = |y_2\rangle\langle y_2|$ and $|x_3\rangle\langle x_3| = |y_4\rangle\langle y_4|$. Thus, this case is reduced to case 4 considered below.

2. $|z_i\rangle = |a_i\rangle \otimes |p\rangle$, $i = \overline{1,4}$. Similarly to case 1, this case is reduced to case 4 by using (35) instead of (34).

3. $|x_1\rangle\langle x_1| = |y_4\rangle\langle y_4|$ and $|z_2\rangle\langle z_2| = |z_3\rangle\langle z_3|$. It follows from (36), (37), and Lemma 6 that this is impossible.

4. $|x_1\rangle\langle x_1| = |y_i\rangle\langle y_i|$ and $|x_{5-i}\rangle\langle x_{5-i}| = |y_4\rangle\langle y_4|$, where $i = 3$ in case (a) and $i = 2$ in case (b).

If $i = 3$, then $y_3 = \alpha x_1$, $y_4 = \beta x_2$, $|\alpha| = |\beta| = 1$, and (34) with $\sigma_* = 1$ implies

$$\bar{\alpha}\langle x_1|U_1 \otimes A|x_1\rangle = -\bar{\beta}\langle x_2|U_1 \otimes A|x_2\rangle, \quad \forall A \in \mathfrak{M}_2. \tag{38}$$

Since x_1 and x_2 are product vectors, it follows from this relation and Lemma 5 that

$$x_1 = a \otimes p \qquad \text{and} \qquad x_2 = b \otimes p$$

for some nonzero vectors a, b, and p. Hence, (36), (37), and Lemma 5 imply

$$\langle b|U_k|a\rangle = \langle b|U_k^*|a\rangle = 0, \quad k = 1, 2.$$

If $\gamma_1 \neq 1$ (i.e., $\theta_1 \neq 0$), then this cannot be valid for nonzero vectors a and b. If $\gamma_1 = 1$, then (38) shows that $\bar{\alpha}\|a\|^2 = -\bar{\beta}\|b\|^2$, while (35) with $\sigma_* = 1$ and Lemma 5 imply $\bar{\beta}\alpha = 1$, i.e., $\alpha = \beta$.

Similarly, if $i = 2$, then by using Lemma 5 one can obtain from (35) that

$$x_1 \parallel y_2 \parallel p \otimes a \qquad \text{and} \qquad x_3 \parallel y_4 \parallel p \otimes b$$

for some nonzero vectors a, b, and p. Hence, (36), (37), and Lemma 5 imply

$$\langle b|V_k|a\rangle = \langle b|V_k^*|a\rangle = 0, \quad k = 1, 2,$$

which cannot be valid for nonzero vectors a and b (since the assumption $\theta_2 \neq 0$ implies $\gamma_2 \neq \bar{\gamma}_2$).

Assume now that the pair (x_1, y_4) have form 3 in Lemma 3, i.e.,

$$x_1 = a\begin{bmatrix}\mu_1\\1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\s\end{bmatrix} + b\begin{bmatrix}\mu_1\\-1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\-s\end{bmatrix}, \qquad y_4 = c\begin{bmatrix}\bar{\mu}_1\\1\end{bmatrix} \otimes \begin{bmatrix}\bar{\mu}_2\\-s\end{bmatrix} + d\begin{bmatrix}\bar{\mu}_1\\-1\end{bmatrix} \otimes \begin{bmatrix}\bar{\mu}_2\\s\end{bmatrix},$$

where $s = \pm 1$; let us show the incompatibility of the system (33)–(37) if the pair (z_2, z_3) has forms 1–3 in Lemma 4. We will do this by reducing to the case of tensor product vectors x_1, z_2, z_3, and y_4 considered above.

