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CODING THEORY

On Classical Capacities of Infinite-Dimensional Quantum Channels[1]

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Abstract—A coding theorem for entanglement-assisted communication via an infinite-dimensional quantum channel with linear constraints is extended to a natural degree of generality. Relations between the entanglement-assisted classical capacity and χ-capacity of constrained channels are obtained, and conditions for their coincidence are given. Sufficient conditions for continuity of the entanglement-assisted classical capacity as a function of a channel are obtained. Some applications of the obtained results to analysis of Gaussian channels are considered. A general (continuous) version of the fundamental relation between coherent information and the measure of privacy of classical information transmission via an infinite-dimensional quantum channel is proved.

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1. INTRODUCTION

A central role in quantum information theory is played by the notion of a quantum channel, a noncommutative analog of a transition probability matrix in classical theory. Informational properties of a quantum channel are characterized by a number of different capacities depending on the type of transmitted information, additional resources used to increase the rate of transmission, security requirements, etc.; see, e.g., [1]. One of the most important of these quantities is the entanglement-assisted classical capacity, which characterizes the ultimate rate of classical information transmission assuming that the transmitter and receiver may use a common entangled state. By definition, this capacity is greater than or equal to the classical (unassisted) capacity of the channel. The Bennett–Shor–Smolin–Thapliyal (BSST) theorem [2] gives an explicit expression for the entanglement-assisted capacity of a finite-dimensional unconstrained channel, showing that this capacity is equal to the maximum of quantum mutual information.

When applying the protocol of entanglement-assisted communication to *infinite-dimensional* channels, one has to impose certain constraints on input states. A typical physically motivated constraint is bounded energy of states used for encoding. This constraint is determined by the linear inequality

$$\operatorname{Tr} \rho F \leq E, \quad E > 0, \tag{1}$$

where F is a positive self-adjoint operator, the Hamiltonian of the input quantum system. An operational definition of the entanglement-assisted classical capacity of an infinite-dimensional quantum channel with linear constraint (1) is given in [3], where a generalization of the BSST theorem is proved under special restrictions on the channel and on the constraint operator. Recent advances in the study of entropy characteristics of infinite-dimensional quantum channels (in particular,

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generalization of the notion of quantum conditional entropy [4]) make it possible to establish a general version of the BSST theorem for a channel with linear constraints without any simplifying assumptions. A proof of this result is the first part of the paper.

The second part is devoted to studying relations between entanglement-assisted and unassisted classical capacities of infinite-dimensional constrained channels and conditions for their (non)coincidence. It is shown that under certain circumstances coincidence of the above capacities implies that the channel is essentially classical-quantum (for a definition, see, e.g., [1]).

We also consider the problem of continuity of the entanglement-assisted classical capacity as a function of a channel. This question has a physical motivation in the fact that preparing a quantum channel in a real experiment is subject to unavoidable imprecisions. In the finite-dimensional case, continuity of the entanglement-assisted classical capacity was proved in [5]. In infinite dimensions, this capacity is not continuous in general (it is only lower semicontinuous); however, we suggest several sufficient conditions for its continuity and consider some applications.

In Section 6 we prove an infinite-dimensional generalization of the identity due to Schumacher and Westmoreland [6], which underlies the fundamental relation between the quantum capacity and privacy of classical information transmission through a quantum channel.

In the Appendix we give auxiliary facts concerning Bosonic Gaussian channels.

2. PRELIMINARIES

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators in $\mathcal{H}$, and $\mathfrak{B}_+(\mathcal{H})$ the positive cone in $\mathfrak{B}(\mathcal{H})$. Let $\mathfrak{T}(\mathcal{H})$ be the Banach space of all trace-class operators in $\mathcal{H}$, and let $\mathfrak{S}(\mathcal{H})$ be the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of positive operators with unit trace, called *states* [1,7]. We denote by $I_{\mathcal{H}}$ the unit operator in a Hilbert space $\mathcal{H}$ and by $\mathrm{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

A linear completely positive trace-preserving map $\Phi\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is called a *channel* [1,7]. By the Stinespring dilation theorem, complete positivity of Φ implies existence of a Hilbert space $\mathcal{H}_E$ and isometry $V\colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that[2]

$$\Phi[\rho] = \mathrm{Tr}_E\, V\rho V^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A). \tag{2}$$

The channel $\widehat{\Phi}\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_E)$ defined as

$$\widehat{\Phi}[\rho] = \mathrm{Tr}_B\, V\rho V^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A), \tag{3}$$

is said to be *complementary* to the channel Φ [8]. The complementary channel is uniquely defined in the following sense: if $\widehat{\Phi}'\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_{E'})$ is a channel defined by (3) via another isometry $V'\colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_{E'}$ for which (2) holds, then there is a partial isometry $W\colon \mathcal{H}_E \to \mathcal{H}_{E'}$ such that

$$\widehat{\Phi}'[\rho] = W\widehat{\Phi}[\rho]W^*, \qquad \widehat{\Phi}[\rho] = W^*\widehat{\Phi}'[\rho]W, \quad \rho \in \mathfrak{T}(\mathcal{H}_A).$$

Let $H(\rho)$ be the von Neumann entropy of the state ρ, and $H(\rho\,\|\,\sigma)$ the quantum relative entropy of the states ρ and σ [7,9,10]. A finite collection of states $\{\rho_i\}$ with the corresponding probability distribution $\{\pi_i\}$ is called an *ensemble* and is denoted by $\{\pi_i, \rho_i\}$. The state $\bar\rho = \sum_i \pi_i\rho_i$ is called the *average state* of the ensemble $\{\pi_i, \rho_i\}$.

The χ-quantity of an ensemble $\{\pi_i, \rho_i\}$ is defined as

$$\chi(\{\pi_i, \rho_i\}) \doteq \sum_i \pi_i H(\rho_i\,\|\,\bar\rho) = H(\bar\rho) - \sum_i \pi_i H(\rho_i),$$

[2] Here and in what follows, we write $\mathrm{Tr}_X(\cdot) = \mathrm{Tr}_{\mathcal{H}_X}(\cdot)$ for brevity.

where the second expression is valid under the condition $H(\bar{\rho}) < +\infty$. We will also use the notation $\chi_\Phi(\{\pi_i, \rho_i\}) = \chi(\{\pi_i, \Phi[\rho_i]\})$. The χ-quantity can be considered as a quantum analog of the Shannon information; it appears in the expression for the classical capacity of a quantum channel (see below).

Let F be a positive self-adjoint operator in $\mathcal{H}_A$. For any state $\rho \in \mathfrak{S}(\mathcal{H}_A)$, the value $\operatorname{Tr} \rho F$ (finite or infinite) is defined as $\sup_n \operatorname{Tr} \rho P_n F P_n$, where P_n is the spectral projector of F corresponding to the interval $[0, n]$.

