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INFORMATION THEORY

Conditions for Coincidence of the Classical Capacity and Entanglement-Assisted Capacity of a Quantum Channel[1]

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Abstract—Several relations between the Holevo capacity and entanglement-assisted classical capacity of a quantum channel are proved; necessary and sufficient conditions for their coincidence are obtained. In particular, it is shown that these capacities coincide if (respectively, only if) the channel (respectively, the χ-essential part of the channel) belongs to the class of classical-quantum channels (the χ-essential part is a restriction of a channel obtained by discarding all states that are useless for transmission of classical information). The obtained conditions and their corollaries are generalized to channels with linear constraints. By using these conditions it is shown that the question of coincidence of the Holevo capacity and entanglement-assisted classical capacity depends on the form of a constraint. Properties of the difference between quantum mutual information and the χ-function of a quantum channel are explored.

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1. INTRODUCTION

Informational properties of a quantum channel are characterized by a number of various capacities, which are defined by a type of transmitted information, additional resources used to increase the rate of this transmission, security requirements, etc.

Central roles in analysis of transmission of classical information through a quantum channel Φ are played by the Holevo capacity $\bar{C}(\Phi)$, classical (unassisted) capacity $C(\Phi)$, and entanglement-assisted (classical) capacity $C_{\mathrm{ea}}(\Phi)$ of this channel. The first of them is defined as the maximum rate of information transmission between transmitter and receiver (generally called Alice and Bob) when nonentangled block coding is used by Alice and arbitrary measurement is used by Bob; the second differs from the first by the possibility of using arbitrary block coding by Alice; while the entanglement-assisted capacity is defined as the maximum rate of information transmission between Alice and Bob under the assumption that they share a common entangled state, which can be used in block coding by Alice to increase the rate of information transmission [1–3].

By operational definitions, we have $\bar{C}(\Phi) \leq C(\Phi) \leq C_{\mathrm{ea}}(\Phi)$. During a long time it was conjectured that $\bar{C}(\Phi) = C(\Phi)$ for any channel Φ, until Hastings showed existence of a counterexample to the additivity conjecture [4]. Nevertheless, the equality $\bar{C}(\Phi) = C(\Phi)$ holds for a large class of channels including the noiseless channel, all unital qubit channels, all entanglement-breaking channels, and many other particular examples. In contract to this, possibility of the strict inequality

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$C(\Phi) < C_{\mathrm{ea}}(\Phi)$ was initially obvious, since superdense coding implies that $C_{\mathrm{ea}}(\Phi) = 2C(\Phi) > 0$ if Φ is a noiseless channel. But there exist channels for which

$$\bar{C}(\Phi) = C(\Phi) = C_{\mathrm{ea}}(\Phi) > 0 \tag{1}$$

(as an example, one can consider the channel $\rho \mapsto \sum_k \langle k|\rho|k\rangle |k\rangle\langle k|$, where $\{|k\rangle\}$ is an orthonormal basis). Hence the question naturally arises of how the class of channels for which (1) holds can be characterized. In contrast to an intuitive point of view, this class does not coincide with the class of entanglement-breaking channels: despite the fact that these channels annihilate entanglement of any state shared by Alice and Bob, their entanglement-assisted capacity may be greater than the classical unassisted capacity [1]. On the other hand, in [5] an example is described of a non-entanglement-breaking channel for which $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$ (see Example 2 in Section 2.3 below). A step towards finding an answer to the above question was recently made in [6], where a criterion of (1) for the class of q-c channels defined by quantum observables was obtained.

In the present paper some relations between the capacities $\bar{C}(\Phi)$ and $C_{\mathrm{ea}}(\Phi)$ are obtained, as well as necessary and sufficient conditions for the equality $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$ (Proposition 1 and Theorems 1 and 2). In particular, it is shown that the equality $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$ holds if (respectively, only if) the channel Φ (respectively, the χ-essential part of the channel Φ) belongs to the class of classical-quantum channels (the χ-essential part is defined as a restriction of the channel to the set of states supported by the minimal subspace containing elements of *all* ensembles optimal for this channel in the sense of the Holevo capacity; see Definition 1).

Since in dealing with infinite-dimensional channels it is necessary to impose particular constraints on the choice of input code states, we also consider conditions for coincidence of the entanglement-assisted capacity with the Holevo capacity for quantum channels with linear constraints (Propositions 4 and 5). By using these conditions it is shown that even in the case of classical-quantum channels the question of coincidence of the above capacities depends on the form of the constraint (Example 3, Proposition 6).

In Section 4, properties of the difference between quantum mutual information and the χ-function (constrained Holevo capacity) of a quantum channel (considered as a function of an input state) are studied (Theorem 3). In particular, the sense of the maximum value of this function as a parameter characterizing the "noise level" of a quantum channel is shown.

2. UNCONSTRAINED CHANNELS

Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}_E$ be finite-dimensional Hilbert spaces. In what follows, $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ is a quantum channel, and $\widehat{\Phi}\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_E)$ is its complementary channel, defined uniquely up to unitary equivalence [7].

Let $H(\rho)$ and $H(\rho\|\sigma)$ be, respectively, the von Neumann entropy of the state ρ and the quantum relative entropy of the states ρ and σ [2,3].

The Holevo capacity of the channel Φ can be defined as follows:

$$\bar{C}(\Phi) = \max_{\rho \in \mathfrak{S}(\mathfrak{H}_A)} \chi_\Phi(\rho), \tag{2}$$

where

$$\chi_\Phi(\rho) = \max_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho)) \tag{3}$$

is the χ-function of the channel Φ [8]. Note that

$$\chi_\Phi(\rho) = H(\Phi(\rho)) - \widehat{H}_\Phi(\rho), \tag{4}$$

where $\widehat{H}_\Phi(\rho) = \min_{\sum_i \pi_i\rho_i=\rho} \sum_i \pi_i H(\Phi(\rho_i))$ is the convex hull of the function $\rho \mapsto H(\Phi(\rho))$. By concavity of this function, the above minimum can be taken over ensembles of pure states. An ensemble $\{\pi_i, \rho_i\}$ of pure states is called *optimal for the channel* Φ if (cf. [9])

$$\bar{C}(\Phi) = \chi_\Phi(\bar{\rho}) = \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\bar{\rho})), \quad \bar{\rho} = \sum_i \pi_i\rho_i.$$

By the Holevo–Schumacher–Westmoreland theorem, the classical capacity of the channel Φ can be expressed by the following regularization formula:

$$C(\Phi) = \lim_{n\to+\infty} n^{-1}\bar{C}(\Phi^{\otimes n}).$$

By the Bennett–Shor–Smolin–Thapliyal theorem, the entanglement-assisted capacity of the channel Φ is determined as follows:

$$C_{\mathrm{ea}}(\Phi) = \max_{\rho\in\mathfrak{S}(\mathfrak{H}_A)} I(\rho, \Phi), \tag{5}$$

where $I(\rho, \Phi) = H(\rho) + H(\Phi(\rho)) - H(\widehat{\Phi}(\rho))$ is the quantum mutual information of the channel Φ at state ρ [2,3].

By the operational definitions, we have $\bar{C}(\Phi) \le C(\Phi) \le C_{\mathrm{ea}}(\Phi)$. Analytically, this follows (by virtue of (2) and (5)) from the following expression for the quantum mutual information:

$$I(\rho, \Phi) = H(\rho) + \chi_\Phi(\rho) - \chi_{\hat{\Phi}}(\rho) = \chi_\Phi(\rho) + \Delta_\Phi(\rho), \tag{6}$$

where $\Delta_\Phi(\rho) = H(\rho) - \chi_{\hat{\Phi}}(\rho)$. This expression is easily derived by using (4) and noting that $\widehat{H}_\Phi \equiv \widehat{H}_{\hat{\Phi}}$ (this follows from coincidence of the functions $\rho \mapsto H(\Phi(\rho))$ and $\rho \mapsto H(\widehat{\Phi}(\rho))$ on the set of pure states).

Since $H(\rho) = \sum_i \pi_i H(\rho_i\|\rho)$ for any ensemble $\{\pi_i, \rho_i\}$ of pure states with average state ρ, we have

$$\Delta_\Phi(\rho) = \min_{\substack{\sum_i \pi_i\rho_i=\rho \\ \operatorname{rank}\rho_i=1}} \sum_i \pi_i \left[H(\rho_i\|\rho) - H(\widehat{\Phi}(\rho_i)\|\widehat{\Phi}(\rho))\right] \ge 0, \tag{7}$$

where the last inequality follows from monotonicity of the relative entropy.