1. The pair (z_2, z_3) has form 1, i.e.,

$$z_2 = \begin{bmatrix}\mu_1\\t\end{bmatrix} \otimes \begin{bmatrix}p\\q\end{bmatrix}, \qquad z_3 = \begin{bmatrix}\bar{\mu}_1\\-t\end{bmatrix} \otimes \begin{bmatrix}x\\y\end{bmatrix}, \qquad t = \pm 1, \qquad |p| + |q| \neq 0, \qquad |x| + |y| \neq 0.$$

By substituting the expressions for x_1, z_2, z_3, and y_4 into (34) and noting that

$$\left\langle \begin{matrix}\bar{\mu}_1\\s\end{matrix} \middle| U_k \middle| \begin{matrix}\mu_1\\-s\end{matrix} \right\rangle = 0, \quad s = \pm 1, \quad k = 1, 2, \tag{39}$$

we obtain

$$b\left\langle \begin{matrix}\bar{\mu}_1\\-1\end{matrix} \middle| U_k \middle| \begin{matrix}\mu_1\\-1\end{matrix} \right\rangle \left\langle \begin{matrix}x\\y\end{matrix} \middle| A \middle| \begin{matrix}\mu_2\\-s\end{matrix} \right\rangle = -\sigma_*\bar{c}\left\langle \begin{matrix}\bar{\mu}_1\\1\end{matrix} \middle| U_k \middle| \begin{matrix}\mu_1\\1\end{matrix} \right\rangle \left\langle \begin{matrix}\bar{\mu}_2\\-s\end{matrix} \middle| A \middle| \begin{matrix}p\\q\end{matrix} \right\rangle \quad \text{if} \quad t = 1,$$

and

$$a\left\langle \begin{matrix}\bar{\mu}_1\\1\end{matrix} \middle| U_k \middle| \begin{matrix}\mu_1\\1\end{matrix} \right\rangle \left\langle \begin{matrix}x\\y\end{matrix} \middle| A \middle| \begin{matrix}\mu_2\\s\end{matrix} \right\rangle = -\sigma_*\bar{d}\left\langle \begin{matrix}\bar{\mu}_1\\-1\end{matrix} \middle| U_k \middle| \begin{matrix}\mu_1\\-1\end{matrix} \right\rangle \left\langle \begin{matrix}\bar{\mu}_2\\s\end{matrix} \middle| A \middle| \begin{matrix}p\\q\end{matrix} \right\rangle \quad \text{if} \quad t = -1.$$

The validity of this equality for all $A \in \mathfrak{M}_2$ implies

$$b\lambda_k^- \left|\begin{matrix}\mu_2\\ -s\end{matrix}\right\rangle \left\langle\begin{matrix}x\\ y\end{matrix}\right| = -\sigma_* \bar{c}\lambda_k^+ \left|\begin{matrix}p\\ q\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ -s\end{matrix}\right| \quad \text{if} \quad t = 1,$$

and

$$a\lambda_k^+ \left|\begin{matrix}\mu_2\\ s\end{matrix}\right\rangle \left\langle\begin{matrix}x\\ y\end{matrix}\right| = -\sigma_* \bar{d}\lambda_k^- \left|\begin{matrix}p\\ q\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ s\end{matrix}\right| \quad \text{if} \quad t = -1,$$

where $\lambda_1^{\pm} = \left\langle\begin{matrix}\bar{\mu}_1\\ \pm 1\end{matrix}\middle|U_1\middle|\begin{matrix}\mu_1\\ \pm 1\end{matrix}\right\rangle = 2\mu_1^2$ and $\lambda_2^{\pm} = \left\langle\begin{matrix}\bar{\mu}_1\\ \pm 1\end{matrix}\middle|U_2\middle|\begin{matrix}\mu_1\\ \pm 1\end{matrix}\right\rangle = \pm 2\mu_1$. Since $\lambda_1^+ = \lambda_1^- \neq 0$ and $\lambda_2^+ = -\lambda_2^- \neq 0$, the validity of the above equalities for $k = 1, 2$ implies $b = c = 0$ if $t = 1$ and $a = d = 0$ if $t = -1$. Hence, x_1, z_2, z_3, and y_4 are product vectors.