We impose a linear constraint on input states $\rho^{(n)}$ of the channel $\Phi^{\otimes n}$ of the form

$$\operatorname{Tr} \rho^{(n)} F^{(n)} \leq nE, \tag{4}$$

where

$$F^{(n)} = F \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes F. \tag{5}$$

An operational definition of the classical capacity of a quantum channel with a linear constraint can be found in [3]. We will need an analytical expression, for which we first introduce the χ-capacity of a channel Φ with constraint (4):

$$C_\chi(\Phi, F, E) = \sup_{\rho \colon \operatorname{Tr} \rho F \leq E} C_\chi(\Phi, \rho),$$

where

$$C_\chi(\Phi, \rho) = \sup_{\sum\limits_i \pi_i \rho_i = \rho} \chi_\Phi(\{\pi_i, \rho_i\}) \tag{6}$$

is the constrained χ-capacity of the channel Φ at state ρ (the supremum is over all ensembles with the average state ρ). If $H(\Phi[\rho]) < +\infty$, then

$$C_\chi(\Phi, \rho) = H(\Phi[\rho]) - \widehat{H}_\Phi(\rho), \tag{7}$$

where $\widehat{H}_\Phi(\rho) = \inf\limits_{\sum\limits_i \pi_i \rho_i = \rho} \sum\limits_i \pi_i H(\Phi[\rho_i])$ is the σ-convex hull of the function $\rho \mapsto H(\Phi[\rho])$. Due to concavity of this function, the infimum can be taken over ensembles of pure states. By the Holevo–Schumacher–Westmoreland (HSW) theorem adapted to constrained channels ([3, Proposition 3]), the classical capacity of a channel Φ with constraint (4) is given by the following regularized expression:

$$C(\Phi, F, E) = \lim_{n \to +\infty} n^{-1} C_\chi(\Phi^{\otimes n}, F^{(n)}, nE),$$

where $F^{(n)}$ is defined in (5).

Another important analog of the Shannon information, which appears in connection with the entanglement-assisted classical capacity (see Section 3), is the *quantum mutual information*. In finite dimensions it is defined for an arbitrary state $\rho \in \mathfrak{S}(\mathcal{H}_A)$ by the expression (cf. [11])

$$I(\rho, \Phi) = H(\rho) + H(\Phi[\rho]) - H((\Phi \otimes \mathrm{Id}_R)[\widehat{\rho}]), \tag{8}$$

where $\mathcal{H}_R$ is a Hilbert space isomorphic to $\mathcal{H}_A$ and $\widehat{\rho}$ is a purification of the state ρ in the space $\mathcal{H}_A \otimes \mathcal{H}_R$ so that $\rho = \operatorname{Tr}_R \widehat{\rho}$. By using the complementary channel, the quantum mutual information can be also expressed as follows:

$$I(\rho, \Phi) = H(\rho) + H(\Phi[\rho]) - H(\widehat{\Phi}[\rho]). \tag{9}$$

In infinite dimensions, expressions (8) and (9) may contain uncertainty of the type $\infty - \infty$, and to avoid this problem they should be modified as

$$I(\rho, \Phi) = H\left((\Phi \otimes \mathrm{Id}_R)[\widehat{\rho}] \,\|\, (\Phi \otimes \mathrm{Id}_R)[\rho \otimes \varrho]\right), \tag{10}$$

where $\varrho = \operatorname{Tr}_A \widehat{\rho}$ is the state in $\mathfrak{S}(\mathcal{H}_R)$ with the same nonzero spectrum as ρ. Analytical properties of the function $(\rho, \Phi) \mapsto I(\rho, \Phi)$ defined by (10) were studied in [12] in the infinite-dimensional case.

3. ENTANGLEMENT-ASSISTED CLASSICAL CAPACITY

Consider the following protocol of classical information transmission through a quantum channel $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_{A'})$.[3] Two parties A and B share an entangled (pure) state ω_{AB}. A makes encodings $\lambda \to \mathcal{E}_\lambda$ of a classical signal λ from a finite alphabet Λ with probabilities π_λ and sends its part of this shared state through the channel Φ to B. Here $\mathcal{E}_\lambda$ are encoding channels depending on the signal λ. Thus B receives states $(\Phi \otimes \mathrm{Id}_B)[\omega_\lambda]$, where $\omega_\lambda = (\mathcal{E}_\lambda \otimes \mathrm{Id}_B)[\omega_{AB}]$, with probabilities π_λ, and B aims to extract maximum information about λ by doing measurements on these states. To enable block encoding, this procedure should be applied to the channel $\Phi^{\otimes n}$. Then signal states $\omega_\lambda^{(n)}$ transmitted through the channel $\Phi^{\otimes n} \otimes \mathrm{Id}_B^{\otimes n}$ have a special form

$$\omega_\lambda^{(n)} = (\mathcal{E}_\lambda^{(n)} \otimes \mathrm{Id}_B^{\otimes n})[\omega_{AB}^{(n)}], \tag{11}$$

where $\omega_{AB}^{(n)}$ is the pure entangled state for n copies of the system AB and $\lambda \to \mathcal{E}_\lambda^{(n)}$ are encodings for n copies of the system A.

Constraint (4) is equivalent to a similar constraint on input states of the channel $\Phi^{\otimes n} \otimes \mathrm{Id}_B^{\otimes n}$ with the constraint operator $F_{AB}^{(n)} = F^{(n)} \otimes I_B^{\otimes n}$. Denote by $\mathcal{P}_{AB}^{(n)}$ the collection of ensembles $\pi^{(n)} = \{\pi_\lambda^{(n)}, \omega_\lambda^{(n)}\}$, where $\omega_\lambda^{(n)}$ are states of the form (11) satisfying

$$\sum_{\lambda \in \Lambda} \pi_\lambda^{(n)} \operatorname{Tr} \omega_\lambda^{(n)} F_{AB}^{(n)} \le nE.$$

The classical capacity of the above protocol is called the *entanglement-assisted classical capacity* of the channel Φ under constraint (4) and is denoted by $C_{\mathrm{ea}}(\Phi, F, E)$ (for more detail of the operational definition, see [3]). By a modification of the proof of Proposition 2 in [3], we obtain

$$C_{\mathrm{ea}}(\Phi, F, E) = \lim_{n\to\infty} \frac{1}{n} C_{\mathrm{ea}}^{(n)}(\Phi, F, E), \tag{12}$$

where

$$C_{\mathrm{ea}}^{(n)}(\Phi, F, E) = \sup_{\pi^{(n)} \in \mathcal{P}_{AB}^{(n)}} \chi_{\Phi^{\otimes n} \otimes \mathrm{Id}_B^{\otimes n}}\bigl(\{\pi_\lambda^{(n)}, \omega_\lambda^{(n)}\}\bigr). \tag{13}$$

These are expressions that we will work with in this paper. The following theorem generalizes Proposition 4 from [3] to the case of an arbitrary channel Φ and arbitrary constraint operator F.