Remark 1. The minimum in (7) is attained at an ensemble $\{\pi_i, \rho_i\}$ of pure states if and only if the maximum in (3) is attained at this ensemble. Indeed, since $\sum_i \pi_i H(\Phi(\rho_i)) = \sum_i \pi_i H(\widehat{\Phi}(\rho_i))$, this can easily be shown by using expression (4) for χ-functions of the channels Φ and $\widehat{\Phi}$.

2.1. General Inequalities

Expression (6) immediately implies the general upper bound

$$C_{\mathrm{ea}}(\Phi) \le \bar{C}(\Phi) + \log\dim\mathfrak{H}_A,$$

proved in [10,11] by different methods. By using this expression and noting that $\chi_\Phi(\rho) - \chi_{\hat{\Phi}}(\rho) = I_c(\rho, \Phi)$ is the coherent information of the channel Φ at the state ρ (see [12]), it is easy to obtain the following inequalities[2]:

$$\begin{aligned} H(\rho_1) - \bar{C}(\widehat{\Phi}) &\le C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \\ &\le H(\rho_2) - \chi_{\hat{\Phi}}(\rho_2) \underset{H(\Phi(\cdot))\ge H(\cdot)}{\le} H(\Phi(\rho_2)) - \chi_{\hat{\Phi}}(\rho_2) = I_c(\rho_2, \Phi) + \widehat{H}_\Phi(\rho_2), \end{aligned} \tag{8}$$

[2] Here and in what follows, a subscript in the third inequality means that it holds under the condition $H(\Phi(\rho)) \ge H(\rho)$ for all $\rho \in \mathfrak{S}(\mathfrak{H}_A)$. This condition is valid, in particular, for all bistochastic channels.

where ρ_1 and ρ_2 are states in $\mathfrak{S}(\mathfrak{H}_A)$ such that $\chi_\Phi(\rho_1) = \bar{C}(\Phi)$ (i.e., ρ_1 is the average state of an optimal ensemble) and $I(\rho_2, \Phi) = C_{\mathrm{ea}}(\Phi)$.

Let $Q_1(\Phi) = \max\limits_{\rho\in\mathfrak{S}(\mathfrak{H}_A)} I_c(\rho, \Phi)$, and let $Q(\Phi) = \lim\limits_{n\to+\infty} n^{-1} Q_1(\Phi^{\otimes n})$ be the quantum capacity of the channel Φ [2,3]. The following proposition contains several estimations derived from (8).

Proposition 1. *Let $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ be a quantum channel and $\widehat{\Phi}\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_E)$ its complementary channel.*

(A) *We have the inequalities*

$$\bar{C}(\Phi) - \bar{C}(\widehat{\Phi}) \le C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \underset{H(\Phi(\cdot))\ge H(\cdot)}{\le} Q_1(\Phi) + \min \sum_i \pi_i H(\Phi(\rho_i)), \tag{9}$$

$$C(\Phi) - C(\widehat{\Phi}) \le C_{\mathrm{ea}}(\Phi) - C(\Phi) \underset{H(\Phi(\cdot))\ge H(\cdot)}{\le} Q(\Phi) + \min \sum_i \pi_i H(\Phi(\rho_i)), \tag{10}$$

where the minimum is over all ensembles $\{\pi_i, \rho_i\}$ of pure states such that $I\Big(\sum\limits_i \pi_i\rho_i, \Phi\Big) = C_{\mathrm{ea}}(\Phi)$. This term can be replaced with $\max\limits_{\rho\in \operatorname{extr}\mathfrak{S}(\mathfrak{H}_A)} H(\Phi(\rho))$.

(B) *If the average state of at least one optimal ensemble for the channel Φ coincides with the chaotic state $\rho_c = (\dim\mathfrak{H}_A)^{-1} I_A$, then*

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \ge \log\dim\mathfrak{H}_A - \bar{C}(\widehat{\Phi})$$

and hence $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi) \ \Rightarrow\ \bar{C}(\widehat{\Phi}) = \log\dim\mathfrak{H}_A$.[3]

(C) *If $C_{\mathrm{ea}}(\Phi) = I(\rho_c, \Phi)$, then $\bar{C}(\widehat{\Phi}) = \log\dim\mathfrak{H}_A \ \Rightarrow\ \bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$. If, in addition, the average state of at least one optimal ensemble for the channel $\widehat{\Phi}$ coincides with the chaotic state ρ_c, then*

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \le \log\dim\mathfrak{H}_A - \bar{C}(\widehat{\Phi}).$$

Proof. (A) Inequality (9) directly follows from (8). To obtain inequality (10) by regularization from (8), it is sufficient to note that the function $\mathfrak{S}(\mathfrak{H}_A^{\otimes n}) \ni \omega \mapsto I(\omega, \Phi^{\otimes n})$ attains its maximum at the state $\rho_2^{\otimes n}$ by subadditivity of the quantum mutual information and to use the obvious inequality $\widehat{H}_{\Phi^{\otimes n}}(\rho_2^{\otimes n}) \le n\widehat{H}_\Phi(\rho_2)$.

(B) This assertion directly follows from inequality (8).

(C) To derive the first part of this assertion from inequality (8), note that $\bar{C}(\widehat{\Phi}) = \log\dim\mathfrak{H}_A$ implies $\bar{C}(\widehat{\Phi}) = \chi_{\widehat{\Phi}}(\rho_c)$. The second part directly follows from the second inequality in (8). △

Remark 2. Since $\bar{C}(\widehat{\Phi}) \le \log\dim\mathfrak{H}_E$, we have

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \ge \log\dim\mathfrak{H}_A - \log\dim\mathfrak{H}_E$$

for any channel Φ satisfying the condition of Proposition 1 (B), and hence $C_{\mathrm{ea}}(\Phi) > \bar{C}(\Phi)$ if the dimension of the environment (i.e., the minimal number of Kraus operators) is less than the dimension of the input space of the channel Φ.

For an arbitrary channel Φ, inequality (8) implies

$$C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) \ge H(\bar{\rho}) - \log\dim\mathfrak{H}_E \ge \bar{C}(\Phi) - \log\dim\mathfrak{H}_E,$$

where $\bar{\rho}$ is the average state of any optimal ensemble for the channel Φ.

[3] Note that $\bar{C}(\widehat{\Phi}) \le \log\dim\mathfrak{H}_A$ for any channel Φ.

2.2. Conditions for the Equality $\bar{C}(\Phi) = C_{\rm ea}(\Phi)$ Based on the Petz Theorem

By using expressions (6) and (7), monotonicity of the relative entropy, and the Petz theorem [13, Theorem 3] characterizing the case in which monotonicity of the relative entropy holds with an equality, the following necessary and sufficient conditions for the equality $\bar{C}(\Phi) = C_{\rm ea}(\Phi)$ can be obtained.

Theorem 1. *Let $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ be a quantum channel and $\widehat{\Phi}\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_E)$ its complementary channel.*

(A) *If there exist a channel $\Theta\colon \mathfrak{S}(\mathfrak{H}_E) \to \mathfrak{S}(\mathfrak{H}_A)$ and an ensemble $\{\pi_i, \rho_i\}$ of pure states such that*

$$\Theta(\widehat{\Phi}(\rho_i)) = \rho_i, \quad \forall i, \tag{11}$$

and $I(\bar{\rho}, \Phi) = C_{\rm ea}(\Phi)$, where $\bar{\rho} = \sum\limits_i \pi_i \rho_i$, then $\bar{C}(\Phi) = C_{\rm ea}(\Phi)$.[4]

(B) *If $\bar{C}(\Phi) = C_{\rm ea}(\Phi)$, then for an arbitrary optimal ensemble $\{\pi_i, \rho_i\}$ of pure states for the channel Φ with average state $\bar{\rho}$ there exists a channel $\Theta\colon \mathfrak{S}(\mathfrak{H}_E) \to \mathfrak{S}(\mathfrak{H}_A)$ such that* (11) *holds. The channel Θ can be defined by means of an arbitrary nondegenerate probability distribution $\{\widehat{\pi}_i\}$ if we set its action on any state σ supported by a subspace* $\operatorname{supp}\widehat{\Phi}(\bar{\rho})$ *as follows*:

$$\Theta(\sigma) = [\widehat{\rho}]^{1/2}\widehat{\Phi}^*\left(\left[\widehat{\Phi}(\widehat{\rho})\right]^{-1/2}\sigma\left[\widehat{\Phi}(\widehat{\rho})\right]^{-1/2}\right)[\widehat{\rho}]^{1/2}, \tag{12}$$

where $\widehat{\rho} = \sum\limits_i \widehat{\pi}_i \rho_i$ and $\widehat{\Phi}^*$ is a dual map to the channel $\widehat{\Phi}$.