2. The pair (z_2, z_3) has form 2, i.e.,

$$z_2 = \begin{bmatrix}p\\ q\end{bmatrix} \otimes \begin{bmatrix}\bar{\mu}_2\\ t\end{bmatrix}, \qquad z_3 = \begin{bmatrix}x\\ y\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ -t\end{bmatrix}, \qquad t = \pm 1, \qquad |p| + |q| \neq 0, \qquad |x| + |y| \neq 0.$$

By substituting the expressions for x_1, z_2, z_3, and y_4 into (35) and noting that

$$\left\langle\begin{matrix}\bar{\mu}_2\\ t\end{matrix}\middle|V_k\middle|\begin{matrix}\mu_2\\ -t\end{matrix}\right\rangle = 0, \quad t = \pm 1, \quad k = 1, 2,$$

we obtain

$$a\left\langle\begin{matrix}p\\ q\end{matrix}\middle|A\middle|\begin{matrix}\mu_1\\ 1\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ t\end{matrix}\middle|V_k\middle|\begin{matrix}\mu_2\\ t\end{matrix}\right\rangle = \sigma_* \bar{c}\left\langle\begin{matrix}\bar{\mu}_1\\ 1\end{matrix}\middle|A\middle|\begin{matrix}x\\ y\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ -t\end{matrix}\middle|V_k\middle|\begin{matrix}\mu_2\\ -t\end{matrix}\right\rangle \quad \text{if} \quad t = s,$$

and

$$b\left\langle\begin{matrix}p\\ q\end{matrix}\middle|A\middle|\begin{matrix}\mu_1\\ -1\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ t\end{matrix}\middle|V_k\middle|\begin{matrix}\mu_2\\ t\end{matrix}\right\rangle = \sigma_* \bar{d}\left\langle\begin{matrix}\bar{\mu}_1\\ -1\end{matrix}\middle|A\middle|\begin{matrix}x\\ y\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ -t\end{matrix}\middle|V_k\middle|\begin{matrix}\mu_2\\ -t\end{matrix}\right\rangle \quad \text{if} \quad t = -s.$$

The validity of this equality for all $A \in \mathfrak{M}_2$ implies

$$a\nu_k^t \left|\begin{matrix}\mu_1\\ 1\end{matrix}\right\rangle \left\langle\begin{matrix}p\\ q\end{matrix}\right| = \sigma_* \bar{c}\nu_k^{-t} \left|\begin{matrix}x\\ y\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_1\\ 1\end{matrix}\right| \quad \text{if} \quad t = s,$$

and

$$b\nu_k^t \left|\begin{matrix}\mu_1\\ -1\end{matrix}\right\rangle \left\langle\begin{matrix}p\\ q\end{matrix}\right| = \sigma_* \bar{d}\nu_k^{-t} \left|\begin{matrix}x\\ y\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_1\\ -1\end{matrix}\right| \quad \text{if} \quad t = -s,$$

where $\nu_1^t = \left\langle\begin{matrix}\bar{\mu}_2\\ t\end{matrix}\middle|V_1\middle|\begin{matrix}\mu_2\\ t\end{matrix}\right\rangle = 2\mu_2^2$ and $\nu_2^t = \left\langle\begin{matrix}\bar{\mu}_2\\ t\end{matrix}\middle|V_2\middle|\begin{matrix}\mu_2\\ t\end{matrix}\right\rangle = 2t\mu_2$. Since $\nu_1^t = \nu_1^{-t} \neq 0$ and $\nu_2^t = -\nu_2^{-t} \neq 0$, the validity of the above equalities for $k = 1, 2$ implies $a = c = 0$ if $t = s$ and $b = d = 0$ if $t = -s$. Hence, x_1, z_2, z_3, and y_4 are product vectors.

3. The pair (z_2, z_3) has form 3, i.e.,

$$z_2 = p\begin{bmatrix}\mu_1\\ 1\end{bmatrix} \otimes \begin{bmatrix}\bar{\mu}_2\\ t\end{bmatrix} + q\begin{bmatrix}\mu_1\\ -1\end{bmatrix} \otimes \begin{bmatrix}\bar{\mu}_2\\ -t\end{bmatrix}, \qquad z_3 = x\begin{bmatrix}\bar{\mu}_1\\ -1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ t\end{bmatrix} + y\begin{bmatrix}\bar{\mu}_1\\ 1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ -t\end{bmatrix},$$