Theorem 1. *Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_{A'})$ be a quantum channel, and let F be a self-adjoint positive operator in the space $\mathcal{H}_A$. The entanglement-assisted classical capacity (finite or infinite) of the channel Φ with constraint (4) is given by the expression*

$$C_{\mathrm{ea}}(\Phi, F, E) = \sup_{\rho\colon \operatorname{Tr} \rho F \le E} I(\rho, \Phi). \tag{14}$$

It follows from Theorem 1 that the entanglement-assisted classical capacity (finite or infinite) of the unconstrained channel Φ is

$$C_{\mathrm{ea}}(\Phi) = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A)} I(\rho, \Phi).$$

[3] In this section the output of a quantum channel will be denoted by A' (instead of B) for convenience of notation.

Proof. To prove the inequality $\geq$ in (14), assume first that the channel Φ has a finite-dimensional output (the system A' is finite-dimensional). In this case the required inequality can be proved by repeating arguments from the corresponding part of the proof of Proposition 4 in [3], based on a special encoding protocol. We only make the following remarks concerning the generalization of that proof:

1. Finite dimensionality of the system A' implies finiteness of the output entropy of the channel Φ on the whole space of input states;
2. Finiteness of $\operatorname{Tr}\rho F$ implies that all the eigenvectors of the state ρ belong to the domain of the operator $\sqrt{F}$;
3. Finite dimensionality of the system A' shows that for any finite-rank state ρ the restriction of the channel $\Phi^{\otimes n}$ to the support of the state $\rho^{\otimes n}$ acts as a finite-dimensional channel for each n;
4. If there are no states satisfying the inequality $\operatorname{Tr}\rho F < E$ but there exists an infinite-rank state ρ_0 such that $\operatorname{Tr}\rho_0 F = E$, then there is a sequence $\{\rho_n\}$ of finite-rank states converging to ρ_0 such that $\operatorname{Tr}\rho_n F = E$ for which
$$\liminf_{n\to+\infty} I(\rho_n, \Phi) \geq I(\rho_0, \Phi)$$
by lower semicontinuity of the quantum mutual information.

Let Φ be an arbitrary channel, and let $\{P_n\}$ be a sequence of finite-dimensional projectors in $\mathcal{H}_{A'}$ strongly converging to the unit operator $I_{A'}$. The channel Φ is approximated in the strong convergence topology (see [13]) by the sequence of channels $\Pi_n \circ \Phi$ with finite-dimensional output, where $\Pi_n(\rho) = P_n\rho P_n + [\operatorname{Tr}\rho(I_{A'} - P_n)]\tau$ and τ is a given state in A'. Since the inequality $\geq$ in (14) is proved for a channel with finite-dimensional output, the chain rule for the entanglement-assisted capacity implies
$$C_{\mathrm{ea}}(\Phi, F, E) \geq C_{\mathrm{ea}}(\Pi_n \circ \Phi, F, E) \geq I(\rho, \Pi_n \circ \Phi).$$
Lower semicontinuity of the function $\Phi \mapsto I(\rho, \Phi)$ in the strong convergence topology and the chain rule for quantum mutual information (see Proposition 1 in [12]) imply
$$\lim_{n\to+\infty} I(\rho, \Pi_n \circ \Phi) = I(\rho, \Phi) \leq +\infty \quad \text{for all} \quad \rho.$$
Hence, the inequality $\geq$ in (14) for the channel Φ follows from the above inequality.

Now we prove the inequality $\leq$ in (14). By the lemma below, the expression $\chi_{\Phi^{\otimes n}\otimes \mathrm{Id}_B^{\otimes n}}(\ldots)$ on the right-hand side of (13) is bounded from above by $I\Big(\sum\limits_{\lambda} \pi_\lambda^{(n)} (\omega_\lambda^{(n)})_A, \Phi^{\otimes n}\Big)$. From (12) we get
$$C_{\mathrm{ea}}(\Phi, F, E) \leq \lim_{n\to\infty} \frac{1}{n} \sup_{\pi^{(n)}\in\mathcal{P}_{AB}^{(n)}} I\left(\sum_\lambda \pi_\lambda^{(n)} (\omega_\lambda^{(n)})_A, \Phi^{\otimes n}\right).$$
The right-hand side is less than or equal to
$$\sup_{\rho^{(n)}:\, \operatorname{Tr}\rho^{(n)}F^{(n)} \leq nE} I\big(\rho^{(n)}, \Phi^{\otimes n}\big) \equiv \bar{I}_n(\Phi).$$
Note that the sequence $\bar{I}_n(\Phi)$ is additive. To show this, it suffices to prove
$$\bar{I}_n(\Phi) \leq n\bar{I}_1(\Phi). \tag{15}$$
By subadditivity of quantum mutual information,
$$I\big(\rho^{(n)}, \Phi^{\otimes n}\big) \leq \sum_{j=1}^{n} I\big(\rho_j^{(n)}, \Phi\big),$$

where $\rho_j^{(n)}$ are partial states, and, by concavity,

$$\sum_{j=1}^{n} I(\rho_j^{(n)}, \Phi) \leq nI\left(\frac{1}{n}\sum_{j=1}^{n}\rho_j^{(n)}, \Phi\right).$$

The inequality $\operatorname{Tr}\rho^{(n)}F^{(n)} \leq nE$ is equivalent to $\operatorname{Tr}\left(\frac{1}{n}\sum_{j=1}^{n}\rho_j^{(n)}\right)F \leq E$, hence (15) follows. Thus,

$$C_{\mathrm{ea}}(\Phi, F, E) \leq \sup_{\rho\colon \operatorname{Tr}\rho F \leq E} I(\rho, \Phi).$$

Lemma. *Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_{A'})$ be a quantum channel and σ an arbitrary state in $\mathfrak{S}(\mathcal{H}_B)$. Then for an arbitrary ensemble $\{\pi_i, \omega_i\}$ of states in $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $(\omega_i)_B = \sigma \in \mathfrak{S}(\mathcal{H}_B)$ for all i, the inequality*

$$\chi_{\Phi\otimes \mathrm{Id}_B}(\{\pi_i, \omega_i\}) \leq I(\omega_A, \Phi) \tag{16}$$

holds, where $\omega = \sum_i \pi_i\omega_i$ is the average state of the ensemble $\{\pi_i, \omega_i\}$.

In the proof of this lemma we will use an infinite-dimensional generalization of the conditional entropy proposed in [4], which is briefly described below.

In finite dimensions, the conditional entropy of a state ρ of a composite system AB is defined as

$$H(A\,|\,B)_\rho \doteq H(\rho) - H(\rho_B). \tag{17}$$

The conditional entropy is finite, but in contrast to the classical case it may be negative.