If $\{\widehat{\pi}_i\}$ is a degenerate probability distribution, then relation (11) *holds for the channel Θ defined by* (12) *for all i such that $\widehat{\pi}_i > 0$.*

Proof. (A) If $\{\pi_i, \rho_i\}$ is an ensemble of pure states with average state $\bar{\rho}$ for which (11) holds, then monotonicity of the relative entropy and (7) imply $\Delta_\Phi(\bar{\rho}) = 0$, and hence $C_{\rm ea}(\Phi) = I(\bar{\rho}, \Phi) = \chi_\Phi(\bar{\rho}) \le \bar{C}(\Phi)$.

(B) Since $\chi_\Phi(\rho) \le I(\rho, \Phi)$ for any state ρ by (6), it is easy to see that $\bar{C}(\Phi) = C_{\rm ea}(\Phi)$ implies $\chi_\Phi(\bar{\rho}) = I(\bar{\rho}, \Phi)$ for any an optimal ensemble $\{\pi_i, \rho_i\}$ of pure states with average state $\bar{\rho}$. It follows from (7) and Remark 1 that

$$H(\rho_i \| \bar{\rho}) = H(\widehat{\Phi}(\rho_i) \| \widehat{\Phi}(\bar{\rho})), \quad \forall i.$$

Hence, the Petz theorem [13, Theorem 3] implies existence of a channel Θ for which (11) holds. By monotonicity of the relative entropy, for an arbitrary probability distribution $\{\widehat{\pi}_i\}$ we have

$$H(\rho_i \| \widehat{\rho}) = H(\widehat{\Phi}(\rho_i) \| \widehat{\Phi}(\widehat{\rho})), \quad \widehat{\rho} = \sum_i \widehat{\pi}_i \rho_i,$$

for all i such that $\widehat{\pi}_i > 0$. Hence, the formula for the channel Θ also follows from the Petz theorem. △

Theorem 1 (A) makes it possible to prove the equality $C_{\rm ea}(\Phi) = \bar{C}(\Phi)$ for all classical-quantum channels (see Theorem 2 in Section 2.3).

Theorem 1 (B) can be used to prove the strict inequality $C_{\rm ea}(\Phi) > \bar{C}(\Phi)$ by showing that (11) cannot be valid for an optimal ensemble $\{\pi_i, \rho_i\}$ and for the channel Θ defined by (12).

Example 1. Consider the entanglement-breaking channel

$$\Phi(\rho) = \sum_k \langle \varphi_k | \rho | \varphi_k \rangle |k\rangle\langle k|,$$

[4] It is sufficient to require that Θ is a trace-preserving positive map for which monotonicity of the relative entropy holds.

where $\{|\varphi_k\rangle\}$ is an overcomplete system of vectors in the space $\mathfrak{H}_A$ (i.e., $\sum_k |\varphi_k\rangle\langle\varphi_k| = I_A$) and $\{|k\rangle\}$ is an orthonormal basis in the space $\mathfrak{H}_B$. It is easy to see that $\Phi = \widehat{\Phi}$. Hence, $I(\rho,\Phi) = H(\rho)$ and $C_{\mathrm{ea}}(\Phi) = \log\dim\mathfrak{H}_A$. Assume that $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi) = \log\dim\mathfrak{H}_A$. Then the average state of any optimal ensemble $\{\pi_i,\rho_i\}$ for the channel Φ coincides with the chaotic state ρ_c in $\mathfrak{S}(\mathfrak{H}_A)$. Since $\widehat{\Phi}^*(A) = \sum_k \langle k|A|k\rangle|\varphi_k\rangle\langle\varphi_k|$ and $\widehat{\Phi}(\rho_c) = \Phi(\rho_c)$ is a full-rank state, relation (11) can be valid for the channel Θ defined by (12) only if $\rho_i = |\varphi_{k_i}\rangle\langle\varphi_{k_i}|$ for some k_i and $\operatorname{rank}\widehat{\Phi}(|\varphi_{k_i}\rangle\langle\varphi_{k_i}|) = \operatorname{rank}\sum_k \langle\varphi_k|\varphi_{k_i}\rangle\langle\varphi_{k_i}|\varphi_k\rangle|k\rangle\langle k| = 1$ for all i. But this can be valid only if $\{|\varphi_k\rangle\}$ is an orthonormal basis. Thus, we conclude that

$$C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi) \quad \Leftrightarrow \quad \{|\varphi_k\rangle\} \text{ is an orthonormal basis.}$$

The same conclusion was obtained in [6] as a corollary of a general criterion for the equality $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$ for the class of channels defined by quantum observables, which is proved by means of the ensemble-measurement duality.

2.3. A Simple Criterion for the Equality $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$

Now we will show that the equality $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$ holds if (respectively, only if) the channel Φ (respectively, the subchannel of Φ determining its classical capacity) belongs to the class of classical-quantum channels.

A channel $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ is said to be *classical-quantum* if it has the following representation:

$$\Phi(\rho) = \sum_{k=1}^{\dim\mathfrak{H}_A} \langle k|\rho|k\rangle\sigma_k, \quad \rho\in\mathfrak{S}(\mathfrak{H}_A), \tag{13}$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathfrak{H}_A$ and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathfrak{H}_B)$ [2,3].

For correct formulation of the above statement, we will need the following notion.

Definition 1. Let $\mathfrak{H}_\Phi^\chi$ be the minimal subspace of $\mathfrak{H}_A$ containing elements of all optimal ensembles for the channel $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$. The restriction Φ_χ of the channel Φ to the set $\mathfrak{S}(\mathfrak{H}_\Phi^\chi)$ is called the χ-*essential* part (subchannel) of the channel Φ.

If $\mathfrak{H}_\Phi^\chi \neq \mathfrak{H}_A$, then pure states corresponding to vectors in $\mathfrak{H}_A \setminus \mathfrak{H}_\Phi^\chi$ cannot be used as elements of an optimal ensemble for the channel Φ. This means, roughly speaking, that these states are useless for nonentangled coding of classical information, and hence it is natural to consider the χ-essential subchannel Φ_χ instead of the channel Φ when dealing with the Holevo capacity of the channel Φ (which coincides with the classical capacity if $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$).

By definition, $\bar{C}(\Phi_\chi) = \bar{C}(\Phi)$. Hence, $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$ implies $C_{\mathrm{ea}}(\Phi_\chi) = C_{\mathrm{ea}}(\Phi)$. Thus, in this case, when speaking about the entanglement-assisted capacity of the channel Φ, we may also consider the χ-essential subchannel Φ_χ instead of the channel Φ.

Theorem 1 makes it possible to prove the following assertions.

Theorem 2. *Let* $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ *be a quantum channel.*

(A) *If* Φ *is a classical-quantum channel, then* $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$.

(B) *If* $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$, *then the* χ*-essential part of the channel* Φ *is a classical-quantum channel.*

Example 2 below shows that in general the χ-essential part of the channel Φ in Theorem 2 (B) cannot be replaced by the channel Φ.

Proof. (A) If Φ has representation (13), then $\Phi = \Phi\circ\Pi$, where $\Pi(\rho) = \sum_k \langle k|\rho|k\rangle|k\rangle\langle k|$ is a channel from $\mathfrak{S}(\mathfrak{H}_A)$ to itself.

It is easy (see [14, proof of Lemma 17]) to show existence of a channel Θ such that $\Theta\circ\widehat{\Phi\circ\Pi} = \widehat{\Pi} = \Pi$.

By the chain rule for the quantum mutual information (see [2, 3]), we have

$$I(\rho,\Phi) = I(\rho,\Phi\circ\Pi) \leq I(\Pi(\rho),\Phi).$$

It follows that the function $\rho\mapsto I(\rho,\Phi)$ attains its maximum at a state diagonizable in the basis $\{|k\rangle\}$. Since $\Theta\circ\widehat{\Phi\circ\Pi}(|k\rangle\langle k|) = \Pi(|k\rangle\langle k|) = |k\rangle\langle k|$ for any k, Theorem 1 (A) implies the equality $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$.

(B) Replacing the channel Φ by its χ-essential subchannel, we may assume that $\mathfrak{H}_\Phi^\chi = \mathfrak{H}_A$.

Let $\Phi(\rho) = \sum\limits_{i=1}^{n} V_i\rho V_i^*$ be a minimal Kraus representation of the channel Φ. Then

$$\widehat{\Phi}(\rho) = \sum_{i,j=1}^{n} \operatorname{Tr} V_i\rho V_j^* |i\rangle\langle j| \qquad \text{and} \qquad \widehat{\Phi}^*(A) = \sum_{i,j=1}^{n} \langle j|A|i\rangle V_j^* V_i,$$

where $\{|i\rangle\}_{i=1}^n$ is an orthonormal basis in the n-dimensional Hilbert space $\mathfrak{H}_E$.

