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INFORMATION THEORY

Mutual and Coherent Information for Infinite-Dimensional Quantum Channels[1]

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Abstract—The paper is devoted to the study of quantum mutual information and coherent information, two important characteristics of a quantum communication channel. Appropriate definitions of these quantities in the infinite-dimensional case are given, and their properties are studied in detail. Basic identities relating the quantum mutual information and coherent information of a pair of complementary channels are proved. An unexpected continuity property of the quantum mutual information and coherent information, following from the above identities, is observed. An upper bound for the coherent information is obtained.

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1. INTRODUCTION

One of achievements in quantum information theory is the discovery of a number of important entropy and information characteristics of quantum systems (see, e.g., [1, 2]). Some of them, such as the χ-capacity and quantum mutual information, have direct classical analogs; others, such as the coherent information and various entanglement measures, either do not have such analogs or they are trivial.

Until recently, the main attention in quantum information theory was paid to finite-dimensional systems, but in recent years considerable interest to infinite-dimensional systems appeared: a broad class, important for applications in quantum optics, is formed by Bosonic Gaussian systems [2, ch. 11]. Note that properties of the entropy and relative entropy were studied in great detail, including the infinite-dimensional case, in connection with quantum statistical mechanics; see, e.g., [3–5]. A study of entropy and information characteristics of quantum communication channels from the general viewpoint of operator theory in a separable Hilbert space was undertaken in [6, 7], where quantities related to the classical capacity—in the first place, the χ-capacity—were investigated. The present work is devoted to two other characteristics, namely, the quantum mutual information and coherent information. The first is closely related to the entanglement-assisted classical capacity, while the second, to the quantum capacity of a channel. One of the authors' goals was to give an appropriate definition of these quantities in the infinite-dimensional case that would not require any additional artificial assumptions. A difficulty, which has been overcome, was in uncertainties in expressions containing differences of entropies, each of which can be infinite in

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the infinite-dimensional case. Our main result is a theorem, which implies that these quantities are naturally defined and finite on the set of input states with finite entropy, where they satisfy identities (22) and (23) for complementary channels (the quantum mutual information is uniquely defined for all input states but can be infinite, still satisfying identity (22)).

In the introductory section (Section 2), a description of the corresponding quantities for a finite quantum system is given. In Section 3 we give a definition and study properties of the quantum mutual information in the infinite-dimensional case. The main identity (22) for complementary channels is proved in Section 4. Section 5 is devoted to the coherent information. In Section 6 we point out a somewhat unexpected continuity property of mutual and coherent informations, implied by identity (22).

2. FINITE-DIMENSIONAL CASE

Consider a quantum system described by a finite-dimensional Hilbert space $\mathcal{H}$, and denote by $\mathfrak{S}(\mathcal{H})$ the convex set of *quantum states*, described by density operators in $\mathcal{H}$, i.e., positive operators with unit trace: $\rho \geq 0$, $\operatorname{Tr}\rho = 1$. The *entropy* of a state ρ (von Neumann entropy) is defined by the relation[3]

$$H(\rho) = \operatorname{Tr}\eta(\rho), \quad \eta(x) = \begin{cases} -x\log x, & x > 0, \\ 0, & x = 0. \end{cases} \tag{1}$$

Let us be given three systems A, B, and E, described by spaces $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_E$, respectively, and an isometric operator $V\colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$. Then the relations

$$\Phi(\rho) = \operatorname{Tr}_E V\rho V^*, \qquad \widetilde{\Phi}(\rho) = \operatorname{Tr}_B V\rho V^*, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \tag{2}$$

where $\operatorname{Tr}_X(\cdot) \doteq \operatorname{Tr}_{\mathcal{H}_X}(\cdot)$, define completely positive trace-preserving maps, i.e., quantum channels $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ and $\widetilde{\Phi}\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$, which are called mutually *complementary* (this construction can be extended to the infinite-dimensional case without changes). The systems A and B describe, respectively, the input and output of the channel Φ, and E describes its "environment" (for details, see [1,2]). The identity operator in $\mathcal{H}_X$ and the identity transformation of the set $\mathfrak{S}(\mathcal{H}_X)$ will be denoted by I_X and Id_X, respectively.

Let $\rho = \rho_A$ be an input state in the space $\mathcal{H}_A$, and ρ_B and ρ_E be results of the action of the channels Φ and $\widetilde{\Phi}$ on the state ρ_A, respectively. The *quantum mutual information* is defined as

$$I(\rho, \Phi) = H(A) + H(B) - H(E), \tag{3}$$

where brief notations $H(A) = H(\rho_A)$, etc., are used [8]. By introducing the reference system $\mathcal{H}_R \cong \mathcal{H}_A$ and the purification vector $\psi_{AR} \in \mathcal{H}_A \otimes \mathcal{H}_R$ for the state ρ_A, the mutual information can be represented in the form

$$I(\rho, \Phi) = H(R) + H(B) - H(BR), \tag{4}$$

where $\rho_{BR} = (\Phi \otimes \mathrm{Id}_R)(|\psi_{AR}\rangle\langle\psi_{AR}|)$.

The mutual information $I(\rho, \Phi)$ have several properties, similar to properties of the Shannon information (see Proposition 1 below). In [9] (see also [2]) it was shown that

$$\max_\rho I(\rho, \Phi) = C_{\mathrm{ea}}(\Phi) \tag{5}$$

is the classical entanglement-assisted capacity of the channel Φ.

[3] In the present paper, log denotes the natural logarithm.

Introducing an analogous characteristic for the complementary channel,

$$\begin{aligned} I(\rho,\widetilde{\Phi}) &= H(A)+H(E)-H(B) \\ &= H(R)+H(E)-H(ER), \end{aligned} \tag{6}$$

we have a fundamental identity

$$I(\rho,\Phi)+I(\rho,\widetilde{\Phi})=2H(\rho). \tag{7}$$

An important component of the quantum mutual information $I(\rho,\Phi)$ is the *coherent information* (see [10])

$$I_c(\rho,\Phi)=H(B)-H(E)=H(B)-H(RB). \tag{8}$$

This notion is closely related to the quantum capacity of the channel Φ. Namely, in [11] (see also [2]) it was shown that

$$Q(\Phi)=\lim_{n\to\infty}\frac{1}{n}\max_{\rho} I_c(\rho,\Phi^{\otimes n}) \tag{9}$$

is the quantum capacity of the channel Φ. Identity (7) is equivalent to the fact that

$$I_c(\rho,\Phi)+I_c(\rho,\widetilde{\Phi})=0. \tag{10}$$

The aim of the present paper is exploring definitions and properties of analogs of the quantities $I(\rho,\Phi)$ and $I_c(\rho,\Phi)$ in an infinite-dimensional Hilbert space. In particular, it will be shown that the coherent information can naturally be defined on the set of states with finite entropy, where an analog of identity (10) holds. Results of this paper can be used for generalizing relations (5) and (9) to the case of infinite-dimensional channels.

3. MUTUAL INFORMATION

In what follows, $\mathcal{H}$ is a separable Hilbert space. Let $\mathfrak{T}(\mathcal{H})$ be the Banach space of trace class operators, so that $\mathfrak{S}(\mathcal{H})\subset\mathfrak{T}(\mathcal{H})$. Consider the natural extension of the von Neumann entropy $H(\rho)=\operatorname{Tr}\eta(\rho)$ of a quantum state $\rho\in\mathfrak{S}(\mathcal{H})$ to the cone $\mathfrak{T}_+(\mathcal{H})$ of all positive trace class operators.

Definition 1 [4]. The *entropy* of an operator $A\in\mathfrak{T}_+(\mathcal{H})$ is defined as follows:

$$H(A)=\operatorname{Tr} AH\left(\frac{A}{\operatorname{Tr} A}\right)=\operatorname{Tr}\eta(A)-\eta(\operatorname{Tr} A). \tag{11}$$

The entropy is a concave lower semicontinuous function on the cone $\mathfrak{T}_+(\mathcal{H})$, taking values in $[0,+\infty]$. Using Definition 1 and well-known properties of the von Neumann entropy (see [5]), it is easy to obtain the following relations:

$$H(\lambda A)=\lambda H(A),\quad \lambda\geq 0, \tag{12}$$

$$H(A)+H(B-A)\leq H(B)\leq H(A)+H(B-A)+\operatorname{Tr} B\, h_2\left(\frac{\operatorname{Tr} A}{\operatorname{Tr} B}\right), \tag{13}$$

where $A,B\in\mathfrak{T}_+(\mathcal{H})$, $A\leq B$, and $h_2(x)=\eta(x)+\eta(1-x)$.

We will also use the function $S(A)=\operatorname{Tr}\eta(A)$ on the cone $\mathfrak{T}_+(\mathcal{H})$, which coincides with the function $H(A)$ on the set $\mathfrak{S}(\mathcal{H})$.

Definition 2. The *relative entropy* of operators $A,B\in\mathfrak{T}_+(\mathcal{H})$ is defined as follows:

$$H(A\,\|\,B)=\begin{cases}\sum\limits_{i=1}^{+\infty}\langle e_i|\,(A\log A-A\log B+B-A)\,|e_i\rangle, & \operatorname{supp} A\subseteq\operatorname{supp} B,\\ +\infty, & \operatorname{supp} A\not\subseteq\operatorname{supp} B,\end{cases}$$

where $\{|e_i\rangle\}_{i=1}^{+\infty}$ is an orthonormal basis of eigenvectors of the operator A, and the series consists of nonnegative terms [4].