where $t = \pm 1$. If we substitute the expressions for x_1, z_2, z_3, and y_4 into (34) (by using (39)), then the left- and right-hand sides of this equality will be equal, respectively, to

$$\bar{x}b\left\langle\begin{matrix}\bar{\mu}_1\\ -1\end{matrix}\middle|U_k\middle|\begin{matrix}\mu_1\\ -1\end{matrix}\right\rangle \left\langle\begin{matrix}\mu_2\\ t\end{matrix}\middle|A\middle|\begin{matrix}\mu_2\\ -s\end{matrix}\right\rangle + \bar{y}a\left\langle\begin{matrix}\bar{\mu}_1\\ 1\end{matrix}\middle|U_k\middle|\begin{matrix}\mu_1\\ 1\end{matrix}\right\rangle \left\langle\begin{matrix}\mu_2\\ -t\end{matrix}\middle|A\middle|\begin{matrix}\mu_2\\ s\end{matrix}\right\rangle$$

and

$$-\sigma_* \bar{c}p\left\langle\begin{matrix}\bar{\mu}_1\\ 1\end{matrix}\middle|U_k\middle|\begin{matrix}\mu_1\\ 1\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ -s\end{matrix}\middle|A\middle|\begin{matrix}\bar{\mu}_2\\ t\end{matrix}\right\rangle - \sigma_* \bar{d}q\left\langle\begin{matrix}\bar{\mu}_1\\ -1\end{matrix}\middle|U_k\middle|\begin{matrix}\mu_1\\ -1\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ s\end{matrix}\middle|A\middle|\begin{matrix}\bar{\mu}_2\\ -t\end{matrix}\right\rangle.$$

Thus, the validity of this equality for all $A \in \mathfrak{M}_2$ implies

$$\left[\bar{y}a \begin{vmatrix}\mu_2\\ s\end{matrix}\right\rangle \left\langle\begin{matrix}\mu_2\\ -t\end{vmatrix} + \sigma_* \bar{c}p \begin{vmatrix}\bar{\mu}_2\\ t\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ -s\end{vmatrix}\right] = \varsigma_k \left[\sigma_* \bar{d}q \begin{vmatrix}\bar{\mu}_2\\ -t\end{matrix}\right\rangle \left\langle\begin{matrix}\bar{\mu}_2\\ s\end{vmatrix} + \bar{x}b \begin{vmatrix}\mu_2\\ -s\end{matrix}\right\rangle \left\langle\begin{matrix}\mu_2\\ t\end{vmatrix}\right],$$

where $\varsigma_k \doteq -\lambda_k^-/\lambda_k^+ = (-1)^k$. This equality can be valid for $k = 1, 2$ only if the operators in the squared brackets are equal to zero. Since $\mu_2 \neq \pm\bar{\mu}_2$ by the assumption $\theta_2 \neq 0$ and the condition $\theta_2 \neq \pi$, we obtain $ya = cp = dq = xb = 0$. This means that x_1, z_2, z_3, and y_4 are product vectors.

Similar argumentation shows the incompatibility of the system (33)–(37) (by reducing to the case of tensor product vectors) if the pair (z_2, z_3) has form 3 and the pair (x_1, y_4) has form 1 or 2.

Assume finally that the pair (x_1, y_4) has form 4, i.e.,

$$x_1 = h \begin{bmatrix}\mu_1\\ s\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\ t\end{bmatrix}, \qquad y_4 = \begin{bmatrix}\bar{\mu}_1\\ -s\end{bmatrix} \otimes \begin{bmatrix}a\\ b\end{bmatrix} + \begin{bmatrix}c\\ d\end{bmatrix} \otimes \begin{bmatrix}\bar{\mu}_2\\ -t\end{bmatrix}, \qquad s, t = \pm 1,$$

and the pair (z_2, z_3) is arbitrary. We will show that (33)–(35) imply that y_4 is a product vector. Thus, in fact the pair (x_1, y_4) has form 1 or 2.

Assume that y_4 is not a product vector and denote the vectors $[\mu_1, s]^\top$ and $[\mu_2, t]^\top$ by $|s\rangle$ and $|t\rangle$. In this notation, $|x_1\rangle = h|s \otimes t\rangle$.