Let $\{\pi_k, |\varphi_k\rangle\langle\varphi_k|\}$ be an optimal ensemble of pure states for the channel Φ with a full-rank average state. We may assume that $\{|\varphi_k\rangle\}_{k=1}^m$, $m = \dim\mathfrak{H}_A$, is a basis in the space $\mathfrak{H}_A$. Let $\widehat{\pi}_k = 1/m$, $k = \overline{1,m}$. Then $\widehat{\rho} = \sum\limits_{k=1}^{m} \widehat{\pi}_k|\varphi_k\rangle\langle\varphi_k|$ is a full-rank state in $\mathfrak{S}(\mathfrak{H}_A)$. Since $\mathfrak{H}_E$ is an environment space of minimal dimension, $\widehat{\Phi}(\widehat{\rho})$ is a full-rank state in $\mathfrak{S}(\mathfrak{H}_E)$.

Let $|\phi_k\rangle = \sqrt{\widehat{\pi}_k\widehat{\rho}^{-1}}|\varphi_k\rangle$ and $B_k = \widehat{\pi}_k\big[\widehat{\Phi}(\widehat{\rho})\big]^{-1/2}\widehat{\Phi}(|\varphi_k\rangle\langle\varphi_k|)\big[\widehat{\Phi}(\widehat{\rho})\big]^{-1/2}$, $k = \overline{1,m}$. Since $\sum\limits_{k=1}^{m} |\phi_k\rangle\langle\phi_k| = I_{\mathfrak{H}_A}$, $\{|\phi_k\rangle\}_{k=1}^m$ is an orthonormal basis in $\mathfrak{H}_A$. By Theorem 1 (B), $|\phi_k\rangle\langle\phi_k| = \widehat{\Phi}^*(B_k)$ for all k. By the spectral theorem, we have $B_k = \sum\limits_p |\psi_k^p\rangle\langle\psi_k^p|$, where $\{|\psi_k^p\rangle\}_p$ is a set of vectors in $\mathfrak{H}_E$, for each k. Since $\widehat{\Phi}(\widehat{\rho})$ is a full-rank state, we have

$$\sum_{k,p} |\psi_k^p\rangle\langle\psi_k^p| = \sum_k B_k = I_E.$$

By Lemma 1 below, we have $\Phi(\rho) = \sum\limits_{k,p} W_{kp}\rho W_{kp}^*$, where $W_{kp} = \sum\limits_{i=1}^{n} \langle\psi_k^p|i\rangle V_i$.

Since $|\phi_k\rangle\langle\phi_k| = \widehat{\Phi}^*\Big(\sum\limits_p |\psi_k^p\rangle\langle\psi_k^p|\Big)$ for each k and

$$\widehat{\Phi}^*(|\psi_k^p\rangle\langle\psi_k^p|) = \sum_{i,j=1}^{n} \langle j|\psi_k^p\rangle\langle\psi_k^p|i\rangle V_j^* V_i = W_{kp}^* W_{kp},$$

there exists a collection $\{|\beta_{kp}\rangle\}$ of vectors in $\mathfrak{H}_B$ such that $W_{kp} = |\beta_{kp}\rangle\langle\phi_k|$ and $\sum\limits_p \|\beta_{kp}\|^2 = 1$ for each k. Hence,

$$\Phi(\rho) = \sum_{k,p} W_{kp}\rho W_{kp}^* = \sum_k \langle\phi_k|\rho|\phi_k\rangle \sum_p |\beta_{kp}\rangle\langle\beta_{kp}|. \quad \triangle$$

Lemma 1. *Let $\Phi(\rho) = \sum\limits_{i=1}^{n} V_i\rho V_i^*$ be a quantum channel and $\{|i\rangle\}_{i=1}^n$ be an orthonormal basis in the n-dimensional Hilbert space $\mathfrak{H}_E$. An arbitrary overcomplete system $\{|\psi_k\rangle\}_k$ of vectors in $\mathfrak{H}_E$ generates the Kraus representation $\Phi(\rho) = \sum\limits_k W_k\rho W_k^*$ of the channel Φ, where $W_k = \sum\limits_{i=1}^{n} \langle\psi_k|i\rangle V_i$.*

Proof. Since $\sum_k |\psi_k\rangle\langle\psi_k| = I_E$, we have

$$\begin{aligned}\sum_k W_k\rho W_k^* &= \sum_{i,j=1}^n V_i\rho V_j^* \sum_k \langle\psi_k|i\rangle\langle j|\psi_k\rangle \\ &= \sum_{i,j=1}^n V_i\rho V_j^* \sum_k \operatorname{Tr}|i\rangle\langle j||\psi_k\rangle\langle\psi_k| \\ &= \sum_{i=1}^n V_i\rho V_i^*. \quad \triangle\end{aligned}$$

Remark 3. The assertions of Theorem 2 agree with the criterion for the equality $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$ obtained in [6] for the quantum-classical channel

$$\Phi(\rho) = \sum_k [\operatorname{Tr} M_k\rho]|k\rangle\langle k|$$

defined by a collection $\{M_k\}$ of positive operators in $\mathfrak{H}_A$ such that $\sum_k M_k = I_A$, where $\{|k\rangle\}$ is an orthonormal basis in $\mathfrak{H}_B$. Indeed, it is easy to see that this channel is classical-quantum if and only if $M_kM_l = M_lM_k$ for all k and l.

Since $\mathfrak{H}_\Phi^\chi = \mathfrak{H}_A$ means existence of an optimal ensemble for the channel Φ with a full-rank average state, Theorem 2 implies the following criterion for coincidence of the capacities.

Corollary 1. *Let Φ be a quantum channel for which there exists an optimal ensemble with a full-rank average state. Then*

$$C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi) \quad \Leftrightarrow \quad \Phi \textit{ is a classical-quantum channel.}$$

The following example, proposed in [5] (as an example of a non-entanglement-breaking channel such that $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$), shows that the condition in Corollary 1 is essential.

Example 2. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_3$ be qubit spaces. Let $\{|k\rangle\}_{k=1}^4$ and $\{|-\rangle, |+\rangle\}$ be orthonormal bases in $\mathfrak{K} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ and in $\mathfrak{H}_3$, respectively. Consider the channel

$$\Phi(\rho) = \sum_{k=1}^4 [(\langle k| \otimes \langle +|]\,\rho\,[|k\rangle \otimes |+\rangle]\,|k\rangle\langle k| + \frac{1}{2} I_{\mathfrak{H}_2} \otimes \operatorname{Tr}_{\mathfrak{H}_2\otimes\mathfrak{H}_3}[I_{\mathfrak{K}} \otimes |-\rangle\langle -|]\rho$$

from $\mathfrak{S}(\mathfrak{K} \otimes \mathfrak{H}_3)$ to $\mathfrak{S}(\mathfrak{K})$. It is easy to show that $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi) = 2$ and $Q(\Phi) = 1$ [5]. Thus, the channel Φ is non-entanglement-breaking, and hence it is not classical-quantum.

Since $\bar{C}(\Phi) = 2 = \log\dim\mathfrak{K}$, any optimal ensemble for the channel Φ cannot contain states with nonzero output entropy. Thus, the subspace $\mathfrak{H}_\Phi^\chi$ consists of vectors $|\varphi\rangle \otimes |+\rangle$, $|\varphi\rangle \in \mathfrak{K}$. Hence, the χ-essential part of Φ is isomorphic to the classical-quantum channel $\rho \mapsto \sum_{k=1}^4 \langle k|\rho|k\rangle|k\rangle\langle k|$ (in accordance with Theorem 2 (B)).