We need the following statement.

Lemma 1 [4, Lemma 4]. *Let $\{P_n\}$ be a nondecreasing sequence of projectors converging to the identity operator I in the strong operator topology, and let A and B be arbitrary positive trace class operators. Then the sequences $\{H(P_nAP_n)\}$ and $\{H(P_nAP_n \,\|\, P_nBP_n)\}$ are nondecreasing,*

$$H(A) = \lim_{n\to\infty} H(P_nAP_n), \qquad \textit{and} \qquad H(A\,\|\,B) = \lim_{n\to\infty} H(P_nAP_n \,\|\, P_nBP_n).$$

Definition 3. A *quantum channel* is a linear trace-preserving map $\Phi\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ such that

$$\Phi \otimes \mathrm{Id}_n(\mathfrak{S}(\mathcal{H}_A \otimes \ell_n^2)) \subseteq \mathfrak{S}(\mathcal{H}_B \otimes \ell_n^2),$$

for every $n = 1, 2, \ldots$ (ℓ_n^2 is the n-dimensional Hilbert space).

This condition is one of equivalent formulations of *complete positivity*. In particular, for $n = 1$ it implies [12] positivity and boundedness of the map Φ.

In what follows, we will use the fundamental *monotonicity* property of the relative entropy, established in [4]:

$$H(\Phi(A)\,\|\,\Phi(B)) \le H(A\,\|\,B) \tag{14}$$

for an arbitrary quantum channel Φ and arbitrary positive trace class operators A and B.

Definition 4. Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel, and ρ be an arbitrary quantum state in $\mathfrak{S}(\mathcal{H}_A)$ with a spectral representation $\rho = \sum\limits_{i=1}^{\infty} \lambda_i |e_i\rangle\langle e_i|$. The *mutual information* of the channel Φ at the state ρ is defined as follows:

$$I(\rho, \Phi) = H(\Phi \otimes \mathrm{Id}_R(|\varphi_\rho\rangle\langle\varphi_\rho|)\,\|\,\Phi(\rho) \otimes \rho),$$

where

$$|\varphi_\rho\rangle = \sum_{i=1}^{\infty} \sqrt{\lambda_i}|e_i\rangle \otimes |e_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R \tag{15}$$

is a purification vector[4] for the state ρ.

Note that, in the case $\dim \mathcal{H}_A < +\infty$ and $\dim \mathcal{H}_B < +\infty$, this definition is equivalent to (3) and (4), since

$$\begin{aligned} H(\Phi \otimes \mathrm{Id}_R(|\varphi_\rho\rangle\langle\varphi_\rho|)\,\|\,\Phi(\rho) \otimes \rho) &= H(\rho_{BR}\,\|\,\rho_B \otimes \rho_R) \\ &= \mathrm{Tr}(\rho_{BR}(\log(\rho_{BR}) - \log(\rho_B \otimes \rho_R))) \\ &= -H(\rho_{BR}) + H(\rho_B) + H(\rho_R) \\ &= -H(BR) + H(B) + H(R). \end{aligned}$$

Remark 1. The above definition of $I(\rho, \Phi)$ does not depend on the choice of a space $\mathcal{H}_R$ and of a purification vector φ_ρ. This can be shown by using well-known relation between different purification vectors of a given state (see [1, 2]) and properties of the relative entropy.

In the finite-dimensional case, concavity of the mutual information as a function of ρ on the set $\mathfrak{S}(\mathcal{H}_A)$ follows from concavity of the conditional entropy $H(EB) - H(E)$ [2, 8]. In the case $\dim \mathcal{H}_A = +\infty$ and $\dim \mathcal{H}_B < +\infty$, this implies concavity of the mutual information as a function of ρ on the set $\mathfrak{S}_{\mathrm{f}}(\mathcal{H}_A) = \{\rho \in \mathfrak{S}(\mathcal{H}_A) \mid \operatorname{rank} \rho < +\infty\}$.

In what follows, convergence of quantum states means convergence of the corresponding density operators to a limit operator in the trace norm, which is equivalent to the weak operator convergence [12] (see also [6, Appendix A]). Note that the entropy and relative entropy are lower semicontinuous in their arguments with respect to this convergence [3].

[4] This means that $\mathrm{Tr}_R\, |\varphi_\rho\rangle\langle\varphi_\rho| = \rho$.

Let $\mathfrak{F}(A,B)$ be the set of all quantum channels from $\mathfrak{S}(\mathcal{H}_A)$ to $\mathfrak{S}(\mathcal{H}_B)$ endowed with the *strong convergence* topology [7]. Strong convergence of a sequence $\{\Phi_n\}\subset\mathfrak{F}(A,B)$ to a channel $\Phi_0\in\mathfrak{F}(A,B)$ means that $\lim\limits_{n\to+\infty}\Phi_n(\rho)=\Phi_0(\rho)$ for any state $\rho\in\mathfrak{S}(\mathcal{H}_A)$.

The following proposition is devoted to generalization of observations made in [8] to the infinite-dimensional case.

Proposition 1. *The function $(\rho,\Phi)\mapsto I(\rho,\Phi)$ is nonnegative and lower semicontinuous on the set $\mathfrak{S}(\mathcal{H}_A)\times\mathfrak{F}(A,B)$. It has the following properties*:

1. Concavity in ρ: $I(\lambda\rho_1+(1-\lambda)\rho_2,\Phi)\geq\lambda I(\rho_1,\Phi)+(1-\lambda)I(\rho_2,\Phi)$;
2. Convexity in Φ: $I(\rho,\lambda\Phi_1+(1-\lambda)\Phi_2)\leq\lambda I(\rho,\Phi_1)+(1-\lambda)I(\rho,\Phi_2)$;
3. The 1st chain rule: *for arbitrary channels $\Phi\colon\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$ and $\Psi\colon\mathfrak{S}(\mathcal{H}_B)\to\mathfrak{S}(\mathcal{H}_C)$, the inequality $I(\rho,\Psi\circ\Phi)\leq I(\rho,\Phi)$ holds for any $\rho\in\mathfrak{S}(\mathcal{H}_A)$*;
4. The 2nd chain rule: *for arbitrary channels $\Phi\colon\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$ and $\Psi\colon\mathfrak{S}(\mathcal{H}_B)\to\mathfrak{S}(\mathcal{H}_C)$, the inequality $I(\rho,\Psi\circ\Phi)\leq I(\Phi(\rho),\Psi)$ holds for any $\rho\in\mathfrak{S}(\mathcal{H}_A)$*;
5. Subadditivity: *for arbitrary channels $\Phi\colon\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$ and $\Psi\colon\mathfrak{S}(\mathcal{H}_C)\to\mathfrak{S}(\mathcal{H}_D)$, the inequality*
$$I(\omega,\Phi\otimes\Psi)\leq I(\omega_A,\Phi)+I(\omega_C,\Psi)\tag{16}$$
holds for any $\omega\in\mathfrak{S}(\mathcal{H}_A\otimes\mathcal{H}_C)$.

Proof. Nonnegativity of the value $I(\rho,\Phi)$ follows from nonnegativity of the relative entropy. By Lemma 2 below and Remark 1, lower semicontinuity of the function $(\rho,\Phi)\mapsto I(\rho,\Phi)$ follows from lower semicontinuity of the relative entropy in both arguments.

To prove concavity of the function $\rho\mapsto I(\rho,\Phi)$, first assume that $\dim\mathcal{H}_B$ is finite. Let $\rho=\alpha\sigma_1+(1-\alpha)\sigma_2$, and let $\{P_n\}$ be an increasing sequence of finite-rank spectral projectors of the state ρ strongly converging to I_A. Let
$$\rho_n=\frac{P_n\rho P_n}{\operatorname{Tr}P_n\rho}=\frac{\alpha P_n\sigma_1P_n+(1-\alpha)P_n\sigma_2P_n}{\alpha\operatorname{Tr}P_n\sigma_1+(1-\alpha)\operatorname{Tr}P_n\sigma_2}=\frac{\mu_1^n\sigma_1^n+\mu_2^n\sigma_2^n}{\mu_1^n+\mu_2^n},$$
where
$$\mu_1^n=\alpha\operatorname{Tr}P_n\sigma_1,\qquad \sigma_1^n=\alpha\frac{P_n\sigma_1P_n}{\mu_1^n},$$
$$\mu_2^n=(1-\alpha)\operatorname{Tr}P_n\sigma_2,\qquad \sigma_2^n=(1-\alpha)\frac{P_n\sigma_2P_n}{\mu_2^n}.$$
By the concavity of the function $\rho\mapsto I(\rho,\Phi)$ on the set $\mathfrak{S}_{\mathrm{f}}(\mathcal{H}_A)$, mentioned before Proposition 1, we have
$$I(\rho_n,\Phi)\geq\frac{\mu_1^n}{\mu_1^n+\mu_2^n}I(\sigma_1^n,\Phi)+\frac{\mu_2^n}{\mu_1^n+\mu_2^n}I(\sigma_2^n,\Phi).$$
Lemma 3 below implies that $\lim\limits_{n\to\infty}I(\rho_n,\Phi)=I(\rho,\Phi)$. By using lower semicontinuity of the function $\rho\mapsto I(\rho,\Phi)$, we obtain
$$\begin{aligned}I(\rho,\Phi)&\geq\liminf_{n\to\infty}\frac{\mu_1^n}{\mu_1^n+\mu_2^n}I(\sigma_1^n,\Phi)+\liminf_{n\to\infty}\frac{\mu_2^n}{\mu_1^n+\mu_2^n}I(\sigma_2^n,\Phi)\\&\geq\alpha I(\sigma_1,\Phi)+(1-\alpha)I(\sigma_2,\Phi).\end{aligned}$$