Following [4], the conditional entropy of a state ρ of an infinite-dimensional composite system AB is defined as

$$H(A\,|\,B)_\rho \doteq H(\rho_A) - H(\rho\,\|\,\rho_A \otimes \rho_B) \tag{18}$$

provided that $H(\rho_A) < +\infty$. It is easy to see that the right-hand sides of (17) and (18) coincide if $H(\rho) < +\infty$ (finiteness of any two values in the triple $H(\rho_A), H(\rho_B), H(\rho)$ implies finiteness of the third).

It is proved in [4] that the above-defined conditional entropy is a concave function on the convex set of all states ρ of the system AB such that $H(\rho_A) < +\infty$, which possesses the following properties:

$$H(A\,|\,B)_{\rho_{AB}} \geq H(A\,|\,BC)_\rho \tag{19}$$

for any state ρ of ABC (monotonicity), and

$$H(A\,|\,B)_{\rho_{AB}} = -H(A\,|\,C)_{\rho_{AC}} \tag{20}$$

for any pure state ρ of ABC, where it is assumed that $H(\rho_A) < +\infty$.

Proof of the lemma. Let $\{\pi_i, \omega_i\}$ be an ensemble of states in $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ with an average state ω such that $(\omega_i)_B = \sigma \in \mathfrak{S}(\mathcal{H}_B)$ for all i. We have to show that

$$\sum_i \pi_i H(\Phi\otimes \mathrm{Id}_B[\omega_i]\,\|\,\Phi\otimes \mathrm{Id}_B[\omega]) \leq I(\omega_A, \Phi). \tag{21}$$

First, let us prove inequality (21) assuming that $\dim\mathcal{H}_{A'} < +\infty$ and $\dim\mathcal{H}_B < +\infty$. In this case the left-hand side of this inequality can be rewritten as

$$L \doteq H(\Phi\otimes \mathrm{Id}_B[\omega]) - \sum_i \pi_i H(\Phi\otimes \mathrm{Id}_B[\omega_i]).$$

By subadditivity of the von Neumann entropy, we have

$$L \le H(\Phi[\rho]) + \sum_i \pi_i \big[H(\sigma) - H(\Phi\otimes \mathrm{Id}_B[\omega_i])\big],$$

where $\rho = \omega_A$. Note that $H(\Phi\otimes \mathrm{Id}_B[\omega_i]) - H(\sigma)$ is the conditional entropy $H(A'|B)$ at the state $\Phi\otimes \mathrm{Id}_B(\omega_i)$. Let $\widehat{\omega}_i$ be a pure state in ABR_i such that $(\widehat{\omega}_i)_{AB} = \omega_i$. By monotonicity of the conditional entropy (property (19)), we have

$$H(\Phi\otimes \mathrm{Id}_B[\omega_i]) - H(\sigma) = H(A'\,|\,B)_{\Phi\otimes\mathrm{Id}_B[\omega_i]} \ge H(A'\,|\,BR_i)_{\Phi\otimes\mathrm{Id}_{BR_i}[\widehat{\omega}_i]}, \tag{22}$$

where $H(A'|BR_i)$ is defined by (18) (the system R_i is infinite-dimensional, but the system A' is finite-dimensional by the assumption). Since $\widehat{\omega}_i$ is a purification of the state $\rho_i \doteq (\omega_i)_A$, i.e., $(\widehat{\omega}_i)_A = \rho_i$, property (20) of the conditional entropy implies

$$\begin{aligned} H(A'\,|\,BR_i)_{\Phi\otimes\mathrm{Id}_{BR_i}[\widehat{\omega}_i]} &= H(A'\,|\,BR_i)_{\mathrm{Tr}_E\, V\otimes I_{BR_i}\cdot\widehat{\omega}_i\cdot V^*\otimes I_{BR_i}} \\ &= -H(A'\,|\,E)_{\mathrm{Tr}_{BR_i}\, V\otimes I_{BR_i}\cdot\widehat{\omega}_i\cdot V^*\otimes I_{BR_i}} \\ &= -H(A'\,|\,E)_{V\rho_i V^*}, \end{aligned} \tag{23}$$

where E is an environment system for the channel Φ and V is the Stinespring isometry (i.e., $\Phi[\rho] = \mathrm{Tr}_E\, V\rho V^*$).

By using concavity of the conditional entropy (defined by (18)) and property (20), we obtain

$$\sum_i \pi_i H(A'\,|\,E)_{V\rho_i V^*} \le H(A'\,|\,E)_{V\rho V^*} = -H(A'\,|\,R)_{\mathrm{Tr}_E\, V\otimes I_R\cdot\widehat{\rho}\cdot V^*\otimes I_R},$$

where R is a reference system for the state ρ and $\widehat{\rho}$ is a pure state in AR such that $\widehat{\rho}_A = \rho$. Hence, (22) and (23) imply

$$L \le H(\Phi[\rho]) - H(A'\,|\,R)_{\Phi\otimes\mathrm{Id}_R[\widehat{\rho}]} = H(\Phi\otimes\mathrm{Id}_R[\widehat{\rho}]\,\|\,\Phi[\rho]\otimes\widehat{\rho}_R) = I(\rho,\Phi),$$

where definitions (10) and (18) were used.

Thus, inequality (21) is proved under the assumptions that $\dim\mathcal{H}_{A'} < +\infty$ and $\dim\mathcal{H}_B < +\infty$. Its proof in the general case can be obtained using approximation techniques as follows.

Let $\{\pi_i,\omega_i\}$ be an ensemble such that $(\omega_i)_B = \sigma \in \mathfrak{S}(\mathcal{H}_B)$, and let Q_n be the spectral projector of the state σ corresponding to its n maximal eigenvalues. Let $\lambda_n = \mathrm{Tr}\, Q_n\sigma$ and $C_n = I_A\otimes Q_n$. For a natural n, consider the ensemble $\{\pi_i,\omega_i^n\}$ with the average state ω^n, where

$$\omega_i^n = \lambda_n^{-1} C_n\omega_i C_n, \qquad \omega^n = \lambda_n^{-1} C_n\omega C_n.$$

Let $\{P_n\}$ be a sequence of finite-rank projectors in the space $\mathcal{H}_{A'}$ strongly converging to the identity operator $I_{A'}$, and let τ be a pure state in $\mathfrak{S}(\mathcal{H}_{A'})$. Consider the sequence of channels $\Phi_n = \Pi_n\circ\Phi$, where

$$\Pi_n[\rho] = P_n\rho P_n + \tau\,\mathrm{Tr}(I_{A'} - P_n)\rho, \quad \rho\in\mathfrak{S}(\mathcal{H}_{A'}).$$

Since $(\omega_i^n)_B = \lambda_n^{-1}Q_n\sigma$ for all i, the first part of the proof implies

$$\sum_i \pi_i H\left(\Phi_n\otimes\mathrm{Id}_B[\omega_i^n]\,\|\,\Phi_n\otimes\mathrm{Id}_B[\omega^n]\right) \le I(\omega_A^n,\Phi_n).$$

Since $\lambda_n\omega_A^n \le \omega_A$, Lemma 4 in [12] shows that $\lim\limits_{n\to+\infty} I(\omega_A^n,\Phi_n) = I(\omega_A,\Phi)$. Hence, the above inequality implies inequality (21) by lower semicontinuity of the relative entropy. This proves the Lemma and completes the proof of Theorem 1. △

4. RELATIONS BETWEEN ENTANGLEMENT-ASSISTED AND UNASSISTED CLASSICAL CAPACITIES

When dealing with infinite-dimensional quantum systems and channels, it is necessary to consider *generalized ensembles* defined as Borel probability measures μ on the set of all quantum states. From this point of view ordinary ensembles are described by finitely supported measures μ. We denote by $\mathcal{P}(\mathfrak{S}(\mathcal{H}))$ the set of all generalized ensembles of states in $\mathfrak{S}(\mathcal{H})$.