2.4. On Covariant Channels

The class of channels for which the conditions of parts (B) and (C) of Proposition 1 and of Corollary 1 hold simultaneously contains any channel Φ *covariant* with respect to representations $\{V_g\}_{g\in G}$ and $\{W_g\}_{g\in G}$ of a compact group G in the sense that

$$\Phi(V_g\rho V_g^*) = W_g\Phi(\rho)W_g^*, \quad \forall g \in G, \tag{14}$$

provided that the representation $\{V_g\}_{g\in G}$ is irreducible. Indeed, irreducibility of the representation $\{V_g\}_{g\in G}$ implies

$$\rho_c \doteq (\dim\mathfrak{H}_A)^{-1} I_A = \int_G V_g \rho V_g^* \,\mu_H(dg), \quad \forall \rho \in \mathfrak{S}(\mathfrak{H}_A), \tag{15}$$

where μ_H is the Haar measure on the group G [10]. Thus, to prove that

$$\bar{C}(\Phi) = \chi_\Phi(\rho_c), \qquad \bar{C}(\widehat{\Phi}) = \chi_{\hat{\Phi}}(\rho_c), \qquad \text{and} \qquad C_{\mathrm{ea}}(\Phi) = I(\rho_c, \Phi), \tag{16}$$

it is sufficient, by concavity of the χ-function and quantum mutual information, to show that

$$\chi_\Phi(\rho) = \chi_\Phi(V_g \rho V_g^*), \qquad \chi_{\hat{\Phi}}(\rho) = \chi_{\hat{\Phi}}(V_g \rho V_g^*), \qquad \text{and} \qquad I(\rho, \Phi) = I(V_g \rho V_g^*, \Phi), \tag{17}$$

for all $g \in G$ and $\rho \in \mathfrak{S}(\mathfrak{H}_A)$.

The first and third equalities in (17) can easily be proved by using (3) and the well-known expression for the quantum mutual information via the relative entropy (by means of invariance of the relative entropy with respect to unitary transformations of both of their arguments). By these equalities, the second equality follows from (6).

The class of covariant channels is large enough; it contains all unital qubit channels and nontrivial classes of channels in higher dimensions [10, 15].

By using (15) and (16), it is easy to show that (cf. [10])

$$\begin{gathered} \bar{C}(\Phi) = H(\Phi(\rho_c)) - H_{\min}(\Phi), \qquad \bar{C}(\widehat{\Phi}) = H(\widehat{\Phi}(\rho_c)) - H_{\min}(\Phi), \\ C_{\mathrm{ea}}(\Phi) = \log\dim\mathfrak{H}_A + H(\Phi(\rho_c)) - H(\widehat{\Phi}(\rho_c)), \end{gathered} \tag{18}$$

for any channel $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ satisfying the above covariance condition, where $H_{\min}(\Phi) = \min\limits_{\rho\in\mathfrak{S}(\mathfrak{H}_A)} H(\Phi(\rho))$ is the minimal output entropy of the channel Φ (coinciding with $H_{\min}(\widehat{\Phi})$). If, in addition, the representation $\{W_g\}_{g\in G}$ is also irreducible, then $H(\Phi(\rho_c))$ in (18) can be replaced by $\log\dim\mathfrak{H}_B$ [10].

Let $Q_1(\Phi) = \max\limits_{\rho\in\mathfrak{S}(\mathfrak{H}_A)} I_c(\rho, \Phi)$ and $Q(\Phi) = \lim\limits_{n\to+\infty} n^{-1} Q_1(\Phi^{\otimes n})$ be the quantum capacity of the channel Φ. By the above observations, Proposition 1 and Corollary 1 imply the following assertions.

Proposition 2. *Let $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ be a channel satisfying the covariance condition* (14)*. Then*

$$C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi) \quad \Leftrightarrow \quad \Phi \text{ is a classical-quantum channel.}$$

If, in addition, $\dim\mathfrak{H}_B \ge \dim\mathfrak{H}_A$ *and the representation* $\{W_g\}_{g\in G}$ *is irreducible, then*

$$\begin{gathered} C_{\mathrm{ea}}(\Phi) - \bar{C}(\Phi) = \log\dim\mathfrak{H}_A - \bar{C}(\widehat{\Phi}) \le Q_1(\Phi) + H_{\min}(\Phi), \\ C_{\mathrm{ea}}(\Phi) - C(\Phi) = \log\dim\mathfrak{H}_A - C(\widehat{\Phi}) \le Q(\Phi) + H_{\min}(\Phi). \end{gathered}$$

Proof. If the representation $\{W_g\}_{g\in G}$ is irreducible, it is easy to show that $\Phi((\dim\mathfrak{H}_A)^{-1} I_A) = (\dim\mathfrak{H}_B)^{-1} I_B$ [10]. This equality and the condition $\dim\mathfrak{H}_B \ge \dim\mathfrak{H}_A$ imply $H(\Phi(\rho)) \ge H(\rho)$ for any $\rho \in \mathfrak{S}(\mathfrak{H}_A)$ by monotonicity of the relative entropy. Coincidence of the last term in (9) and (10) with $H_{\min}(\Phi)$ follows from (15) and (16). △

2.5. On Degradable and Anti-degradable Channels

Expression (6) and the chain rule for the χ-function (i.e., $\chi_{\Psi\circ\Phi} \le \chi_\Phi$) show that

$$C_{\mathrm{ea}}(\Phi_1) \le \log\dim\mathfrak{H}_A \le C_{\mathrm{ea}}(\Phi_2) \tag{19}$$

for any anti-degradable channel Φ_1 and any degradable channel Φ_2.[5] By using the Petz theorem [13, Theorem 3], one can show that if the first (respectively, second) inequality in (19) holds with an equality, then the anti-degradable channel Φ_1 is degradable (respectively, the degradable channel Φ_2 is anti-degradable).

The second inequality in (19) and Theorem 2 imply the following assertion.

Proposition 3. *If* $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ *is a degradable channel, then one of the following alternatives holds*:

- $\bar{C}(\Phi) < C_{\mathrm{ea}}(\Phi)$;
- Φ *is a classical-quantum channel having the representation*

$$\Phi(\rho) = \sum_{k=1}^{\dim \mathfrak{H}_A} \langle k|\rho|k\rangle \sigma_k, \quad \rho \in \mathfrak{S}(\mathfrak{H}_A), \tag{20}$$

where $\{|k\rangle\}$ *is an orthonormal basis in* $\mathfrak{H}_A$ *and* $\{\sigma_k\}$ *is a collection of states in* $\mathfrak{S}(\mathfrak{H}_B)$ *with mutually orthogonal supports.*

Proof. Assume that $\bar{C}(\Phi) = C_{\mathrm{ea}}(\Phi)$. Since $\bar{C}(\Phi) \le \log\dim\mathfrak{H}_A$ for any channel Φ, the second inequality in (19) shows that $\bar{C}(\Phi) = \log\dim\mathfrak{H}_A$ and hence the average state of any optimal ensemble for the channel Φ coincides with the chaotic state in $\mathfrak{S}(\mathfrak{H}_A)$. By Corollary 1, Φ is a classical-quantum channel having representation (20) in which $\{|k\rangle\}$ is an orthonormal basis in $\mathfrak{H}_A$ and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathfrak{H}_B)$. We will show that supports of these states are mutually orthogonal.

Let $\sigma_k = \sum_{i=1}^{\dim\mathfrak{H}_B} |\psi_{ki}\rangle\langle\psi_{ki}|$. Then $\Phi(\rho) = \sum_{k,i} W_{ki}\rho W_{ki}^*$, where $W_{ki} = |\psi_{ki}\rangle\langle k|$, and by using a standard representation for a complementary channel (cf. [7]) we obtain

$$\widehat{\Phi}(\rho) = \sum_{k,l=1}^{\dim\mathfrak{H}_A} \langle k|\rho|l\rangle |k\rangle\langle l| \otimes \sum_{i,j=1}^{\dim\mathfrak{H}_B} \langle\psi_{lj}|\psi_{ki}\rangle |i\rangle\langle j| \in \mathfrak{S}(\mathfrak{H}_A\otimes\mathfrak{H}_B).$$

Since Φ is a degradable channel having representation (20), $\widehat{\Phi}(|k\rangle\langle l|) = \Psi\circ\Phi(|k\rangle\langle l|) = 0$ for all $k \ne l$. Hence, the above expression for the channel $\widehat{\Phi}$ implies $\langle\psi_{lj}|\psi_{ki}\rangle = 0$ for all i and j and all $k \ne l$. It follows that $\operatorname{supp}\sigma_k \perp \operatorname{supp}\sigma_l$ for all $k \ne l$. △

3. ON CHANNELS WITH LINEAR CONSTRAINTS

When defining various capacities of channels between finite-dimensional quantum systems, we may use any states for information coding. But when dealing with real infinite-dimensional channels, we have to impose particular constraints on the choice of input code states to avoid infinite values of the capacities and provide consistence with the physical implementation of the process of information transmission. A typical physically motivated constraint is defined by the requirement of bounded energy of states used for information coding. This constraint can be called linear, since it is determined by the linear inequality

$$\operatorname{Tr} H\rho \le h, \quad h > 0, \tag{21}$$

where H is a positive operator, the Hamiltonian of the input quantum system. Operational definitions of the Holevo capacity and of unassisted and entanglement-assisted classical capacities of

[5] A channel Φ is said to be degradable if $\widehat{\Phi} = \Psi\circ\Phi$ for some channel Ψ, and anti-degradable if $\widehat{\Phi}$ is a degradable channel [14].

a quantum channel with linear constraints are given in [16], where the corresponding generalizations of the Holevo–Schumacher–Westmoreland and Bennett–Shor–Smolin–Thapliyal theorems are proved.