Let Φ be an arbitrary quantum channel. Consider the sequence of channels $\Phi_n=\Pi_n\circ\Phi$ with a finite-dimensional output, where
$$\Pi_n(\rho)=P_n\rho P_n+[\operatorname{Tr}((I-P_n)\rho)]|\psi\rangle\langle\psi|$$

is a quantum channel from $\mathfrak{S}(\mathcal{H}_B)$ to itself for each n, $\{P_n\}$ is an increasing sequence of finite-rank projectors strongly converging to I_B, and $|\psi\rangle\langle\psi|$ is a fixed pure state in $\mathfrak{S}(\mathcal{H}_B)$. Then for each n the function $\rho\mapsto I(\rho,\Phi_n)$ is concave by the above observation. Since

$$I(\rho,\Phi_n)\le I(\rho,\Phi)\ \forall n \qquad \text{and} \qquad \liminf_{n\to\infty} I(\rho,\Phi_n)\ge I(\rho,\Phi)$$

by monotonicity of the relative entropy and lower semicontinuity of the function $\Phi\mapsto I(\rho,\Phi)$, we have

$$I(\rho,\Phi)=\sup_n I(\rho,\Phi_n).$$

Hence, the function $\rho\mapsto I(\rho,\Phi)$ is concave as a pointwise limit of a sequence of concave functions.

Convexity of the function $\Phi\mapsto I(\rho,\Phi)$ follows from joint convexity of the relative entropy in its arguments [3].

The 1st chain rule immediately follows from Definition 4 and monotonicity of the relative entropy.

The 2nd chain rule is also proved by using monotonicity of the relative entropy in the following way.

Let $|\varphi\rangle\langle\varphi|$ be a purification of the state $\rho\in\mathfrak{S}(\mathcal{H}_A)$ in the space $\mathcal{H}_A\otimes\mathcal{H}_R$; then $|\psi\rangle\langle\psi| = V\otimes I_R|\varphi\rangle\langle\varphi|V^*\otimes I_R$ is a purification of the state $\Phi(\rho)\in\mathfrak{S}(\mathcal{H}_B)$ in the space $\mathcal{H}_B\otimes\mathcal{H}_E\otimes\mathcal{H}_R$ (here V is the isometry from representation (2) of the channel Φ). Hence,

$$I(\Phi(\rho),\Psi)=H\left(\Psi\otimes\mathrm{Id}_{ER}(|\psi\rangle\langle\psi|)\,\|\,\Psi(\mathrm{Tr}_{ER}|\psi\rangle\langle\psi|)\otimes\mathrm{Tr}_B|\psi\rangle\langle\psi|\right).$$

A direct verification shows that taking the partial trace over the space $\mathcal{H}_E$ with respect to each argument of the relative entropy in the above expression transforms the right-hand side of this expression into

$$H\left((\Psi\circ\Phi)\otimes\mathrm{Id}_R(|\varphi\rangle\langle\varphi|)\,\|\,(\Psi\circ\Phi)(\mathrm{Tr}_R|\varphi\rangle\langle\varphi|)\otimes\mathrm{Tr}_A|\varphi\rangle\langle\varphi|\right)=I(\rho,\Psi\circ\Phi).$$

The subadditivity property of the mutual information is derived from the corresponding property of this characteristic for finite-dimensional channels [2, 8].

Let $\{Q_n^X\}$ be an increasing sequence of finite-rank projectors in the space $\mathcal{H}_X$ strongly converging to the operator I_X, where $X=B,D$. The sequence of channels

$$\Pi_n^X(\rho)=Q_n^X\rho Q_n^X+(\mathrm{Tr}(I_X-Q_n^X)\rho)\tau_X$$

from $\mathfrak{S}(\mathcal{H}_X)$ to itself, where τ_X is an arbitrary pure state in the space $\mathcal{H}_X$, strongly converges to the channel Id_X.

Let ω be an arbitrary state in $\mathfrak{S}(\mathcal{H}_A\otimes\mathcal{H}_C)$. Let $\{P_n^X\}$ be an increasing sequence of finite-rank spectral projectors of the state ω_X strongly converging to the operator I_X, where $X=A,C$.

Consider the sequence of states

$$\omega^n=\big(\mathrm{Tr}\big((P_n^A\otimes P_n^C)\cdot\omega\big)\big)^{-1}(P_n^A\otimes P_n^C)\cdot\omega\cdot(P_n^A\otimes P_n^C),$$

which converges to the state ω.

A direct verification shows that

$$\lambda_n\omega_X^n\le\omega_X,\quad X=A,C,\quad \text{where}\quad \lambda_n=\mathrm{Tr}\big((P_n^A\otimes P_n^C)\cdot\omega\big).$$

By Lemma 4 below we have

$$\lim_{n\to+\infty} I(\omega_A^n,\Pi_n^B\circ\Phi)=I(\omega_A,\Phi) \qquad \text{and} \qquad \lim_{n\to+\infty} I(\omega_C^n,\Pi_n^D\circ\Psi)=I(\omega_C,\Psi). \tag{17}$$

Subadditivity of the mutual information for finite-dimensional channels implies

$$I(\omega^n,(\Pi_n^B\circ\Phi)\otimes(\Pi_n^D\circ\Psi))\le I(\omega_A^n,\Pi_n^B\circ\Phi)+I(\omega_C^n,\Pi_n^D\circ\Psi).$$

By (17) and lower semicontinuity of the mutual information as a function of a pair (state, channel), passing to the limit in this inequality implies (16). △

In the proof of Proposition 1, the following lemmas were used.

Lemma 2. *Let $\mathcal{H}$ be a separable Hilbert space. For an arbitrary sequence $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H})$ converging to a state ρ_0, there exists a corresponding purification sequence $\{\widehat{\rho}_n\} \subset \mathfrak{S}(\mathcal{H}\otimes\mathcal{H})$ converging to a purification $\widehat{\rho}_0$ of the state ρ_0.*

Proof. The assertion of the lemma follows from the inequality (see [1, 2])

$$\beta(\rho,\sigma)^2 \le \|\rho-\sigma\|_1$$

for the Bures distance $\beta(\rho,\sigma) = \inf\|\varphi_\rho - \varphi_\sigma\|$, where the infimum is over all purification vectors φ_ρ and φ_σ of the states ρ and σ. △

Lemma 3. *Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel such that $\dim\mathcal{H}_B < +\infty$, and ρ_0 be a state in $\mathfrak{S}(\mathcal{H}_A)$ with a spectral representation $\rho_0 = \sum\limits_{i=1}^{\infty}\lambda_i|e_i\rangle\langle e_i|$. Let*

$$\rho_n = \frac{1}{\mu_n}\sum_{i=1}^{n}\lambda_i|e_i\rangle\langle e_i|, \quad \text{where} \quad \mu_n = \sum_{i=1}^{n}\lambda_i, \tag{18}$$

for every n. Then $\lim\limits_{n\to\infty} I(\rho_n,\Phi) = I(\rho_0,\Phi)$.

Proof. Let $P_n = \sum\limits_{i=1}^{n}|e_i\rangle\langle e_i|$, $n = 1, 2, \ldots$. Since $\dim\mathcal{H}_B < \infty$, the value

$$\begin{aligned} I_n &= H(\Phi\otimes \mathrm{Id}_R(\widehat{\rho}_n)\,\|\,\Phi(\rho_0)\otimes\rho_n) \\ &= \mu_n^{-1} H(Q_n\,(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_0))\,Q_n\,\|\,Q_n\,(\Phi(\rho_0)\otimes\rho_0)\,Q_n), \end{aligned}$$

where

$$\widehat{\rho}_0 = \sum_{i,j=1}^{+\infty}\sqrt{\lambda_i\lambda_j}|e_i\rangle\langle e_j|\otimes|e_i\rangle\langle e_j|, \qquad \widehat{\rho}_n = \mu_n^{-1}\sum_{i,j=1}^{n}\sqrt{\lambda_i\lambda_j}|e_i\rangle\langle e_j|\otimes|e_i\rangle\langle e_j|$$

and $Q_n = I_B\otimes P_n$, is finite. By Lemma 1 we have

$$\lim_{n\to\infty} I_n = H(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_0)\,\|\,\Phi(\rho_0)\otimes\rho_0) = I(\rho_0,\Phi) \le +\infty. \tag{19}$$