In case (a) it follows from (34) and Lemma 11 (see the Appendix) that $|x_2\rangle = |p \otimes t\rangle$ for some vector $|p\rangle$. Hence, the left-hand side of (33) has the form

$$|h|^2|s\rangle\langle s| \otimes |t\rangle\langle t| + |p\rangle\langle p| \otimes |t\rangle\langle t| = \left[|h|^2|s\rangle\langle s| + |p\rangle\langle p|\right] \otimes |t\rangle\langle t|,$$

and (33) implies $|y_4\rangle\langle y_4| \leq \left[|h|^2|s\rangle\langle s| + |p\rangle\langle p|\right] \otimes |t\rangle\langle t|$. This operator inequality can only be valid if y_4 is a product vector.

In case (b) it follows from (35) and Lemma 11 that $|x_3\rangle = |s \otimes q\rangle$ for some vector $|q\rangle$. Hence, the left-hand side of (33) has the form

$$|h|^2|s\rangle\langle s| \otimes |t\rangle\langle t| + |s\rangle\langle s| \otimes |q\rangle\langle q| = |s\rangle\langle s| \otimes \left[|h|^2|t\rangle\langle t| + |q\rangle\langle q|\right],$$

and similarly to case (a) we conclude that y_4 is a product vector.

By using the same argumentation exploiting (33)–(35) and Lemma 11, one can show that neither (x_1, y_4) nor (z_2, z_3) can be of form 4 or 5 (different from forms 1 and 2).

Thus, we have shown that the system (16)–(27) has no nontrivial solutions. This completes the proof of assertion C. △

APPENDIX

Proofs of Lemmas 3 and 4

Proof of Lemma 3. A. Let $\langle z_4| = [a, b, c, d]$ and

$$W = \begin{bmatrix} a & \gamma_2 b & \gamma_1 c & \gamma_1\gamma_2 d\\ b & a & \gamma_1 d & \gamma_1 c\\ c & \gamma_2 d & a & \gamma_2 b\\ d & c & b & a \end{bmatrix}, \qquad S = \begin{bmatrix} \mu_1\mu_2 & \mu_1\mu_2 & \mu_1\mu_2 & \mu_1\mu_2\\ \mu_1 & -\mu_1 & \mu_1 & -\mu_1\\ \mu_2 & \mu_2 & -\mu_2 & -\mu_2\\ +1 & -1 & -1 & +1 \end{bmatrix},$$

where $\mu_k = \sqrt{\gamma_k}$, $k = 1, 2$. By identifying $A \otimes B$ with the matrix $\|a_{ij}B\|$ one can write the equalities $\langle z_4|U_k \otimes V_l|z_1\rangle = 0$, $k, l = 1, 2$, as the system of linear equations

$$W|z_1\rangle = 0. \tag{40}$$

It is easy to see that $S^{-1}WS = \operatorname{diag}\{p_1, p_2, p_3, p_4\}$, where

$$\begin{aligned} p_1 &= a + \mu_2 b + \mu_1 c + \mu_1\mu_2 d, \qquad & p_2 &= a - \mu_2 b + \mu_1 c - \mu_1\mu_2 d, \\ p_3 &= a + \mu_2 b - \mu_1 c - \mu_1\mu_2 d, \qquad & p_4 &= a - \mu_2 b - \mu_1 c + \mu_1\mu_2 d. \end{aligned} \tag{41}$$

Thus, system (40) is equivalent to the system $p_k u_k = 0$, $k = \overline{1,4}$, where $[u_1, u_2, u_3, u_4]^\top = S^{-1}|z_1\rangle$. Hence, this system has nontrivial solutions if and only if $p_1p_2p_3p_4 = 0$ and

$$\{p_k = 0\} \iff \{W|q_k\rangle = 0\},$$

where $|q_k\rangle$ is the kth column of the matrix S.