The aim of this section is studying relations between the above capacities of a quantum channel with linear constraints, in particular showing that the question of coincidence of these capacities for a given channel depends on the form of the constraint.

For simplicity we restrict our attention to a finite-dimensional case.

The Holevo capacity of the channel Φ with constraint (21) can be defined as follows:

$$\bar{C}(\Phi, H, h) = \max_{\operatorname{Tr} H\rho \le h} \chi_\Phi(\rho),$$

where χ_Φ is the χ-function of the channel Φ defined in (3). An ensemble $\{\pi_i, \rho_i\}$ of pure states with average state $\bar{\rho}$ is called *optimal* for the channel Φ with constraint (21) if

$$\bar{C}(\Phi, H, h) = \chi_\Phi(\bar{\rho}) = \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})) \qquad \text{and} \qquad \operatorname{Tr} H\bar{\rho} \le h.$$

By the generalized Holevo–Schumacher–Westmoreland theorem [16, Proposition 3], the classical capacity of the channel Φ with constraint (21) can be expressed by the following regularization formula:

$$C(\Phi, H, h) = \lim_{n \to +\infty} n^{-1} \bar{C}(\Phi^{\otimes n}, H_n, nh),$$

where $H_n = H \otimes I \otimes \ldots \otimes I + I \otimes H \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes H$ (each of the n terms consists of n factors).

By the generalized Bennett–Shor–Smolin–Thapliyal theorem [16, Proposition 4], the entanglement-assisted capacity of the channel Φ with constraint (21) is given by

$$C_{\text{ea}}(\Phi, H, h) = \max_{\operatorname{Tr} H\rho \le h} I(\rho, \Phi),$$

where $I(\rho, \Phi)$ is the quantum mutual information of the channel Φ at the state ρ, defined after (5).

Almost all the results of Section 2 concerning relations between the capacities $\bar{C}(\Phi)$ and $C_{\text{ea}}(\Phi)$ can be reformulated for the corresponding capacities of a constrained channel. For example, instead of (8) we have

$$\begin{aligned} H(\rho_1) - \bar{C}(\widehat{\Phi}, H, h) &\le C_{\text{ea}}(\Phi, H, h) - \bar{C}(\widehat{\Phi}, H, h) \\ &\le H(\rho_2) - \chi_{\hat{\Phi}}(\rho_2) \underset{H(\Phi(\cdot)) \ge H(\cdot)}{\le} H(\Phi(\rho_2)) - \chi_{\hat{\Phi}}(\rho_2) = I_c(\rho_2, \Phi) + \widehat{H}_\Phi(\rho_2), \end{aligned}$$

where ρ_1 and ρ_2 are states in $\mathfrak{S}(\mathfrak{H}_A)$ such that $\operatorname{Tr} H\rho_i \le h$, $i = 1, 2$, $\chi_\Phi(\rho_1) = \bar{C}(\Phi, H, h)$, and $I(\rho_2, \Phi) = C_{\text{ea}}(\Phi, H, h)$.

By repeating the corresponding proofs, it is easy to obtain the following proposition.

Proposition 4. *The assertions of Proposition* 1, *Theorem* 1, *and Theorem* 2 (B) *remain valid with* $\bar{C}(\Phi)$ *and* $C_{\text{ea}}(\Phi)$ *replaced, respectively, by* $\bar{C}(\Phi, H, h)$ *and* $C_{\text{ea}}(\Phi, H, h)$ (*under the natural definition of the* χ*-essential part of the channel* Φ *with constraint* (21)). *The assertions of Theorem* 2 (A) *remains valid under this replacement if the basis* $\{|k\rangle\}$ *in representation* (13) *of the channel* Φ *consists of eigenvectors of the operator* H.

The following example shows that the assertion of Theorem 2 (A) without the above-given additional condition is not valid for constrained channels.

Example 3. Consider the classical-quantum channel

$$\Pi(\rho)=\sum_k\langle k|\rho|k\rangle|k\rangle\langle k|,$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathfrak{H}_A=\mathfrak{H}_B$. Let $h<(\dim\mathfrak{H}_A)^{-1}\operatorname{Tr}H$.

By using the generalized version of Theorem 1, we will show that $C_{\mathrm{ea}}(\Pi,H,h)=\bar{C}(\Pi,H,h)$ *if and only if the operator* H *is diagonizable in the basis* $\{|k\rangle\}$.

Since $\Pi=\widehat{\Pi}$, we have $I(\rho,\Pi)=H(\rho)$ and $C_{\mathrm{ea}}(\Pi,H,h)=\max\limits_{\operatorname{Tr}H\rho\leq h}H(\rho)$. By using the Lagrange method, it is easy to show that the above maximum is attained at a unique state $\rho_*=(\operatorname{Tr}\exp(-\lambda H))^{-1}\exp(-\lambda H)$, where λ is given by the equation $\operatorname{Tr}H\exp(-\lambda H)=h\operatorname{Tr}\exp(-\lambda H)$. If $C_{\mathrm{ea}}(\Pi,H,h)=\bar{C}(\Pi,H,h)$, then Theorem 1 implies existence of an ensemble $\{\pi_i,\rho_i\}$ of pure states with average state ρ_* such that

$$\rho_i=\rho_*^{1/2}\Pi^*\left([\Pi(\rho_*)]^{-1/2}\Pi(\rho_i)[\Pi(\rho_*)]^{-1/2}\right)\rho_*^{1/2},\quad\forall i.$$

Since $\Pi^*=\Pi$ and ρ_* is a full-rank state, this equality can be valid only if $\rho_i=|k\rangle\langle k|$ for some k. Thus, $\{|k\rangle\}$ is a basis of eigenvectors for the state ρ_* and hence for the operator H.

If the operator H is diagonizable in the basis $\{|k\rangle\}$, then $\rho_*=\sum_k\pi_k|k\rangle\langle k|$, and hence

$$\bar{C}(\Pi,H,h)\geq\sum_k\pi_kH(\Pi(|k\rangle\langle k|)\|\Pi(\rho_*))=H(\rho_*)=C_{\mathrm{ea}}(\Pi,H,h).$$

Proposition 3 is generalized as follows.

Proposition 5. *Let* $\Phi\colon\mathfrak{S}(\mathfrak{H}_A)\to\mathfrak{S}(\mathfrak{H}_B)$ *be a degradable channel*, H *a positive operator*, $h>0$, *and* $h_*=(\dim\mathfrak{H}_A)^{-1}\operatorname{Tr}H$. *Then one of the following alternatives holds*:
- $\bar{C}(\Phi,H,h)<C_{\mathrm{ea}}(\Phi,H,h)$;
- Φ *is a classical-quantum channel having the representation*

$$\Phi(\rho)=\sum_{k=1}^{\dim\mathfrak{H}_A}\langle k|\rho|k\rangle\sigma_k,\quad\rho\in\mathfrak{S}(\mathfrak{H}_A),\tag{22}$$

where $\{\sigma_k\}$ *is a collection of states in* $\mathfrak{S}(\mathfrak{H}_B)$ *with mutually orthogonal supports*, *and* $\{|k\rangle\}$ *is*
  - *an orthonormal basis in* $\mathfrak{H}_A$ *if* $h\geq h_*$;
  - *the orthonormal basis of eigenvectors of the operator* H *if* $h<h_*$.

Proof. Since $\chi_\Phi(\rho)\leq H(\rho)$ and $I(\rho,\Phi)\geq H(\rho)$ (Φ is a degradable channel), the equality $\bar{C}(\Phi,H,h)=C_{\mathrm{ea}}(\Phi,H,h)$ can be valid only if

$$\bar{C}(\Phi,H,h)=C_{\mathrm{ea}}(\Phi,H,h)=\max_{\operatorname{Tr}H\rho\leq h}H(\rho).$$

If $h\geq h_*$, then this maximum coincides with $\log\dim\mathfrak{H}_A$, which means that the constraint has no effect and hence the second alternative in Proposition 3 holds.

If $h<h_*$, then the above maximum is always attained at a full-rank state, and the generalized version of Theorem 2 (B) implies that Φ is a classical-quantum channel having representation (22). Similarly to the proof of Proposition 3, one can show that the states in the collection $\{\sigma_k\}$ have mutually orthogonal supports.