Now we prove that $\lim\limits_{n\to\infty} I_n = \lim\limits_{n\to\infty} I(\rho_n,\Phi)$ by considering the difference $I_n - I(\rho_n,\Phi)$. Since $H(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n)) < +\infty$, we have

$$\begin{aligned} I_n - I(\rho_n,\Phi) &= H(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n)\,\|\,\Phi(\rho_0)\otimes\rho_n) - H(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n)\,\|\,\Phi(\rho_n)\otimes\rho_n) \\ &= -H(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n)) - \mathrm{Tr}(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n))(\log\Phi(\rho_0)\otimes\rho_n) \\ &\qquad + H(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n)) + \mathrm{Tr}(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n))\log(\Phi(\rho_n)\otimes\rho_n) = A - B, \end{aligned}$$

where

$$\begin{aligned} A &= -\,\mathrm{Tr}(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n))\log(\Phi(\rho_0)\otimes\rho_n), \\ B &= -\,\mathrm{Tr}(\Phi\otimes\mathrm{Id}_R(\widehat{\rho}_n))\log(\Phi(\rho_n)\otimes\rho_n). \end{aligned}$$

We use the property of the logarithm

$$\log(\rho\otimes\sigma) = \log(\rho)\otimes I + I\otimes\log(\sigma), \tag{20}$$

where, in the case of non-full-rank states ρ and σ, restrictions to the subspaces $\mathrm{supp}(\rho)$ and $\mathrm{supp}(\sigma)$ are considered, i.e.,

$$P_\rho\otimes P_\sigma\,(\log(\rho\otimes\sigma)) = (P_\rho\log(\rho)P_\rho)\otimes P_\sigma + P_\rho\otimes(P_\sigma\log(\sigma)P_\sigma), \tag{21}$$

where P_ρ and P_σ are, respectively, the projectors onto $\mathrm{supp}(\rho)$ and $\mathrm{supp}(\sigma)$. Since $P_{\Phi(\rho_n)} \le P_{\Phi(\rho_0)}$, we have

$$\begin{aligned} A &= -\operatorname{Tr}(\Phi\otimes \mathrm{Id}_R(\widehat{\rho}_n))(\log\Phi(\rho_0)\otimes I_R) - \operatorname{Tr}(\Phi\otimes \mathrm{Id}_R(\widehat{\rho}_n))(I_B\otimes \log(\rho_n)) \\ &= -\operatorname{Tr}\Phi(\rho_n)\log\Phi(\rho_0) + H(\rho_n). \end{aligned}$$

In a similar way, we obtain

$$\begin{aligned} B &= -\operatorname{Tr}(\Phi\otimes \mathrm{Id}_R(\widehat{\rho}_n))(\log\Phi(\rho_n)\otimes I_R) - \operatorname{Tr}(\Phi\otimes \mathrm{Id}_R(\widehat{\rho}_n))(I_B\otimes \log(\rho_n)) \\ &= H(\Phi(\rho_n)) + H(\rho_n). \end{aligned}$$

Hence,

$$I_n - I(\rho_n,\Phi) = A - B = -\operatorname{Tr}\Phi(\rho_n)\log\Phi(\rho_0) - H(\Phi(\rho_n)) = H(\Phi(\rho_n)\,\|\,\Phi(\rho_0)).$$

By monotonicity of the relative entropy, we have

$$H(\Phi(\rho_n)\,\|\,\Phi(\rho_0)) \le H(\rho_n\,\|\,\rho_0) = -\sum_{i=1}^{n}\frac{\lambda_i}{\mu_n}\log\mu_n = -\log\mu_n.$$

Since $\mu_n \to 1$ as $n\to\infty$, we get $\lim\limits_{n\to\infty}(I_n - I(\rho_n,\Phi)) = 0$. Taking into account (19), this implies

$$\lim_{n\to\infty} I(\rho_n,\Phi) = I(\rho_0,\Phi) \le +\infty. \quad \triangle$$

Lemma 4. *Let Φ be an arbitrary channel from $\mathfrak{S}(\mathcal{H}_A)$ to $\mathfrak{S}(\mathcal{H}_B)$, and $\{\Pi_n\}$ be a sequence of channels from $\mathfrak{S}(\mathcal{H}_B)$ to $\mathfrak{S}(\mathcal{H}_B)$ strongly converging to the identity channel. Let $\{\rho_n\}$ be a sequence of states in $\mathfrak{S}(\mathcal{H}_A)$ converging to a state ρ_0 such that $\lambda_n\rho_n \le \rho_0$ for some sequence $\{\lambda_n\}$ converging to* 1. *Then*

$$\lim_{n\to+\infty} I(\rho_n,\Pi_n\circ\Phi) = I(\rho_0,\Phi).$$

Proof. It follows from the inequality $\lambda_n\rho_n \le \rho_0$ that $\rho_0 = \lambda_n\rho_n + (1-\lambda_n)\sigma_n$, where σ_n is a state in $\mathfrak{S}(\mathcal{H}_A)$. Hence, concavity and nonnegativity of the mutual information and the 1st chain rule imply the inequality

$$\lambda_n I(\rho_n,\Pi_n\circ\Phi) \le I(\rho_0,\Pi_n\circ\Phi) \le I(\rho_0,\Phi),$$

showing that $\limsup\limits_{n\to+\infty} I(\rho_n,\Pi_n\circ\Phi) \le I(\rho_0,\Phi)$. This and lower semicontinuity of the function $(\rho,\Phi)\mapsto I(\rho,\Phi)$ imply the assertion of the lemma. $\triangle$

4. RELATION BETWEEN MUTUAL INFORMATIONS OF COMPLEMENTARY CHANNELS

The main result of this section is an infinite-dimensional generalization of relation (7) between mutual informations of a pair of complementary channels (nontriviality of this result is connected with a possible uncertainty "$\infty-\infty$" in expressions (4) and (6)).

Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_E$ be separable Hilbert spaces, and $V\colon \mathcal{H}_A\to\mathcal{H}_B\otimes\mathcal{H}_E$ be an isometry; then relations (2) define a pair of complementary channels Φ and $\widetilde{\Phi}$, similarly to the finite-dimensional case.

Theorem. *For an arbitrary state $\rho\in\mathfrak{S}(\mathcal{H}_A)$, we have the relation*

$$I(\rho,\Phi) + I(\rho,\widetilde{\Phi}) = 2H(\rho). \tag{22}$$

Proof. Let $\{|h_i\rangle\}_{i=1}^{+\infty}$ be an orthonormal basis in the space $\mathcal{H}_E$. Then

$$V|\varphi\rangle = \sum_{i=1}^{\infty} V_i|\varphi\rangle \otimes |h_i\rangle,$$

where $V_i\colon \mathcal{H}_A \to \mathcal{H}_B$ is a sequence of bounded operators satisfying the condition $\sum_{i=1}^{\infty} V_i^* V_i = I_A$, the channel Φ has the Kraus representation $\Phi(\rho) = \sum_{i=1}^{\infty} V_i \rho V_i^*$, and the complementary channel $\widetilde{\Phi}$ has the representation $\widetilde{\Phi}(\rho) = \sum_{i,j=1}^{+\infty} [\operatorname{Tr} V_i \rho V_j^*] |h_i\rangle\langle h_j|$ (cf. [13]).

Let $\rho = \sum_{i=1}^{m} \lambda_i |e_i\rangle\langle e_i|$ be a finite-rank state in $\mathfrak{S}(\mathcal{H}_A)$, and $\widehat{\rho}$ be its purification in $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_R)$. Consider a sequence of quantum operations[5] $\Phi_n(\rho) = \sum_{i=1}^{n} V_i \rho V_i^*$. The sequence $\{\Phi_n\}$ strongly and monotonously converges to the channel Φ (i.e., $\Phi_n(\rho) \le \Phi_{n+1}(\rho)$ for all n and $\rho \in \mathfrak{S}(\mathcal{H}_A)$).

Since $\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho})$ is a finite-rank state, we have

$$\begin{aligned} X_n &= H\left(\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho}) \,\|\, \Phi(\rho) \otimes \rho\right) \\ &= -S(\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho})) - \operatorname{Tr}(\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho})) \log(\Phi(\rho) \otimes \rho) + R_n, \end{aligned}$$

where $R_n = 1 - \operatorname{Tr} \Phi_n(\rho) \to 0$ as $n \to \infty$. By Lemma 7 in the Appendix, we have $\lim\limits_{n\to\infty} X_n = I(\rho, \Phi)$. Since

$$\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho}) = \operatorname{Tr}_E \left(I_B \otimes P_n \otimes I_R\right) \cdot (V \otimes I_R) \cdot \widehat{\rho} \cdot (V^* \otimes I_R) \cdot (I_B \otimes P_n \otimes I_R),$$

where $P_n = \sum_{i=1}^{n} |h_i\rangle\langle h_i|$ is a finite-dimensional projector in the space $\mathcal{H}_E$, and the partial trace is taken in the space $\mathcal{H}_B \otimes \mathcal{H}_E \otimes \mathcal{H}_R$, the operator $\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho})$ is isomorphic to the operator

$$\widetilde{\Phi}_n(\rho) = \operatorname{Tr}_{BR} \left(I_B \otimes P_n \otimes I_R\right) \cdot (V \otimes I_R) \cdot \widehat{\rho} \cdot (V^* \otimes I_R) \cdot (I_B \otimes P_n \otimes I_R),$$

where $\widetilde{\Phi}_n(\cdot) = P_n \widetilde{\Phi}(\cdot) P_n$ is the quantum operation complementary to Φ_n. Thus, $S(\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho})) = S(\widetilde{\Phi}_n(\rho))$. By using (20) and noting that $\Phi_n(\cdot) \le \Phi(\cdot)$, we obtain

$$\begin{aligned} &- \operatorname{Tr}(\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho})) \log(\Phi(\rho) \otimes \rho) \\ &\qquad = - \operatorname{Tr}(\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho}))(\log(\Phi(\rho)) \otimes I_R) - \operatorname{Tr}(\Phi_n \otimes \mathrm{Id}_R(\widehat{\rho}))(I_B \otimes \log(\rho)) \\ &\qquad = - \operatorname{Tr} \Phi_n(\rho) \log(\Phi(\rho)) - \operatorname{Tr}\left(\operatorname{Tr}_B \Phi_n \otimes \mathrm{Id}_R(\widehat{\rho})\right) \log(\rho). \end{aligned}$$