Thus, by choosing some of the variables $p_1, \ldots, p_4$ equal to zero we obtain all pairs (z_1, z_4) such that $\langle z_4|U_k \otimes V_l|z_1\rangle = 0$, $k, l = 1, 2$. We have

(a) $C_4^2 = 6$ variants to chose $p_k = p_l = 0$ and $p_i \neq 0$, $i \neq k, l$;
(b) $C_4^1 = 4$ variants to chose $p_k = 0$ and $p_i \neq 0$, $i \neq k$;
(c) $C_4^3 = 4$ variants to chose $p_k = p_l = p_j = 0$ and $p_i \neq 0$, $i \neq k, l, j$

(the case $p_1 = p_2 = p_3 = p_4 = 0$ means that $a = b = c = d = 0$, i.e., gives a trivial solution only).

By identifying the vectors $x \otimes y$ and $[x_1y, x_2y]^\top$ it is easy to see that

$$|q_1\rangle = \begin{bmatrix}\mu_1\\1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\1\end{bmatrix}, \qquad |q_2\rangle = \begin{bmatrix}\mu_1\\1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\-1\end{bmatrix},$$

$$|q_3\rangle = \begin{bmatrix}\mu_1\\-1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\1\end{bmatrix}, \qquad |q_4\rangle = \begin{bmatrix}\mu_1\\-1\end{bmatrix} \otimes \begin{bmatrix}\mu_2\\-1\end{bmatrix}$$

and that

$$p_1 = 0 \iff |z_4\rangle = \begin{bmatrix}c_1\\c_2\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\-1\end{bmatrix} + \begin{bmatrix}\bar\mu_1\\-1\end{bmatrix} \otimes \begin{bmatrix}c_3\\c_4\end{bmatrix}, \quad c_1, \ldots, c_4 \in \mathbb{C},$$

$$p_2 = 0 \iff |z_4\rangle = \begin{bmatrix}c_1\\c_2\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\1\end{bmatrix} + \begin{bmatrix}\bar\mu_1\\-1\end{bmatrix} \otimes \begin{bmatrix}c_3\\c_4\end{bmatrix}, \quad c_1, \ldots, c_4 \in \mathbb{C},$$

$$p_3 = 0 \iff |z_4\rangle = \begin{bmatrix}c_1\\c_2\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\-1\end{bmatrix} + \begin{bmatrix}\bar\mu_1\\1\end{bmatrix} \otimes \begin{bmatrix}c_3\\c_4\end{bmatrix}, \quad c_1, \ldots, c_4 \in \mathbb{C},$$

$$p_4 = 0 \iff |z_4\rangle = \begin{bmatrix}c_1\\c_2\end{bmatrix} \otimes \begin{bmatrix}\bar\mu_2\\1\end{bmatrix} + \begin{bmatrix}\bar\mu_1\\1\end{bmatrix} \otimes \begin{bmatrix}c_3\\c_4\end{bmatrix}, \quad c_1, \ldots, c_4 \in \mathbb{C}.$$

Hence, the above six possibilities in (a) correspond to forms 1–3 in Lemma 3 (for example, the choice $p_1 = p_2 = 0$ and $p_3, p_4 \neq 0$ corresponds to form 1 with $s = 1$), while the four possibilities in (b) and (c) correspond, respectively, to forms 4 and 5.

B. Denote the above matrix W with $z_4 = x_4$ and $z_4 = y_4$, respectively, by W_x and W_y. Then the equalities in (24) and (25) can be rewritten as the system

$$W_x|y_1\rangle = W_y|x_1\rangle = 0, \qquad W_x|x_1\rangle = W_y|y_1\rangle = |c\rangle, \quad |c\rangle \in \mathbb{C}^4. \tag{42}$$

Since $S^{-1}W_xS = \operatorname{diag}\{p_1^x, p_2^x, p_3^x, p_4^x\}$ and $S^{-1}W_yS = \operatorname{diag}\{p_1^y, p_2^y, p_3^y, p_4^y\}$, where $p_1^x, p_2^x, p_3^x, p_4^x$ and $p_1^y, p_2^y, p_3^y, p_4^y$ are defined in (41) with $z_4 = x_4$ and $z_4 = y_4$, respectively, system (42) is equivalent to

$$p_k^x v_k = p_k^y u_k = 0, \qquad p_k^x u_k = p_k^y v_k = \widetilde{c}_k, \quad k = \overline{1,4}, \tag{43}$$