Let us show that the equality $\bar{C}(\Phi,H,h)=C_{\mathrm{ea}}(\Phi,H,h)$ can be valid in the case $h<h_*$ if and only if the operator H is diagonizable in the basis $\{|k\rangle\}$ from representation (22) of the channel Φ. For the channel $\Pi(\rho)=\sum\limits_k\langle k|\rho|k\rangle|k\rangle\langle k|$, this assertion is proved in Example 3. To prove it in the general case, it suffices to note that $\bar{C}(\Phi,H,h)=\bar{C}(\Pi,H,h)$ and $C_{\mathrm{ea}}(\Phi,H,h)=C_{\mathrm{ea}}(\Pi,H,h)$. These equalities follow from the chain rules for the capacities, since it is easy to construct channels Ψ_1 and Ψ_2 such that $\Pi=\Psi_1\circ\Phi$ and $\Phi=\Psi_2\circ\Pi$. $\triangle$

The following proposition shows that coincidence of $\bar{C}(\Phi, H, h)$ and $C_{\rm ea}(\Phi, H, h)$ for any constraint parameters (H, h) is a very strong requirement.

Proposition 6. *If* $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ *is a quantum channel such that* $C_{\rm ea}(\Phi, H, h) = \bar{C}(\Phi, H, h)$ *for any operator* $H \geq 0$ *and* $h > 0$, *then* Φ *is a classical-quantum channel such that* $\chi_{\hat{\Phi}}(\rho) = H(\rho)$ *for all* $\rho \in \mathfrak{S}(\mathfrak{H}_A)$. *If the conjecture below is true, then* Φ *is a completely depolarizing channel.*

Proof. By Lemma 1 in [8], an arbitrary full-rank state ρ in $\mathfrak{S}(\mathfrak{H}_A)$ can be made the average state of an optimal ensemble for the channel Φ with constraint (21) by an appropriate choice of the operator H. Hence, the condition of the proposition and continuity arguments imply $I(\rho, \Phi) = \chi_\Phi(\rho)$ for any state ρ in $\mathfrak{S}(\mathfrak{H}_A)$. By virtue of (6), this means that $\chi_{\hat{\Phi}}(\rho) = H(\rho)$ for any state ρ in $\mathfrak{S}(\mathfrak{H}_A)$. By the generalized version of Theorem 2 (B), Φ is a classical-quantum channel. △

Conjecture. *If* $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ *is a quantum channel such that* $\chi_\Phi(\rho) = H(\rho)$ *for all* $\rho \in \mathfrak{S}(\mathfrak{H}_A)$, *then the channel* Φ *coincides* (*up to unitary equivalence*) *with the channel* $\rho \mapsto \rho \otimes \sigma$ *for some state* σ.

4. THE FUNCTION $\Delta_\Phi(\rho) = I(\rho, \Phi) - \chi_\Phi(\rho)$ AND ITS MAXIMUM VALUE

A central role in the analysis of relations between entanglement-assisted and unassisted classical capacities of a quantum channel Φ is played by the function

$$\Delta_\Phi(\rho) = I(\rho, \Phi) - \chi_\Phi(\rho)$$

introduced in Section 2, where it was mentioned that

$$\Delta_\Phi(\rho) = H(\rho) - \chi_{\hat{\Phi}}(\rho) = \min_{\substack{\sum_i \pi_i \rho_i = \rho \\ \operatorname{rank} \rho_i = 1}} \sum_i \pi_i \left[H(\rho_i \| \rho) - H(\widehat{\Phi}(\rho_i) \| \widehat{\Phi}(\rho)) \right]$$

and that the above minimum is attained at an ensemble $\{\pi_i, \rho_i\}$ of pure states if and only if this ensemble is χ_Φ-optimal in the sense of the following definition.

Definition 2. An ensemble $\{\pi_i, \rho_i\}$ of pure states is said to be χ_Φ-*optimal* if the maximum in definition (3) of the χ-function of the channel Φ is attained at this ensemble.

Since $\widehat{H}_\Phi \equiv \widehat{H}_{\hat{\Phi}}$, any χ_Φ-optimal ensemble is $\chi_{\hat{\Phi}}$-optimal, and vice versa.

The above formula for the function Δ_Φ and monotonicity of the relative entropy imply the following observation.

Lemma 2. *If* Φ *is a degradable channel, then* $\Delta_\Phi(\rho) \geq \Delta_{\hat{\Phi}}(\rho)$ *for all* ρ.

The following theorem describes properties of the function Δ_Φ.

Theorem 3. *Let* $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$ *be a quantum channel and* $\widehat{\Phi}\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_E)$ *its complementary channel. Then* Δ_Φ *is a nonnegative continuous function on the set* $\mathfrak{S}(\mathfrak{H}_A)$ *vanishing on the subset* $\operatorname{extr} \mathfrak{S}(\mathfrak{H}_A)$ *of pure states. It has the following properties*:

(1) *If there exists a channel* $\Theta\colon \mathfrak{S}(\mathfrak{H}_E) \to \mathfrak{S}(\mathfrak{H}_A)$ *such that*

$$\Theta(\widehat{\Phi}(\rho_i)) = \rho_i, \quad \forall i, \tag{23}$$

for some ensemble $\{\pi_i, \rho_i\}$ *of pure states with average state* ρ, *then* $\Delta_\Phi(\rho) = 0$ *and the ensemble* $\{\pi_i, \rho_i\}$ *is* χ_Φ-*optimal*;

(2) *If* $\Delta_\Phi(\rho) = 0$, *then*

- (23) *holds for any* χ_Φ*-optimal ensemble* $\{\pi_i, \rho_i\}$ *with average state* ρ, *where* Θ *is a channel acting on a state* σ *supported by the subspace* $\operatorname{supp}\widehat{\Phi}(\rho)$ *as follows*: $\Theta(\sigma) = A\widehat{\Phi}^*(B\sigma B)A$, $A = \rho^{1/2}$, $B = \widehat{\Phi}(\rho)^{-1/2}$;
- $\Phi|_{\mathfrak{S}(\mathfrak{H}_\rho)}$ *is a classical-quantum subchannel of the channel* Φ, *where* $\mathfrak{H}_\rho$ *is the support of the state* ρ;
- $\Delta_\Phi(\sum\limits_i \lambda_i\rho_i) = 0$ *for any* χ_Φ*-optimal ensemble* $\{\pi_i, \rho_i\}$ *with average state* ρ *and any probability distribution* $\{\lambda_i\}$;

(3) *The function* Δ_Φ *is concave on the set*[6] $\left\{\sum\limits_i \lambda_i\rho_i \mid \sum\limits_i \lambda_i = 1,\ \lambda_i \geq 0\right\}$ *for any* χ_Φ*-optimal ensemble* $\{\pi_i, \rho_i\}$;

(4) Monotonicity: *for an arbitrary channel* $\Psi\colon \mathfrak{S}(\mathfrak{H}_B) \to \mathfrak{S}(\mathfrak{H}_C)$ *we have the inequality*

$$\Delta_{\Psi\circ\Phi}(\rho) \leq \Delta_\Phi(\rho), \quad \rho \in \mathfrak{S}(\mathfrak{H}_A);$$

(5) Subadditivity for tensor product states: *for an arbitrary channel* $\Psi\colon \mathfrak{S}(\mathfrak{H}_C) \to \mathfrak{S}(\mathfrak{H}_D)$ *we have the inequality*

$$\Delta_{\Phi\otimes\Psi}(\rho\otimes\sigma) \leq \Delta_\Phi(\rho) + \Delta_\Psi(\sigma), \quad \rho \in \mathfrak{S}(\mathfrak{H}_A), \quad \sigma \in \mathfrak{S}(\mathfrak{H}_C),$$

which is satisfied with equality if the strong additivity of the Holevo capacity holds for the channels Φ *and* Ψ (*see* [8]).

Proof. (1) This property follows from monotonicity of the relative entropy and the remark before Definition 2.

(2) The first assertion follows from the Petz theorem [13, Theorem 3], which characterizes the case in which monotonicity of the relative entropy holds with equality. The second assertion is derived from the first by using arguments from the proof of Theorem 2 (B). The third assertion follows from the first and property (1).