Consider the quantity

$$\begin{aligned} Y_n &= H\big(\widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho}) \,\|\, \widetilde{\Phi}_n(\rho) \otimes \rho\big) \\ &= -S\big(\widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho})\big) - \operatorname{Tr}\big(\widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho})\big) \log\big(\widetilde{\Phi}_n(\rho) \otimes \rho\big). \end{aligned}$$

Then $\lim\limits_{n\to\infty} Y_n = I(\rho, \widetilde{\Phi})$ by Lemma 1. Similarly to the computation of the summands of X_n, we obtain

$$S\big(\widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho})\big) = S(\Phi_n(\rho))$$

and

$$\begin{aligned} &- \operatorname{Tr}\big(\widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho})\big) \log\big(\widetilde{\Phi}_n(\rho) \otimes \rho\big) \\ &\qquad = - \operatorname{Tr}\big(\widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho})\big)\big(\log(\widetilde{\Phi}_n(\rho)) \otimes I_R\big) - \operatorname{Tr}\big(\widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho})\big)\big(I_E \otimes \log(\rho)\big) \\ &\qquad = S\big(\widetilde{\Phi}_n(\rho)\big) - \operatorname{Tr}\big(\operatorname{Tr}_E \widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho})\big) \log(\rho). \end{aligned}$$

[5] A *quantum operation* is a linear completely positive trace-nonincreasing map [1, 2].

Let us show that $\lim_{n\to\infty}(X_n + Y_n) = 2H(\rho)$. By the definition of the relative entropy, we have

$$\begin{aligned} X_n + Y_n &= -\operatorname{Tr}\Phi_n(\rho)\log(\Phi(\rho)) - S(\Phi_n(\rho)) + C_n + D_n + R_n \\ &= H(\Phi_n(\rho)\,\|\,\Phi(\rho)) + C_n + D_n, \end{aligned}$$

where

$$C_n = -\operatorname{Tr}\left(\operatorname{Tr}_B \Phi_n \otimes \mathrm{Id}_R(\widehat{\rho})\right)\log(\rho), \qquad D_n = -\operatorname{Tr}\left(\operatorname{Tr}_E \widetilde{\Phi}_n \otimes \mathrm{Id}_R(\widehat{\rho})\right)\log(\rho).$$

Let us prove that $\lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n = H(\rho)$. By noting that

$$\operatorname{Tr}_B \Phi_n \otimes \mathrm{Id}_R(\widehat{\rho}) = \sum_{i,j=1}^{m} \sqrt{\lambda_i\lambda_j}\operatorname{Tr}\Phi_n(|e_i\rangle\langle e_j|)|e_i\rangle\langle e_j|,$$

we obtain

$$C_n = \sum_{i=1}^{m}(-\lambda_i\log\lambda_i)\operatorname{Tr}\Phi_n(|e_i\rangle\langle e_i|),$$

and hence $\lim_{n\to\infty} C_n = H(\rho)$, since $\lim_{n\to\infty}\operatorname{Tr}\Phi_n(|e_i\rangle\langle e_i|) = 1$. In a similar way one can prove that $\lim_{n\to\infty} D_n = H(\rho)$. Lemma 7 in the Appendix implies

$$\lim_{n\to\infty} H(\Phi_n(\rho)\,\|\,\Phi(\rho)) = 0.$$

Thus, we have $\lim_{n\to\infty}(X_n + Y_n) = 2H(\rho)$. Since $\lim_{n\to\infty} X_n = I(\rho,\Phi)$ and $\lim_{n\to\infty} Y_n = I(\rho,\widetilde{\Phi})$, the assertion of the theorem is proved for finite-rank states. Since the left- and right-hand sides of relation (22) are concave lower semicontinuous nonnegative functions (by Proposition 1), validity of this relation for all states follows from Lemma 6 in [14], which states that any concave lower semicontinuous lower bounded function on a set of quantum states is uniquely determined by its restriction to the set of finite-rank states. △

5. COHERENT INFORMATION

Since in the infinite-dimensional case the right-hand side in definition (8) of the coherent information $I_c(\rho,\Phi)$ can be indefinite even for a state ρ with finite entropy, while the results of Section 4 show finiteness of the mutual information $I(\rho,\Phi)$ for any such state ρ and any channel Φ, it seems natural to use the following definition of the coherent information for an infinite-dimensional quantum channel.

Definition 5. Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel, and ρ be a state in $\mathfrak{S}(\mathcal{H}_A)$ with finite entropy. The *coherent information* of the channel Φ at the state ρ is defined as follows:

$$I_c(\rho,\Phi) = I(\rho,\Phi) - H(\rho).$$

The above-defined quantity inherits properties 2 and 3 of the mutual information (see Proposition 1). The theorem also implies the inequalities

$$-H(\rho) \le I_c(\rho,\Phi) \le H(\rho)$$

and a generalization of identity (10) to the infinite-dimensional case.

Corollary 1. *Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel, and $\widetilde{\Phi}\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ be the channel complementary to Φ. For an arbitrary state $\rho \in \mathfrak{S}(\mathcal{H}_A)$ with finite entropy, we have the relation*

$$I_c(\rho,\Phi) + I_c(\rho,\widetilde{\Phi}) = 0. \tag{23}$$

Remark 2. In the case $H(\rho) < +\infty$ and $H(\Phi(\rho)) < +\infty$, we have $I(\rho, \Phi) = H(\rho) + H(\Phi(\rho)) - H(\widetilde{\Phi}(\rho))$, and hence

$$I_c(\rho, \Phi) = H(\Phi(\rho)) - H(\widetilde{\Phi}(\rho)). \tag{24}$$

An alternative expression for the coherent information of a channel Φ at a state ρ with finite entropy can be obtained by using the relation of this quantity to the secret classical capacity of a channel, mentioned in [15]. Consider the χ-function of a channel Φ defined as

$$\chi_\Phi(\rho) = \sup \sum_i \pi_i H(\Phi(\rho_i) \,\|\, \Phi(\rho)), \quad \rho \in \mathfrak{S}(\mathcal{H}),$$

where the supremum is over all convex decompositions $\rho = \sum_i \pi_i \rho_i$, $\rho_i \in \mathfrak{S}(\mathcal{H})$. This function is closely connected to the classical capacity of the channel Φ (cf. [2]). If $H(\Phi(\rho)) < +\infty$, then $\chi_\Phi(\rho) = H(\Phi(\rho)) - \overline{\mathrm{co}}\, H_\Phi(\rho)$, where $\overline{\mathrm{co}}\, H_\Phi(\rho)$ is the convex closure of the output entropy of the channel Φ (cf. [7]). Taking into account that $\overline{\mathrm{co}}\, H_\Phi \equiv \overline{\mathrm{co}}\, H_{\widetilde{\Phi}}$ and that $|H(\Phi(\rho)) - H(\widetilde{\Phi}(\rho))| \le H(\rho)$ by the triangle inequality [1], for an arbitrary state ρ such that $H(\rho) < +\infty$ and $H(\Phi(\rho)) < +\infty$, we have

$$I_c(\rho, \Phi) = \chi_\Phi(\rho) - \chi_{\widetilde{\Phi}}(\rho). \tag{25}$$

Since $\max\{\chi_\Phi(\rho), \chi_{\widetilde{\Phi}}(\rho)\} \le H(\rho)$ by monotonicity of the relative entropy, the right-hand side in (25) is well defined and takes values in $[-H(\rho), H(\rho)]$ under a single condition $H(\rho) < +\infty$. By using the approximation method for quantum channels proposed in [7] and Proposition 5 below, one can show that expression (25) is *equivalent* to Definition 5 provided that $H(\rho) < +\infty$.

In finite dimensions, the equality $H(\rho) = I_c(\rho, \Phi)$ is a necessary and sufficient condition for perfect reversibility of the channel Φ on the state ρ (see [1, Theorem 12.10]). We generalize this to the infinite-dimensional case.

Definition 6. A channel Φ is called *perfectly reversible on a state* $\rho \in \mathfrak{S}(\mathcal{H}_A)$ if there exists a channel $\mathcal{D}\colon \mathfrak{S}(\mathcal{H}_B) \to \mathfrak{S}(\mathcal{H}_A)$ such that

$$\mathcal{D} \circ \Phi(\widetilde{\rho}) = \widetilde{\rho},$$

for all states $\widetilde{\rho}$ with $\operatorname{supp} \widetilde{\rho} \subset \mathcal{L} \equiv \operatorname{supp} \rho$.