where $[u_1, u_2, u_3, u_4]^\top = S^{-1}|x_1\rangle$, $[v_1, v_2, v_3, v_4]^\top = S^{-1}|y_1\rangle$ and $[\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3, \widetilde{c}_4]^\top = S^{-1}|c\rangle$. System (43) has a solution only if $\widetilde{c}_k = 0$ for all k. Indeed, if $p_k^y \neq 0$ for some k, then the first equality in (43) implies $u_k = 0$ and the second equality in (43) shows that $\widetilde{c}_k = 0$. Hence, $|c\rangle = S|\widetilde{c}\rangle = 0$. △

Lemma 4 follows from Lemma 3 with γ_2 replaced by $\bar\gamma_2$.

Auxiliary Lemmas

Lemma 9. *If* $|a\rangle\langle x| + |b\rangle\langle y| + |c\rangle\langle z| = 0$, *then either* $a \parallel b \parallel c$ *or* $x \parallel y \parallel z$.

Proof. We may assume that all the vectors are nonzero (since otherwise the assertion is trivial).

Let $p \perp x$. Then $\langle y|p\rangle|b\rangle + \langle z|p\rangle|c\rangle = 0$, and hence either $b \parallel c$ or $\langle y|p\rangle = \langle z|p\rangle = 0$.

If $b \parallel c$, then we have $|a\rangle\langle x| = -|b\rangle\langle y + \lambda z|$, $\lambda \in \mathbb{C}$, and hence $a \parallel b \parallel c$.

If $\langle y|p\rangle = \langle z|p\rangle = 0$, then $x \parallel y \parallel z$, since the vector p is arbitrary. △

Lemma 10. *The equality*

$$X_1 \otimes Y_1 + X_2 \otimes Y_2 = X_3 \otimes Y_3 + X_4 \otimes Y_4, \tag{44}$$

where $X_i = |x_i\rangle\langle x_i|$, $Y_i = |y_i\rangle\langle y_i|$, $i = \overline{1,4}$, *can be valid in the following cases only*:

1. $x_i \parallel x_j$ *for all* i *and* j, *and* $Y_1\|x_1\|^2 + Y_2\|x_2\|^2 = Y_3\|x_3\|^2 + Y_4\|x_4\|^2$;
2. $y_i \parallel y_j$ *for all* i *and* j, *and* $X_1\|y_1\|^2 + X_2\|y_2\|^2 = X_3\|y_3\|^2 + X_4\|y_4\|^2$;
3. $X_1 \otimes Y_1 = X_4 \otimes Y_4$ *and* $X_2 \otimes Y_2 = X_3 \otimes Y_3$;
4. $X_1 \otimes Y_1 = X_3 \otimes Y_3$ *and* $X_2 \otimes Y_2 = X_4 \otimes Y_4$.

Proof. We may assume that all the vectors x_i and y_i are nonzero (since otherwise the assertion is trivial).

Let $p \perp x_1$. By multiplying both sides of (44) by $|p\rangle\langle p| \otimes I$, we obtain

$$|\langle x_2|p\rangle|^2 Y_2 = |\langle x_3|p\rangle|^2 Y_3 + |\langle x_4|p\rangle|^2 Y_4. \tag{45}$$

If $x_2 \parallel x_1$, then $\langle x_3|p\rangle = \langle x_4|p\rangle = 0$, and hence $x_1 \parallel x_2 \parallel x_3 \parallel x_4$, since the vector p is arbitrary; i.e., case 1 holds.

If $x_2 \nparallel x_1$, then one can choose p such that $\langle x_2|p\rangle \neq 0$. Thus, (45) implies that either $x_3 \nparallel x_1$ or $x_4 \nparallel x_1$. We have the following possibilities:

(a) If $x_i \nparallel x_1$ for $i = 2, 3, 4$, then one can choose p such that $\langle x_i|p\rangle \neq 0$, $i = 2, 3, 4$. It follows from (45) that $y_2 \parallel y_3 \parallel y_4$. Hence, (44) leads to the equality $X_1 \otimes Y_1 = [\ldots] \otimes Y_2$, which gives $y_1 \parallel y_2$. Thus, we have $y_1 \parallel y_2 \parallel y_3 \parallel y_4$; i.e., case 2 holds.