The χ-quantity of a generalized ensemble μ is defined as

$$\chi(\mu) = \int\limits_{\mathfrak{S}(\mathcal{H})} H(\rho \,\|\, \bar{\rho}(\mu))\mu(d\rho) = H(\bar{\rho}(\mu)) - \int\limits_{\mathfrak{S}(\mathcal{H})} H(\rho)\mu(d\rho), \tag{24}$$

where $\bar{\rho}(\mu) = \int\limits_{\mathfrak{S}(\mathcal{H})} \rho\mu(d\rho)$ is the average state of μ (the Bochner integral) and the second formula is valid under the condition $H(\bar{\rho}(\mu)) < +\infty$ [14]. For an arbitrary generalized ensemble $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ and a channel $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$, one can define a new ensemble $\mu \circ \Phi^{-1} \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_B))$ (the image of the ensemble μ under the action of the channel Φ) as follows:

$$\mu \circ \Phi^{-1}(B) = \mu(\{\rho \in \mathfrak{S}(\mathcal{H}_A) \,|\, \Phi[\rho] \in B\}).$$

The χ-quantity of the ensemble $\mu \circ \Phi^{-1}$ will be denoted by $\chi_\Phi(\mu)$. We have

$$\chi_\Phi(\mu) = \int\limits_{\mathfrak{S}(\mathcal{H}_A)} H(\Phi[\rho] \,\|\, \Phi[\bar{\rho}(\mu)])\mu(d\rho) = H(\Phi[\bar{\rho}(\mu)]) - \int\limits_{\mathfrak{S}(\mathcal{H}_A)} H(\Phi[\rho])\mu(d\rho), \tag{25}$$

where the second equality is valid under the condition $H(\Phi[\bar{\rho}(\mu)]) < +\infty$.

It is shown in [14] that the constrained χ-capacity defined by (6) can be expressed as

$$C_\chi(\Phi, \rho) = \sup_{\mu\colon \bar{\rho}(\mu)=\rho} \chi_\Phi(\mu) \tag{26}$$

(the supremum is over all generalized ensembles in $\mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ with the average state ρ), and hence

$$C_\chi(\Phi, F, E) = \sup_{\mu\colon \operatorname{Tr}\bar{\rho}(\mu)F \le E} \chi_\Phi(\mu). \tag{27}$$

In this section we study general relations between the capacities $C_\chi(\Phi, F, E)$, $C(\Phi, F, E)$, and $C_{\mathrm{ea}}(\Phi, F, E)$ and give conditions for their coincidence under the assumption[4]

$$H(\rho) < +\infty, \quad \text{for all } \rho \text{ such that } \operatorname{Tr}\rho F \le E, \tag{28}$$

which implies, in particular, finiteness of all these values. A basic role in this analysis is played by the following expression for the quantum mutual information:

$$I(\Phi, \rho) = H(\rho) + C_\chi(\Phi, \rho) - C_\chi(\widehat{\Phi}, \rho), \tag{29}$$

which is valid under the condition $H(\rho) < +\infty$ (since $C_\chi(\Phi, \rho) \le H(\rho)$ for any channel Φ, this condition implies finiteness of all terms on the right-hand side of (29)).

If $H(\Phi[\rho])$ and $H(\widehat{\Phi}[\rho])$ are finite, then expression (29) directly follows from (7) and (9), since $\widehat{H}_\Phi \equiv \widehat{H}_{\widehat{\Phi}}$ (this follows from the coincidence of $H(\Phi[\rho])$ and $H(\widehat{\Phi}[\rho])$ for pure states ρ); in the general case, it can be proved by using Proposition 4 in Section 6.

[4] One can show that this assumption holds if and only if $\operatorname{Tr}\exp(-\lambda F) < +\infty$ for some $\lambda > 0$.

By subadditivity of the quantum mutual information, expression (29) implies a formal proof of the inequality

$$C(\Phi, F, E) \le C_{\mathrm{ea}}(\Phi, F, E), \tag{30}$$

which looks obvious from the operational definitions of the capacities. It also implies the following inequalities.

Proposition 1. *Let* $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *be a quantum channel, and let* F *be a positive operator such that condition* (28) *is valid. The inequalities*

$$\begin{aligned} C_{\mathrm{ea}}(\Phi, F, E) &\ge 2C_\chi(\Phi, F, E) - C_\chi(\widehat{\Phi}, F, E),\\ C_{\mathrm{ea}}(\Phi, F, E) &\ge 2C(\Phi, F, E) - C(\widehat{\Phi}, F, E) \end{aligned} \tag{31}$$

hold, where $\widehat{\Phi}\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ *is the complementary channel to* Φ.

Note that, in contrast to (30), both inequalities in (31) hold with equality if Φ is a noiseless channel. These inequalities show that coincidence of $C_{\mathrm{ea}}(\Phi, F, E)$ and $C_\chi(\Phi, F, E)$ (or $C(\Phi, F, E)$) can take place only if $C(\Phi, F, E) \le C(\widehat{\Phi}, F, E)$ (respectively, if $C(\Phi, F, E) \le C(\widehat{\Phi}, F, E)$).

Proof. For an arbitrary $\varepsilon > 0$, let ρ_ε be a state in $\mathfrak{S}(\mathcal{H}_A)$ such that

$$C_\chi(\Phi, F, E) < C_\chi(\Phi, \rho_\varepsilon) + \varepsilon, \quad \operatorname{Tr} \rho_\varepsilon F \le E.$$

Since $C_\chi(\Phi, \rho_\varepsilon) \le H(\rho_\varepsilon) < +\infty$, Theorem 1 and formula (29) show that

$$C_{\mathrm{ea}}(\Phi, F, E) \ge I(\rho_\varepsilon, \Phi) \ge 2C_\chi(\Phi, \rho_\varepsilon) - C_\chi(\widehat{\Phi}, \rho_\varepsilon) \ge 2C_\chi(\Phi, F, E) - C_\chi(\widehat{\Phi}, F, E) - 2\varepsilon,$$

which implies the first inequality in (31).