In other words, the subspace $\mathcal{L}$ is a quantum code correcting errors of the channel Φ; see [1]. Introduce a reference system $\mathcal{H}_R$ and consider a purification $\rho_{AR} = |\varphi_{AR}\rangle\langle\varphi_{AR}| \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_R)$ of the state ρ.

Lemma 5. *A channel Φ is perfectly reversible on a state $\rho \in \mathfrak{S}(\mathcal{H}_A)$ if and only if there exists a channel $\mathcal{D}\colon \mathfrak{S}(\mathcal{H}_B) \to \mathfrak{S}(\mathcal{H}_A)$ such that*

$$(\mathcal{D} \circ \Phi \otimes \mathrm{Id}_R)(\rho_{AR}) = \rho_{AR}. \tag{26}$$

A proof of the lemma is presented in the Appendix.

Proposition 2. *Let $H(\rho) < \infty$. A channel $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ is perfectly reversible on the state ρ if and only if one of the following equivalent conditions holds*: $I_c(\rho, \Phi) = H(\rho)$, $I(\rho, \widetilde{\Phi}) = 0$.

Proof. By the theorem and Definition 5, we have

$$H(\rho) - I_c(\rho, \Phi) = I(\rho, \widetilde{\Phi}) \ge 0,$$

where the equality holds if and only if $\rho_{RE} = \rho_R \otimes \rho_E$, since $I(\rho, \widetilde{\Phi}) = H(\rho_{RE} \| \rho_R \otimes \rho_E)$. The rest of the proof mainly follows the proof given in [2] and is presented here for completeness.

Necessity. Let $V\colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ be the isometry from representation (2) of the channel Φ. Consider the pure state $\rho_{BRE} = |\varphi_{BRE}\rangle\langle\varphi_{BRE}|$, where $|\varphi_{BRE}\rangle = (V \otimes I_R)|\varphi_{AR}\rangle$. Since the channel Φ is perfectly reversible, there exists a channel $\mathcal{D}$ such that (26) holds, and hence

$$(\mathcal{D} \otimes \mathrm{Id}_{RE})(\rho_{BRE}) = \rho_{ARE}.$$

Since ρ_{AR} is a pure state, we have $\rho_{ARE} = \rho_{AR} \otimes \rho_E$. By taking partial traces over the space $\mathcal{H}_A$, we obtain $\rho_{RE} = \rho_R \otimes \rho_E$.

Sufficiency. Consider the vector $|\varphi_{BRE}\rangle = (V \otimes I_R)|\varphi_{AR}\rangle$. Then $|\varphi_{BRE}\rangle$ is a purification vector for the state ρ_{RE}. Since $\rho_{RE} = \rho_R \otimes \rho_E$, we may take $|\varphi_{AR}\rangle \otimes |\varphi_{EE'}\rangle$ as a purification vector for the state ρ_{RE}, where E' is a reference system for the system E.

Without loss of generality, we may assume that the Hilbert spaces of both purifications are infinite-dimensional, so that there exists an isometry $W\colon \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_{E'}$ such that

$$(I_{RE} \otimes W)|\varphi_{BRE}\rangle = |\varphi_{AR}\rangle \otimes |\varphi_{EE'}\rangle$$

and, correspondingly,

$$(I_{RE} \otimes W)|\varphi_{BRE}\rangle\langle\varphi_{BRE}|(I_{RE} \otimes W^*) = |\varphi_{AR}\rangle\langle\varphi_{AR}| \otimes |\varphi_{EE'}\rangle\langle\varphi_{EE'}|.$$

By taking partial traces over the spaces $\mathcal{H}_E$ and $\mathcal{H}_{E'}$, we obtain the perfect reversibility condition (26), where

$$\mathcal{D}(\sigma) = \mathrm{Tr}_{E'}\, W\sigma W^*, \quad \sigma \in \mathfrak{S}(\mathcal{H}_B). \quad \triangle$$

As is mentioned above, the entropy $H(\rho)$ of the state ρ is an upper bound for the coherent information $I_c(\rho, \Phi)$ of an arbitrary channel Φ at this state. The following proposition gives a more precise upper bound for $I_c(\rho, \Phi)$, expressed via the Kraus operators of the channel Φ.

Proposition 3. *Let* $\Phi(\cdot) = \sum\limits_{i=1}^{\infty} V_i(\cdot)V_i^*$ *be a quantum channel. Then, for an arbitrary state* ρ *with finite entropy, we have the inequality*

$$I_c(\rho, \Phi) \le \sum_{i=1}^{\infty} H(V_i\rho V_i^*). \tag{27}$$

The equality holds in (27) *if* $\mathrm{Ran}\, V_i \perp \mathrm{Ran}\, V_j$ *for all* $i \ne j$.

The expression on the right-hand side of (27) can be considered as the mean entropy of an a posteriori state in the quantum measurement described by the collection of operators $\{V_i\}_{i=1}^{+\infty}$ at an a priori state ρ [2, Section 6.5]. By the Groenevold–Lindblad–Ozawa inequality, this value does not exceed $H(\rho)$ [16].

Proof. First we show that the equality holds in (27) if $\mathrm{Ran}V_i \perp \mathrm{Ran}V_j$ for all $i \ne j$.

Let ρ be a state in $\mathfrak{S}(\mathcal{H}_A)$ with finite entropy, and $|\varphi\rangle\langle\varphi|$ be its purification in $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_R)$. By using well-known properties of the relative entropy (see [4]) and noting that $\sum\limits_{i=1}^{\infty} V_i^* V_i = I_A$, we obtain

$$\begin{aligned} I(\rho, \Phi) &= H\left(\Phi \otimes \mathrm{Id}_R(|\varphi\rangle\langle\varphi|) \,\|\, \Phi \otimes \mathrm{Id}_R(\rho \otimes \rho)\right) \\ &= H\left(\sum_{i=1}^{\infty} V_i \otimes I_R|\varphi\rangle\langle\varphi|V_i^* \otimes I_R \,\Big\|\, \sum_{i=1}^{\infty}(V_i \otimes I_R)(\rho \otimes \rho)(V_i^* \otimes I_R)\right) \\ &= \sum_{i=1}^{\infty} H\left(V_i \otimes I_R|\varphi\rangle\langle\varphi|V_i^* \otimes I_R \,\|\, (V_i \otimes I_R)(\rho \otimes \rho)(V_i^* \otimes I_R)\right) \end{aligned}$$

$$= H(\rho) + \sum_{i=1}^{\infty} [S(V_i\rho V_i^*) - \eta(\operatorname{Tr} V_i\rho V_i^*)]$$
$$= H(\rho) + \sum_{i=1}^{\infty} H(V_i\rho V_i^*).$$

Let $\Phi(\cdot) = \sum\limits_{i=1}^{\infty} V_i(\cdot)V_i^*$ be an arbitrary channel, and let $\mathcal{H}_C = \bigoplus\limits_{i=1}^{\infty} \mathcal{H}_B^i$, where $\mathcal{H}_B^i \cong \mathcal{H}_B$. Let U_i be an isometric embedding of $\mathcal{H}_B$ in $\mathcal{H}_C$ such that $U_i\mathcal{H}_B = \mathcal{H}_B^i$ for each i.

As is proved above, for the quantum channel $\widehat{\Phi}(\cdot) = \sum\limits_{i=1}^{\infty} U_iV_i(\cdot)V_i^*U_i^*$ we have the equality

$$I_c(\rho, \widehat{\Phi}) = \sum_{i=1}^{\infty} H(U_iV_i\rho V_i^*U_i^*) = \sum_{i=1}^{\infty} H(V_i\rho V_i^*),$$

whence we obtain (27) by applying the 1st chain rule for the coherent information to the composition $\Psi \circ \widehat{\Phi} = \Phi$, where $\Psi(\cdot) = \sum\limits_{i=1}^{\infty} U_i^*(\cdot)U_i$ is a channel from $\mathfrak{S}(\mathcal{H}_C)$ to $\mathfrak{S}(\mathcal{H}_B)$. △

6. ON CONTINUITY OF THE MUTUAL AND COHERENT INFORMATION

Proposition 1 and the theorem provide the following continuity condition for the mutual and coherent information.

Proposition 4. *For an arbitrary quantum channel* $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$, *the functions*

$$\rho \mapsto I(\rho, \Phi) \qquad \textit{and} \qquad \rho \mapsto I_c(\rho, \Phi)$$

are continuous on any subset $\mathcal{A} \subset \mathfrak{S}(\mathcal{H}_A)$ *on which the von Neumann entropy is continuous.*

Proof. By Proposition 1, the functions $\rho \mapsto I(\rho, \Phi)$ and $\rho \mapsto I(\rho, \widetilde{\Phi})$ are lower semicontinuous, while by the theorem their sum coincides with the doubled von Neumann entropy, which is continuous on the set $\mathcal{A}$ by the condition. Hence, these functions are continuous on the set $\mathcal{A}$. The function $\rho \mapsto I_c(\rho, \Phi)$ is continuous on the set $\mathcal{A}$ as a difference of two functions that are continuous on this set. △

Example. Let H be the Hamiltonian of a quantum system A. Then the subset $\mathcal{K}_{H,h}$ of $\mathfrak{S}(\mathcal{H}_A)$ consisting of the states ρ such that $\operatorname{Tr} H\rho \leq h$ can be treated as the set of states with mean energy not greater than h. If the operator H is such that $\operatorname{Tr} e^{-\lambda H} < +\infty$ for all $\lambda > 0$, then the von Neumann entropy is continuous on $\mathcal{K}_{H,h}$ [3, 5]. This holds, for example, for the Hamiltonian of a system of quantum oscillators [3]. By Proposition 4, for an arbitrary quantum channel Φ, the mutual information $I(\rho, \Phi)$ and coherent information $I_c(\rho, \Phi)$ are continuous functions of a state $\rho \in \mathcal{K}_{H,h}$ for each finite $h > 0$. Hence, these functions are bounded and attain their supremum on the set $\mathcal{K}_{H,h}$ (by compactness of this set [6]).