(b) If $x_i \nparallel x_1$ for $i = 2, 3$, but $x_4 \parallel x_1$, then one can choose p such that $\langle x_i|p\rangle \neq 0$, $i = 2, 3$. It follows from (45) that $y_2 \parallel y_3$. Hence, $x_4 = \alpha x_1$ and $y_3 = \beta y_2$, $\alpha, \beta \in \mathbb{C}$. It follows from (44) that

$$X_1 \otimes [Y_1 - |\alpha|^2 Y_4] = [X_3|\beta|^2 - X_2] \otimes Y_2,$$

and hence $Y_1 - |\alpha|^2 Y_4 = \lambda Y_2$, $\lambda \in \mathbb{C}$. If $\lambda \neq 0$, then Lemma 9 implies $y_1 \parallel y_2 \parallel y_3 \parallel y_4$; i.e., case 2 holds. If $\lambda = 0$, then $y_1 \parallel y_4$ and $x_2 \parallel x_3$. Thus, we have

$$X_4 \otimes Y_4 = \gamma X_1 \otimes Y_1, \qquad X_3 \otimes Y_3 = \delta X_2 \otimes Y_2, \quad \gamma, \delta \in \mathbb{C},$$

and (44) implies $(1 - \gamma) X_1 \otimes Y_1 = (\delta - 1) X_2 \otimes Y_2$. Since $x_1 \nparallel x_2$, we have $\gamma = \delta = 1$; i.e., case 3 holds.

(c) If $x_i \nparallel x_1$ for $i = 2, 4$, but $x_3 \parallel x_1$, then similar arguments (with the interchange $3 \leftrightarrow 4$) show that case 4 holds. △

Lemma 11. *Let* $U = \mathrm{diag}\{1, \gamma\}$, *and let* x *and* y *be nonzero vectors in* $\mathbb{C}^2$. *If* $\langle a|U \otimes A|x \otimes y\rangle = \langle c|U \otimes A|d\rangle$ *for all* $A \in \mathfrak{M}_2$, *then either* $|d\rangle = |z\rangle \otimes |y\rangle$ *or* $|c\rangle = |p\rangle \otimes |q\rangle$ *for some vectors* p, q, *and* z *in* $\mathbb{C}^2$.

Proof. By using the isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^2 \ni u \otimes v \longleftrightarrow [u_1 v, u_2 v]^\top \in \mathbb{C}^2 \oplus \mathbb{C}^2$, the condition of the lemma can be rewritten as follows:

$$\left\langle \begin{matrix} a_1 \\ a_2 \end{matrix} \middle| \begin{matrix} A & 0 \\ 0 & \gamma A \end{matrix} \middle| \begin{matrix} x_1 y \\ x_2 y \end{matrix} \right\rangle = \left\langle \begin{matrix} c_1 \\ c_2 \end{matrix} \middle| \begin{matrix} A & 0 \\ 0 & \gamma A \end{matrix} \middle| \begin{matrix} d_1 \\ d_2 \end{matrix} \right\rangle, \quad \forall A \in \mathfrak{M}_2,$$

where a_1 and a_2 are components of the vector a, etc. Thus, we have

$$x_1\langle a_1|A|y\rangle + x_2\gamma\langle a_2|A|y\rangle = \langle c_1|A|d_1\rangle + \gamma\langle c_2|A|d_2\rangle, \quad \forall A \in \mathfrak{M}_2,$$

which is equivalent to the equality $|y\rangle\langle \bar{x}_1 a_1 + \bar{x}_2\bar{\gamma} a_2| = |d_1\rangle\langle c_1| + \gamma|d_2\rangle\langle c_2|$. By Lemma 9 this is possible if either $d_1 \parallel d_2 \parallel y$, which means $|d\rangle = |z\rangle \otimes |y\rangle$, or $c_1 \parallel c_2$, which means $|c\rangle = |p\rangle \otimes |q\rangle$. $\triangle$

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