The second inequality in (31) is obtained from the first by regularization. △

Now we consider the question of coincidence of the capacities $C_{\mathrm{ea}}(\Phi, F, E)$ and $C_\chi(\Phi, F, E)$.

We call a channel $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *classical-quantum* (briefly, *c-q channel*) if the image of the dual channel $\Phi^*\colon \mathfrak{B}(\mathcal{H}_B) \to \mathfrak{B}(\mathcal{H}_A)$ consists of commuting operators. If all these operators are diagonal in a fixed orthonormal basis $\{|k\rangle\}$ in $\mathcal{H}_A$, we say that the c-q channel is of *discrete type*. In this case it has the following representation:

$$\Phi[\rho] = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k|\rho|k\rangle \sigma_k, \tag{32}$$

where $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$. Any finite-dimensional c-q channel is of discrete type. An example of a c-q channel which is not of discrete type is provided by a Bosonic Gaussian c-q channel (see the Appendix).

It is shown in [15] that $C_\chi(\Phi) = C_{\mathrm{ea}}(\Phi)$ for any finite-dimensional unconstrained c-q channel Φ and, moreover, that this equality implies that the restriction of the channel Φ to the support of the average state of any optimal ensemble is a c-q channel (an ensemble is said to be optimal [16] if its χ-quantity coincides with $C_\chi(\Phi)$). An example from [17] shows that the words "the restriction of" in the last assertion can not be dropped.

To generalize the above assertion to infinite dimensions, we have to consider the notion of a generalized optimal ensemble for a constrained infinite-dimensional channel [14]. A generalized ensemble μ_* is said to be optimal for the channel Φ with constraint (4) if

$$\operatorname{Tr} \bar{\rho}(\mu_*) F \le E \qquad \text{and} \qquad C_\chi(\Phi, F, E) = \chi_\Phi(\mu_*),$$

which means that the supremum in (27) is achieved on μ_*.

This is a natural generalization of the notion of an optimal ensemble for a finite-dimensional (constrained or unconstrained) channel. In contrast to the finite-dimensional case, an optimal generalized ensemble for an infinite-dimensional constrained channel need not always exist, but one can prove the following sufficient condition.

Proposition 2 [14]. *If the subset of* $\mathfrak{S}(\mathcal{H}_A)$ *defined by the inequality* $\operatorname{Tr}\rho F \leq E$ *is compact*[5] *and the function* $\rho \mapsto H(\Phi[\rho])$ *is continuous on this subset, then there exists a generalized optimal ensemble for the channel* Φ *with constraint* (4).

This condition holds for an arbitrary Bosonic Gaussian channel with energy constraint; see [14, remark after Proposition 3]. It also holds for any channel having the Kraus representation with a finite number of terms provided that the operator F satisfies the condition $\operatorname{Tr}\exp(-\lambda F) < +\infty$ for all $\lambda > 0$ (this can be proved by using Proposition 6.6 in [10]).

The following theorem gives a necessary condition for coincidence of the capacities $C_\chi(\Phi, F, E)$ and $C_{\rm ea}(\Phi, F, E)$.

Theorem 2. *Assume that a generalized optimal ensemble* μ_* *for a channel* $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ *with constraint* (4) *exists* (*in particular, the condition of Proposition* 2 *holds*) *and that condition* (28) *is valid. Let* $\mathcal{H}_*$ *be the support of the average state of* μ_*, *i.e.,* $\mathcal{H}_* = \operatorname{supp}\bar{\rho}(\mu_*)$.

If $C_\chi(\Phi, F, E) = C_{\rm ea}(\Phi, F, E)$, *then the restriction of the channel* Φ *to the set* $\mathfrak{S}(\mathcal{H}_*)$ *is a c-q channel of discrete type.*

Proof. Without loss of generality, we may assume that the optimal generalized ensemble μ_* is supported by pure states. This follows from convexity of the function $\sigma \mapsto H(\Phi[\sigma] \,\|\, \Phi[\rho])$, since for an arbitrary measure $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ there exists a measure $\widehat{\mu} \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ supported by pure states such that $\bar{\rho}(\widehat{\mu}) = \bar{\rho}(\mu)$ and $\int f(\sigma)\widehat{\mu}(d\sigma) \geq \int f(\sigma)\mu(d\sigma)$ for any convex lower semicontinuous nonnegative function f on $\mathfrak{S}(\mathcal{H}_A)$ (this measure $\widehat{\mu}$ can be constructed by using arguments from the proof of the theorem in [14]).

The equality $C_\chi(\Phi, F, E) = C_{\rm ea}(\Phi, F, E)$ implies $C_\chi(\Phi, \bar{\rho}(\mu_*)) = I(\Phi, \bar{\rho}(\mu_*))$. It follows from condition (28) and representation (29) that this is equivalent to the equality $H(\bar{\rho}(\mu_*)) = C_\chi(\widehat{\Phi}, \bar{\rho}(\mu_*)) < +\infty$. Since $C_\chi(\Phi, \bar{\rho}(\mu_*)) = \chi_\Phi(\mu_*)$, the remark after Proposition 4 in Section 6 and condition (28) imply that $C_\chi(\widehat{\Phi}, \bar{\rho}(\mu_*)) = \chi_{\widehat{\Phi}}(\mu_*)$. Since $H(\bar{\rho}(\mu_*)) = \chi(\mu_*)$, the equality $H(\bar{\rho}(\mu_*)) = \chi_{\widehat{\Phi}}(\mu_*)$ shows that the channel $\widehat{\Phi}$ preserves the χ-quantity of the ensemble μ_*, i.e., $\chi_{\widehat{\Phi}}(\mu_*) = \chi(\mu_*)$. By Theorem 5 in [18], the restriction of the channel $\widehat{\widehat{\Phi}} \cong \Phi$ to the set $\mathfrak{S}(\mathcal{H}_*)$ is a c-q channel of discrete type. △

Remark. In contrast to unconstrained channels, the assertion of Theorem 2 is not reversible even in finite dimensions: the entanglement-assisted classical capacity of a discrete-type c-q channel with a linear constraint may be greater than its unassisted classical capacity [15, Example 3]. By repeating arguments from the proof of Theorem 2 in [15] and using condition (28) one can show that $C_\chi(\Phi, F, E) = C_{\rm ea}(\Phi, F, E)$ *for any discrete-type c-q channel* Φ *with constraint* (4) *provided that the operator* F *is diagonal in the basis* $\{|k\rangle\}$ *from representation* (32) *of the channel* Φ.

For an arbitrary nontrivial subspace $\mathcal{H}_0$ of $\mathcal{H}_A$, the restriction of the channel $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ to the subset $\mathfrak{S}(\mathcal{H}_0)$ will be called the *subchannel* of Φ corresponding to the subspace $\mathcal{H}_0$.

Theorem 2 implies the following sufficient condition for noncoincidence of $C_{\rm ea}(\Phi, F, E)$ and $C_\chi(\Phi, F, E)$.