(3) Since $\widehat{H}_\Phi \equiv \widehat{H}_{\hat{\Phi}}$, representation (4) for the function $\chi_{\hat{\Phi}}$ implies

$$\Delta_\Phi(\rho) = \left[H(\rho) - H(\widehat{\Phi}(\rho))\right] + \widehat{H}_\Phi(\rho).$$

By the identity $H(\bar{\rho}) - \sum\limits_i \pi_i H(\rho_i) = \sum\limits_i \pi_i H(\rho_i\|\bar{\rho})$, where $\bar{\rho} = \sum\limits_i \pi_i\rho_i$, concavity of the term in the square brackets on the set $\mathfrak{S}(\mathfrak{H}_A)$ follows from monotonicity of the relative entropy. Thus, to prove this assertion, it suffices to show that the function $\widehat{H}_\Phi$ is affine on the set $\left\{\sum\limits_i \lambda_i\rho_i \mid \sum\limits_i \lambda_i = 1, \lambda_i \geq 0\right\}$. This can be done by noting that the function $\widehat{H}_\Phi$ coincides with the double Fenchel transform of the function $H\circ\Phi$ and then using Proposition 1 from [17].

(4) By using the Stinespring representation, it is easy to show (see [14, proof of Lemma 17]) that there exists a channel Θ such that $\widehat{\Phi} = \Theta\circ\widehat{\Psi\circ\Phi}$. Hence, the chain rule for the χ-function implies

$$\Delta_{\Psi\circ\Phi}(\rho) = H(\rho) - \chi_{\widehat{\Psi\circ\Phi}}(\rho) \leq H(\rho) - \chi_{\hat{\Phi}}(\rho) = \Delta_\Phi(\rho).$$

(5) Since $\widehat{\Phi\otimes\Psi} = \widehat{\Phi}\otimes\widehat{\Psi}$ (see [7]), this assertion follows from an obvious inequality $\chi_{\hat{\Phi}\otimes\hat{\Psi}}(\rho\otimes\sigma) \geq \chi_{\hat{\Phi}}(\rho) + \chi_{\hat{\Psi}}(\sigma)$, which is satisfied with equality if the strong additivity of the Holevo capacity holds for the channels Φ and Ψ [8]. △

The following proposition shows the sense of the maximum value of the function Δ_Φ.

[6] The function Δ_Φ is not concave on $\mathfrak{S}(\mathfrak{H}_A)$ in general, since otherwise we would obtain $\Delta_\Phi(\rho) \leq \Delta_\Phi(\rho_c) = 0$ for any covariant channel Φ such that $C_{\mathrm{ea}}(\Phi) = \bar{C}(\Phi)$.

Proposition 7. *Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel. Then*

$$\max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) = \sup_{H,h} \left[C_{\mathrm{ea}}(\Phi,H,h) - \bar{C}(\Phi,H,h)\right], \tag{24}$$

where the supremum is over all pairs of the form (positive operator $H \in \mathfrak{B}(\mathcal{H}_A)$, $h > 0$).

Proof. For given H and h, let ρ be a state in $\mathfrak{S}(\mathcal{H}_A)$ such that $\operatorname{Tr} H\rho \le h$ and $C_{\mathrm{ea}}(\Phi,H,h) = I(\rho,\Phi)$. Since $\bar{C}(\Phi,H,h) \ge \chi_\Phi(\rho)$, we have

$$\Delta_\Phi(\rho) = I(\rho,\Phi) - \chi_\Phi(\rho) \ge C_{\mathrm{ea}}(\Phi,H,h) - \bar{C}(\Phi,H,h).$$

This implies "$\ge$" in (24).

Let $\varepsilon > 0$ be arbitrary and ρ_ε be a full-rank state in $\mathfrak{S}(\mathcal{H}_A)$ such that $\Delta_\Phi(\rho_\varepsilon) \ge \max\limits_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) - \varepsilon$. By Lemma 1 in [8], there exists a pair (H,h) such that $\operatorname{Tr} H\rho_\varepsilon \le h$ and $\bar{C}(\Phi,H,h) = \chi_\Phi(\rho_\varepsilon)$. Since $C_{\mathrm{ea}}(\Phi,H,h) \ge I(\rho_\varepsilon,\Phi)$, we have

$$C_{\mathrm{ea}}(\Phi,H,h) - \bar{C}(\Phi,H,h) \ge I(\rho_\varepsilon,\Phi) - \chi_\Phi(\rho_\varepsilon) = \Delta_\Phi(\rho_\varepsilon) \ge \max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) - \varepsilon,$$

which implies "$\le$" in (24). △

It is easy to see that $\max\limits_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) \in [0, \log\dim\mathcal{H}_A]$. If $\Delta_\Phi(\rho) \equiv 0$, then the condition of Proposition 6 holds. If $\max\limits_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta_\Phi(\rho) = \log\dim\mathcal{H}_A$, then Φ is unitary equivalent to the channel $\rho \mapsto \rho\otimes\sigma$, where σ is a given state. Indeed, this implies $\chi_{\hat{\Phi}}(\rho_c) = 0$, where ρ_c is the chaotic state in $\mathfrak{S}(\mathcal{H}_A)$, and hence $\chi_{\hat{\Phi}}(\rho) \equiv 0$ by concavity and nonnegativity of the χ-function, which means that $\widehat{\Phi}$ is a completely depolarizing channel.

Remark 4. Subadditivity of the function Δ_Φ (property (5) in Theorem 3) implies existence of the regularization $\Delta^*_\Phi(\rho) = \lim\limits_{n\to+\infty} n^{-1}\Delta_{\Phi^{\otimes n}}(\rho^{\otimes n})$. By repeating arguments from the proof of Proposition 7 and using subadditivity of the quantum mutual information, it is easy to show that

$$\max_{\rho\in\mathfrak{S}(\mathcal{H}_A)} \Delta^*_\Phi(\rho) \ge \sup_{H,h} \left[C_{\mathrm{ea}}(\Phi,H,h) - C(\Phi,H,h)\right].$$

The equality in this inequality obviously takes place if the strong additivity of the Holevo capacity holds for the channel Φ (see [8]), but seemingly this is not the case in general.

Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ and $\Psi\colon \mathfrak{S}(\mathcal{H}_B) \to \mathfrak{S}(\mathcal{H}_C)$ be quantum channels. Monotonicity of the function Δ_Φ (property (4) in Theorem 3) shows that the inequality

$$C_{\mathrm{ea}}(\Psi\circ\Phi,H,h) - \bar{C}(\Psi\circ\Phi,H,h) \le C_{\mathrm{ea}}(\Phi,H,h) - \bar{C}(\Phi,H,h)$$

is valid if the functions $\rho \mapsto I(\rho,\Psi\circ\Phi)$ and $\rho \mapsto \chi_\Phi(\rho)$ have a common maximum point under the condition $\operatorname{Tr} H\rho \le h$ (this holds for unconstrained channels Φ and Ψ satisfying the covariance condition (14) with $\mathcal{H}_A = \mathcal{H}_B$ and $V_g = W_g$).

In general, validity of the above inequality is an interesting open question, but monotonicity of the function Δ_Φ and Proposition 7 imply the following observation.

Corollary 2. *Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ and $\Psi\colon \mathfrak{S}(\mathcal{H}_B) \to \mathfrak{S}(\mathcal{H}_C)$ be arbitrary quantum channels. Then*

$$\sup_{H,h} \left[C_{\mathrm{ea}}(\Psi\circ\Phi,H,h) - \bar{C}(\Psi\circ\Phi,H,h)\right] \le \sup_{H,h} \left[C_{\mathrm{ea}}(\Phi,H,h) - \bar{C}(\Phi,H,h)\right].$$

If we introduce the parameter

$$D(\Phi) = \sup_{H,h} \left[C_{\mathrm{ea}}(\Phi, H, h) - \bar{C}(\Phi, H, h)\right]$$

for the channel $\Phi\colon \mathfrak{S}(\mathfrak{H}_A) \to \mathfrak{S}(\mathfrak{H}_B)$, the above observations can be reformulated as follows:

- $D(\Phi) = \max\limits_{\rho\in\mathfrak{S}(\mathfrak{H}_A)} \Delta_\Phi(\rho)$;
- $D(\Psi\circ\Phi) \le D(\Phi)$ for any channel $\Psi\colon \mathfrak{S}(\mathfrak{H}_B) \to \mathfrak{S}(\mathfrak{H}_C)$;
- $D(\Phi) \in [0, \log\dim\mathfrak{H}_A]$;
- $D(\Phi) = \log\dim\mathfrak{H}_A$ if and only if Φ is unitary equivalent to the noiseless channel $\rho \mapsto \rho\otimes\sigma$, where σ is a given state;
- $D(\Phi) = 0$ if Φ is a completely depolarizing channel ("if and only if" provided the conjecture at the end of Section 3 is true).

The above properties show that the parameter $D(\Phi)$ can be considered as one of characteristics of the channel Φ describing its "level of noise." Unfortunately, a way for computing this parameter for nontrivial examples of quantum channels is unclear.

Generalizations of the results obtained in this paper to infinite-dimensional constrained channels are presented in the second part of [18].

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