Remark 3. Proposition 4 and identity (22) show that, for an arbitrary channel Φ, the function $\rho \mapsto I(\rho, \Phi)$ is continuous and bounded on the set

$$\mathfrak{S}_k(\mathcal{H}_A) = \{\rho \in \mathfrak{S}(\mathcal{H}_A) \mid \operatorname{rank} \rho \leq k\}$$

for each $k = 1, 2, \ldots$. Hence, properties of the function $\rho \mapsto I(\rho, \Phi)$ can be explored by using the approximation method considered in [14, Section 4]. This method makes it possible to clarify the sense of the continuity condition for the function $\rho \mapsto I(\rho, \Phi)$ in Proposition 4 and to show its necessity for a particular class of channels.

By Proposition 3 in [14], the function $\rho \mapsto I(\rho,\Phi)$ is a pointwise limit of the increasing sequence

$$\rho \mapsto I_k(\rho,\Phi) = \sup_{\{\pi_i,\rho_i\}\in\mathcal{E}^k_\rho} \sum_i \pi_i I(\rho_i,\Phi) \tag{28}$$

of concave functions continuous on the set $\mathfrak{S}(\mathcal{H}_A)$, where $\mathcal{E}^k_\rho$ is the set of all ensembles[6] $\{\pi_i,\rho_i\}$ such that $\sum_i \pi_i\rho_i = \rho$ and $\operatorname{rank}\rho_i \le k$ for all i. Hence, a necessary and sufficient condition of continuity of the function $\rho \mapsto I(\rho,\Phi)$ on a set $\mathcal{A} \subset \mathfrak{S}(\mathcal{H}_A)$ can be expressed as follows:

$$\lim_{k\to+\infty} \sup_{\rho\in\mathcal{A}_c} \Delta^I_k(\rho,\Phi) = 0 \quad \text{for any compact set } \mathcal{A}_c \subseteq \mathcal{A}, \tag{29}$$

where $\Delta^I_k(\rho,\Phi) = I(\rho,\Phi) - I_k(\rho,\Phi)$. One can show that

$$\Delta^I_k(\rho,\Phi) = \inf_{\{\pi_i,\rho_i\}\in\mathcal{E}^k_\rho} \sum_i \pi_i \left[H(\rho_i\,\|\,\rho) + H(\Phi(\rho_i)\,\|\,\Phi(\rho)) - H(\widetilde{\Phi}(\rho_i)\,\|\,\widetilde{\Phi}(\rho))\right]$$

for any state ρ with finite entropy. By monotonicity and nonnegativity of the relative entropy, the expression in square brackets is not greater than $2H(\rho_i\,\|\,\rho)$. Thus, (29) holds if

$$\lim_{k\to+\infty} \sup_{\rho\in\mathcal{A}_c} \inf_{\{\pi_i,\rho_i\}\in\mathcal{E}^k_\rho} \sum_i \pi_i H(\rho_i\,\|\,\rho) = 0, \quad \text{for any compact set } \mathcal{A}_c \subseteq \mathcal{A},$$

which means uniform convergence of the sequence $\{H_k\}$ of continuous approximators of the entropy (defined by formula (28) with $H(\rho_i)$ instead of $I(\rho_i,\Phi)$ on the right-hand side) on compact subsets of $\mathcal{A}$, and this is equivalent to continuity of the entropy on this set [14].

Thus, *the assertion of Proposition 4 is explained by the implication*

$$H_k(\rho) \overset{\mathcal{A}}{\rightrightarrows} H(\rho) < +\infty \quad\Longrightarrow\quad I_k(\rho,\Phi) \overset{\mathcal{A}}{\rightrightarrows} I(\rho,\Phi) < +\infty, \quad \forall \mathcal{A} \subset \mathfrak{S}(\mathcal{H}_A), \tag{30}$$

which is valid for any channel Φ by monotonicity of the relative entropy.

If Φ is a degradable channel, i.e., $\widetilde{\Phi} = \Lambda \circ \Phi$ for some channel Λ, then we have $I(\rho,\Phi) < +\infty \Rightarrow H(\rho) < +\infty$ by the theorem and by the 1st chain rule from Proposition 1, while the expression in square brackets in the above formula for $\Delta^I_k(\rho,\Phi)$ is not less than $H(\rho_i\,\|\,\rho)$ by monotonicity of the relative entropy. Thus, for a degradable channel Φ, two-sided implication holds in (30), and hence the continuity condition for the function $\rho \mapsto I(\rho,\Phi)$ in Proposition 4 is necessary and sufficient:

$$\lim_{n\to+\infty} H(\rho_n) = H(\rho_0) < +\infty \quad\Longleftrightarrow\quad \lim_{n\to+\infty} I(\rho_n,\Phi) = I(\rho_0,\Phi) < +\infty,$$

for any sequence $\{\rho_n\}$ converging to a state ρ_0.

To explore continuity of capacities as functions of a channel, it is necessary to consider the corresponding entropy characteristics as functions of a pair (input state, channel), i.e., as functions on the Cartesian product of the set of all input states $\mathfrak{S}(\mathcal{H}_A)$ and the set of all channels $\mathfrak{F}(A,B)$ from A to B endowed with an appropriate (sufficiently weak) topology. As is shown in [7], for this purpose it is reasonable to use the strong convergence topology on the set $\mathfrak{F}(A,B)$ described before Proposition 1 in Section 3. By using an obvious modification of the arguments in the proof of Proposition 4, one can derive the following result from Proposition 1 and the theorem.

Proposition 5. *Let $\{\Phi_n\}$ be a sequence of channels in $\mathfrak{F}(A,B)$ strongly converging to a channel Φ_0, and assume that there exists a sequence $\{\widetilde{\Phi}_n\}$ of channels in $\mathfrak{F}(A,E)$ strongly converging*

[6] An ensemble $\{\pi_i,\rho_i\}$ is a collection of states $\{\rho_i\}$ with the corresponding probability distribution $\{\pi_i\}$.

to a channel $\widetilde{\Phi}_0$ *such that* $(\Phi_n, \widetilde{\Phi}_n)$ *is a complementary pair for each* $n = 0, 1, 2, \ldots$ *. Then the relations*

$$\lim_{n\to+\infty} I(\rho_n, \Phi_n) = I(\rho_0, \Phi_0) \qquad \text{and} \qquad \lim_{n\to+\infty} I_c(\rho_n, \Phi_n) = I_c(\rho_0, \Phi_0) \tag{31}$$

hold for any sequence $\{\rho_n\}$ *of states in* $\mathfrak{S}(\mathcal{H}_A)$ *converging to the state* ρ_0 *such that* $\lim\limits_{n\to+\infty} H(\rho_n) = H(\rho_0) < +\infty$*.*

Let $\mathfrak{V}_1(A, B)$ be the set of all sequences $\overline{V} = \{V_i\}_{i=1}^{+\infty}$ of operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ such that $\sum\limits_{i=1}^{+\infty} V_i^* V_i = I_A$, endowed with the topology of coordinatewise strong operator convergence.

Corollary 2. *For an arbitrary subset* $\mathcal{A} \subset \mathfrak{S}(\mathcal{H}_A)$ *on which the von Neumann entropy is continuous, the functions*

$$(\rho, \overline{V}) \mapsto I(\rho, \Phi[\overline{V}]), \qquad (\rho, \overline{V}) \mapsto I_c(\rho, \Phi[\overline{V}]), \qquad (\rho, \overline{V}) \mapsto \sum_{i=1}^{+\infty} H(V_i \rho V_i^*),$$

where $\Phi[\overline{V}](\cdot) = \sum\limits_{i=1}^{+\infty} V_i(\cdot)V_i^*$*, are continuous on the set* $\mathcal{A} \times \mathfrak{V}_1(A, B)$*.*

Proof. By Proposition 5, continuity of the first two functions follows from continuity of the maps

$$\mathfrak{V}_1(A, B) \ni \overline{V} \mapsto \Phi[\overline{V}] \in \mathfrak{F}(A, B) \qquad \text{and} \qquad \mathfrak{V}_1(A, B) \ni \overline{V} \mapsto \widetilde{\Phi}[\overline{V}] \in \mathfrak{F}(A, E),$$

where $\widetilde{\Phi}[\overline{V}](\cdot) = \sum\limits_{i,j=1}^{+\infty} [\operatorname{Tr} V_i(\cdot)V_j^*] |h_i\rangle\langle h_j|$, and $\{|h_i\rangle\}$ is an orthonormal basis in $\mathcal{H}_E$.