Corollary. *Let the assumptions of Theorem* 2 *hold. Then* $C_{\rm ea}(\Phi, F, E) > C_\chi(\Phi, F, E)$ *if one of the following conditions is valid*:

1. *The channel* Φ *is not a c-q channel of discrete type and the optimal measure* μ_* *has a nondegenerate average state*;
2. *The channel* Φ *has no c-q subchannels of discrete type.*

[5] This subset is compact if and only if the spectrum of operator F consists of eigenvalues of finite multiplicity accumulating at infinity; see the lemma in [3] and Lemma 3 in [14].

As is mentioned above, the assumptions of Theorem 2 hold for an arbitrary Gaussian channel $\Phi_{K,l,\alpha}$ if F is the Hamiltonian of the input system (K, l, and α are parameters of the channel; see the Appendix). By the corollary and Proposition 5 in the Appendix, the strict inequality $C_{\mathrm{ea}}(\Phi_{K,l,\alpha}, F, E) > C_\chi(\Phi_{K,l,\alpha}, F, E)$ holds if one of the following conditions is valid:

1. $K \neq 0$, and the optimal measure μ_* has a nondegenerate average state;
2. The rank of K coincides with the dimension $2k$ of the input symplectic space (k is the number of input modes).

Condition 1 holds if the conjecture on Gaussian optimizers (see [1, Ch. 12]) is valid for the channel $\Phi_{K,l,\alpha}$.

5. ON CONTINUITY OF THE ENTANGLEMENT-ASSISTED CAPACITY

Since a physical channel is always determined with some finite accuracy, it is necessary to explore the question of continuity of its information capacity with respect to small perturbations of a channel. Mathematically, this means that we have to study continuity of the capacity as a function of a channel assuming that the set of all channels is equipped with some appropriate topology.

In this section we consider continuity properties of the entanglement-assisted capacity with respect to the strong convergence topology on the set of all channels [13]. Strong convergence of a sequence of channels $\Phi_n\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ to a channel $\Phi_0\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ means that $\lim\limits_{n\to+\infty} \Phi_n[\rho] = \Phi_0[\rho]$ for any state $\rho \in \mathfrak{S}(\mathcal{H}_A)$.

Theorem 1 and lower semicontinuity of quantum mutual information as a function of a channel in the strong convergence topology imply that $\Phi \mapsto C_{\mathrm{ea}}(\Phi, F, E)$ is a lower semicontinuous function in this topology on the set of all quantum channels; i.e.,

$$\liminf_{n\to+\infty} C_{\mathrm{ea}}(\Phi_n, F, E) \geq C_{\mathrm{ea}}(\Phi_0, F, E) \ \ (\leq +\infty)$$

for any sequence $\{\Phi_n\}$ of channels strongly converging to the channel Φ_0.

The following proposition gives sufficient conditions for the continuity.

Proposition 3. *Let F be a self-adjoint positive operator such that* $\operatorname{Tr}\exp(-\lambda F) < +\infty$ *for all* $\lambda > 0$, *and let $\{\Phi_n\}$ be a sequence of channels strongly converging to a channel Φ_0. The relation*

$$\lim_{n\to+\infty} C_{\mathrm{ea}}(\Phi_n, F, E) = C_{\mathrm{ea}}(\Phi_0, F, E) < +\infty \tag{33}$$

holds if one of following conditions is valid:

1. $\lim\limits_{n\to+\infty} H(\Phi_n[\rho_n]) = H(\Phi_0[\rho_0])$ *for an arbitrary sequence $\{\rho_n\}$ converging to a state ρ_0 such that* $\operatorname{Tr}\rho_n F \leq E$, $n = 0, 1, 2, \ldots$;
2. *There exists a sequence $\{\widehat{\Phi}_n\}$ of channels strongly converging to a channel $\widehat{\Phi}_0$ such that $(\Phi_n, \widehat{\Phi}_n)$ is a complementary pair for each $n = 0, 1, 2, \ldots$.*

Condition 1 in Proposition 3 holds for any converging sequence of Gaussian channels provided that F is an oscillator Hamiltonian of a Bosonic system.

Condition 2 in Proposition 3 holds for the sequence of the channels

$$\Phi_n[\rho] = \sum_{i=1}^{+\infty} V_i^n \rho (V_i^n)^*,$$

where $\{V_i^n\}_n$ is a sequence of operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ strongly converging to the operator V_i^0 for each i such that $\sum\limits_{i=1}^{+\infty}(V_i^n)^*V_i^n = I_A$ for all n. Indeed,

$$\widehat{\Phi}_n[\rho] = \sum_{i,j=1}^{+\infty}[\operatorname{Tr} V_i^n\rho(V_j^n)^*]|i\rangle\langle j|,$$

where $\{|i\rangle\}_{i=1}^{+\infty}$ is an orthonormal basis in $\mathcal{H}_E$, and it is easy to see that the sequences $\{\Phi_n\}$ and $\{\widehat{\Phi}_n\}$ strongly converge to the channels Φ_0 and $\widehat{\Phi}_0$ (defined by the same formulas with $n = 0$).

Proof. First note that the set

$$\mathcal{A} = \{\rho \in \mathfrak{S}(\mathcal{H}_A) \mid \operatorname{Tr}\rho F \le E\}$$

is compact (by the lemma in [3]) and the function $\rho \mapsto H(\rho)$ is continuous on this set (by Proposition 6.6 in [10]).

By Proposition 4 in [12], for each n the function $\rho \mapsto I(\rho, \Phi_n)$ is continuous on the compact set $\mathcal{A}$, and hence

$$C_{\mathrm{ea}}(\Phi_n, F, E) = \sup_{\rho\in\mathcal{A}} I(\rho, \Phi_n) = I(\rho_n, \Phi_n) < +\infty$$

for a particular state ρ_n in $\mathcal{A}$.

Assume that there exists

$$\lim_{n\to+\infty} C_{\mathrm{ea}}(\Phi_n, F, E) > C_{\mathrm{ea}}(\Phi_0, F, E). \tag{34}$$

By the remark before Proposition 3, to prove (33) it suffices to show that (34) leads to a contradiction.

Since $\mathcal{A}$ is a compact set, we may assume (by passing to a subsequence) that the sequence $\{\rho_n\}$ converges to a particular state $\rho_0 \in \mathcal{A}$. Hence, to obtain a contradiction to (34), it suffices to prove that

$$\lim_{n\to+\infty} I(\rho_n, \Phi_n) = I(\rho_0, \Phi_0). \tag{35}$$

Conditions 1 and 2 of Proposition 3 provide two different ways to prove (35). If condition 1 holds, then

$$I(\rho_n, \Phi_n) = H(\rho_n) + H(\Phi_n[\rho_n]) - H(\Phi_n \otimes \mathrm{Id}_R[|\varphi_n\rangle\langle\varphi_n|]),$$

where $|\varphi_n\rangle$ is any purification for the state ρ_n, $n = 0, 1, 2, \ldots$.