To prove continuity of these maps, it suffices to show that

$$\lim_{n\to+\infty} \Phi[\overline{V}_n](|\varphi\rangle\langle\varphi|) = \Phi[\overline{V}_0](|\varphi\rangle\langle\varphi|) \tag{32}$$

and

$$\lim_{n\to+\infty} \widetilde{\Phi}[\overline{V}_n](|\varphi\rangle\langle\varphi|) = \widetilde{\Phi}[\overline{V}_0](|\varphi\rangle\langle\varphi|), \tag{33}$$

for any sequence $\{\overline{V}_n\} \subset \mathfrak{V}_1(A, B)$ converging to a vector $\overline{V}_0 \in \mathfrak{V}_1(A, B)$ and for any unit vector $\varphi \in \mathcal{H}_A$.

Let $\overline{V}_n = \{V_i^n\}_{i=1}^{+\infty}$ for each $n \geq 0$. Relation (32) can be proved by noting that the condition $\sum\limits_{i=1}^{+\infty} \|V_i^n|\varphi\rangle\|^2 = 1$ for all $n \geq 0$ implies

$$\lim_{m\to+\infty} \sup_{n\geq 0} \operatorname{Tr} \sum_{i>m} V_i^n |\varphi\rangle\langle\varphi| (V_i^n)^* = \lim_{m\to+\infty} \sup_{n\geq 0} \sum_{i>m} \|V_i^n|\varphi\rangle\|^2 = 0.$$

Relation (33) is easily proved by using the result from [12] mentioned before Proposition 1.

To prove continuity of the third function, consider the following construction. Let $\mathcal{H}_C = \bigoplus\limits_{i=1}^{+\infty} \mathcal{H}_B^i$, where $\mathcal{H}_B^i \cong \mathcal{H}_B$, and let U_i be an isometric embedding of $\mathcal{H}_B$ in $\mathcal{H}_C$ such that $U_i\mathcal{H}_B = \mathcal{H}_B^i$ for each i.

For an arbitrary sequence $\{V_i\}_{i=1}^{+\infty}$ in $\mathfrak{V}_1(A, B)$, one can take the sequence $\{\widehat{V}_i = U_i V_i\}_{i=1}^{+\infty}$ in $\mathfrak{V}_1(A, C)$ such that $\operatorname{Ran} \widehat{V}_i \perp \operatorname{Ran} \widehat{V}_j$ for all $i \neq j$. Since the above correspondence is continuous (as a map from $\mathfrak{V}_1(A, B)$ to $\mathfrak{V}_1(A, C)$), the above observation shows continuity of the function

$$(\rho, \overline{V}) \mapsto I_c(\rho, \widehat{\Phi}[\overline{V}]) = \sum_{i=1}^{+\infty} H(\widehat{V}_i \rho \widehat{V}_i^*) = \sum_{i=1}^{+\infty} H(V_i \rho V_i^*)$$

on the set $\mathcal{A} \times \mathfrak{V}_1(A, B)$, where $\widehat{\Phi}[\overline{V}](\cdot) = \sum\limits_{i=1}^{+\infty} \widehat{V}_i(\cdot)\widehat{V}_i^*$ and where the first equality follows from the last assertion of Proposition 3. △

As was mentioned in Section 5, the quantity $\sum_{i=1}^{+\infty} H(V_i\rho V_i^*)$ can be considered as the mean entropy of an a posteriori state in the quantum measurement described by the collection of operators $\{V_i\}_{i=1}^{+\infty}$. Corollary 2 shows that continuity of the entropy $H(\rho)$ of an a priori state ρ implies continuity of the mean entropy of the a posteriori state as a function of a pair (a priori state, measurement) provided that the strong operator topology is used in the definition of convergence of a sequence of measurements. This assertion strengthens an analogous assertion in Example 3 in [14], where a *stronger* topology (the so-called $*$-strong operator topology) was used in the definition of convergence of a sequence of measurements.[7] Hence, with the use of Corollary 2, one can strengthen all the assertions in Example 3 in [14] by inserting the strong operator topology in the definition of convergence of a sequence of measurements, which seems more natural in this context.

APPENDIX

Lemma 6. *Let ρ and σ be states in $\mathfrak{S}(\mathcal{H})$, and let C be an operator in $\mathfrak{T}_+(\mathcal{H})$. Then*

$$H(\lambda\rho+(1-\lambda)\sigma\,\|\,C)\geq\lambda H(\rho\,\|\,C)+(1-\lambda)H(\sigma\,\|\,C)-h_2(\lambda),\quad\forall\lambda\in[0,1],$$

where $h_2(\lambda)=\eta(\lambda)+\eta(1-\lambda)$.

Proof. Let $\{P_n\}$ be an increasing sequence of finite-rank projectors strongly converging to the identity operator. Then $A_n=P_n\rho P_n$, $B_n=P_n\sigma P_n$, and $C_n=P_nCP_n$ are finite-rank operators for each n, and hence

$$\begin{aligned}H(\lambda A_n+(1-\lambda)B_n\,\|\,C_n)&=\operatorname{Tr}(\lambda A_n+(1-\lambda)B_n)(-\log C_n)\\&-S(\lambda A_n+(1-\lambda)B_n)+\operatorname{Tr}C_n-\operatorname{Tr}(\lambda A_n+(1-\lambda)B_n)\\&\geq\lambda\operatorname{Tr}A_n(-\log C_n)+(1-\lambda)\operatorname{Tr}B_n(-\log C_n)+\operatorname{Tr}C_n-\lambda\operatorname{Tr}A_n-(1-\lambda)\operatorname{Tr}B_n\\&-\lambda S(A_n)-(1-\lambda)S(B_n)-\eta(\operatorname{Tr}(\lambda A_n+(1-\lambda)B_n))+\lambda\eta(\operatorname{Tr}A_n)\\&+(1-\lambda)\eta(\operatorname{Tr}B_n)-x_nh_2(x_n^{-1}\lambda\operatorname{Tr}A_n)=\lambda H(A_n\,\|\,C_n)+(1-\lambda)H(B_n\,\|\,C_n)\\&-\eta(\operatorname{Tr}(\lambda A_n+(1-\lambda)B_n))+\lambda\eta(\operatorname{Tr}A_n)+(1-\lambda)\eta(\operatorname{Tr}B_n)-x_nh_2(x_n^{-1}\lambda\operatorname{Tr}A_n),\end{aligned}$$

where $x_n=\operatorname{Tr}(\lambda A_n+(1-\lambda)B_n)$ and where we used the inequality

$$H(\lambda A_n+(1-\lambda)B_n)\leq\lambda H(A_n)+(1-\lambda)H(B_n)+x_nh_2(x_n^{-1}\lambda\operatorname{Tr}A_n),$$

following from (12) and (13). By Lemma 1, passing to the limit as $n\to+\infty$ implies the desired inequality. △

Lemma 7. *Let $\{A_n\}$ be a sequence of operators in $\mathfrak{T}_+(\mathcal{H})$ converging in the trace norm to an operator A_0 such that $A_n\leq A_0$ for all n. Then*

$$\lim_{n\to\infty}H(A_n\,\|\,B)=H(A_0\,\|\,B),\quad\textit{for any operator } B\in\mathfrak{T}_+(\mathcal{H}).$$

Proof. We can assume that A_0 is a state. It can be represented in the form

$$A_0=\lambda_n\rho_n+(1-\lambda_n)\sigma_n,$$

where

$$\lambda_n=\operatorname{Tr}A_n,\qquad\rho_n=\frac{A_n}{\operatorname{Tr}A_n},\qquad\sigma_n=\frac{A-A_n}{1-\lambda_n}.$$

[7] Note that this stronger version cannot be proved by the method used in [14].

By Lemma 6 and by nonnegativity of the relative entropy, we have

$$\begin{aligned} H(A_0 \,\|\, B) &\geq \lambda_n H(\rho_n \,\|\, B) + (1-\lambda_n) H(\sigma_n \,\|\, B) - h_2(\lambda_n) \\ &\geq H(A_n \,\|\, \lambda_n B) - h_2(\lambda_n) \\ &= H(A_n \,\|\, B) - \operatorname{Tr} B(1-\lambda_n) - \lambda_n \log(\lambda_n) - h_2(\lambda_n), \end{aligned}$$

and hence $\limsup_{n\to\infty} H(A_n \,\|\, B) \leq H(A_0 \,\|\, B)$. By lower semicontinuity of the relative entropy, this implies the assertion of the lemma. △

Proof of Lemma 5. Let $T = \mathcal{D} \circ \Phi$. Consider the set of conditions

$$T(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|, \quad \forall |\psi\rangle \in \operatorname{supp} \rho, \tag{34}$$

$$T(|\psi\rangle\langle\varphi|) = |\psi\rangle\langle\varphi|, \quad \forall |\psi\rangle, |\varphi\rangle \in \operatorname{supp} \rho, \tag{35}$$

$$T(|e_i\rangle\langle e_j|) = |e_i\rangle\langle e_j|, \quad \forall i, j, \tag{36}$$

where $|e_i\rangle$ is the set of eigenvectors of the state ρ corresponding to nonzero eigenvalues. Then we have the following implications: Definition 6 ⇔ (34) follows from the spectral representation, (34) ⇔ (35) follows from the polarization identity, (35) ⇔ (36) is obvious, and (36) ⇔ (26) follows from formula (15). △

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