By lower semicontinuity of the function $(\Phi, \rho) \mapsto I(\rho, \Phi)$, continuity of the entropy on the set $\mathcal{A}$, and condition 1, to prove (35) it suffices to show that

$$\liminf_{n\to+\infty} H(\Phi_n \otimes \mathrm{Id}_R[|\varphi_n\rangle\langle\varphi_n|]) \ge H(\Phi_0 \otimes \mathrm{Id}_R[|\varphi_0\rangle\langle\varphi_0|]).$$

This relation follows from lower semicontinuity of the relative entropy, since strong convergence of the sequence $\{\Phi_n\}$ to the channel Φ_0 implies strong convergence of the sequence $\{\Phi_n \otimes \mathrm{Id}_R\}$ to the channel $\Phi_0 \otimes \mathrm{Id}_R$, and we can choose a sequence $\{|\varphi_n\rangle\}$ that converges to the vector $|\varphi_0\rangle$ [12, Lemma 2].

If condition 2 holds, then (35) directly follows from Proposition 5 in [12]. △

6. COHERENT INFORMATION AND A MEASURE OF PRIVATE CLASSICAL INFORMATION

Let $\Phi\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel and $\widehat{\Phi}\colon \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ be its complementary channel. In finite dimensions, *coherent information* of the channel Φ at any state ρ is defined as the difference between $H(\Phi[\rho])$ and $H(\widehat{\Phi}[\rho])$ [6, 7]. Coherent information is still another quantum analog of the Shannon information relevant to the quantum capacity of a channel [1, 7]. In infinite dimensions, the values $H(\Phi[\rho])$ and $H(\widehat{\Phi}[\rho])$ may be infinite even for a state ρ with finite entropy; therefore, coherent information can be defined via quantum mutual information as a function with values in $(-\infty, +\infty]$ as follows (cf. [12]):

$$I_c(\rho, \Phi) = I(\rho, \Phi) - H(\rho).$$

Let ρ be a state in $\mathfrak{S}(\mathcal{H}_A)$ with finite entropy. By monotonicity of the χ-quantity, the values $\chi_\Phi(\mu)$ and $\chi_{\widehat{\Phi}}(\mu)$ do not exceed $H(\rho) = \chi(\mu)$ for any measure $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ supported by pure states with barycenter ρ. The following proposition can be considered as a generalization of the basic relation in [6] which underlies a fundamental connection between quantum capacity and private transmission of classical information through a quantum channel. A measure for the latter is given by the difference $\chi_\Phi(\mu) - \chi_{\widehat{\Phi}}(\mu)$ between the χ-quantities of the receiver and environment (eavesdropper).

Proposition 4. *Let μ be a measure in $\mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ supported by pure states with barycenter ρ. Then*

$$\chi_\Phi(\mu) - \chi_{\widehat{\Phi}}(\mu) = I(\rho, \Phi) - H(\rho) = I_c(\rho, \Phi). \tag{36}$$

This proposition shows, in particular, that the difference $\chi_\Phi(\mu) - \chi_{\widehat{\Phi}}(\mu)$ does not depend on μ. Thus, if the supremum in expression (26) for the value $C_\chi(\Phi, \rho)$ is achieved at some measure μ_*, then the supremum in a similar expression for $C_\chi(\widehat{\Phi}, \rho)$ is achieved at this measure μ_*, and vice versa.

Proof. If $H(\Phi[\rho]) < +\infty$, then $H(\widehat{\Phi}[\rho]) < +\infty$ by the triangle inequality (see [7]), and (36) can be derived from (9) by using the second formula in (25) and noting that the functions $\rho \mapsto H(\Phi[\rho])$ and $\rho \mapsto H(\widehat{\Phi}[\rho])$ coincide on the set of pure states. In the general case, we have to use the approximation method to prove (36). To realize this method, it is necessary to introduce some additional notions.

Let $\mathfrak{T}_1(\mathcal{H}) = \{A \in \mathfrak{T}(\mathcal{H}) \mid A \geq 0,\ \operatorname{Tr} A \leq 1\}$. We will use the following two extensions of the von Neumann entropy to the set $\mathfrak{T}_1(\mathcal{H})$ (cf.[9]):

$$S(A) = -\operatorname{Tr} A \log A, \qquad H(A) = S(A) + \operatorname{Tr} A \log \operatorname{Tr} A, \quad \forall A \in \mathfrak{T}_1(\mathcal{H}).$$

Nonnegativity, concavity, and lower semicontinuity of the von Neumann entropy imply similar properties of the functions S and H on the set $\mathfrak{T}_1(\mathcal{H})$.

The relative entropy for two operators $A, B \in \mathfrak{T}_1(\mathcal{H})$ is defined as follows (for details, see [9]):

$$H(A \,\|\, B) = \sum_i \langle i|\,(A \log A - A \log B + B - A)\,|i\rangle,$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of A. By means of this extension of the relative entropy, the χ-quantity of a measure μ in $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}))$ is defined by the first expression in (24).[6]

A completely positive linear map $\Phi\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ which does not increase trace is called a *quantum operation* [7]. For any quantum operation Φ, the Stinespring representation (2) holds,

[6] $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}))$ is the set of all probability measures on $\mathfrak{T}_1(\mathcal{H})$ equipped with the weak convergence topology.

where V is a contraction. The complementary operation $\widehat{\Phi}\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_E)$ is defined via this representation by relation (3).

By an obvious modification of arguments used in the proof of Proposition 1 in [14], it is easy to show that the function $\mu \mapsto \chi(\mu)$ is lower semicontinuous on the set $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}))$ and that for an arbitrary quantum operation Φ and a measure $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ such that $S(\Phi[\bar{\rho}(\mu)]) < +\infty$ the χ-quantity of the measure $\mu \circ \Phi^{-1} \in \mathcal{P}(\mathfrak{T}_1(\mathcal{H}_B))$ can be expressed as follows:

$$\chi_\Phi(\mu) = S(\Phi[\bar{\rho}(\mu)]) - \int\limits_{\mathfrak{S}(\mathcal{H}_A)} S(\Phi[\rho])\, \mu(d\rho). \tag{37}$$

Now we are in a position to prove (36) in the general case. Note that for a given measure $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ the function $\Phi \mapsto \chi_\Phi(\mu)$ is lower semicontinuous on the set of all quantum operations equipped with the strong convergence topology (for which $\Phi_n \to \Phi$ means $\Phi_n[\rho] \to \Phi[\rho]$ for all ρ [13]). This follows from the lower semicontinuity of the functional $\mu \mapsto \chi(\mu)$ on the set $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}_B))$, since for an arbitrary sequence $\{\Phi_n\}$ of quantum operations strongly converging to a quantum operation Φ the sequence $\{\mu \circ \Phi_n^{-1}\}$ weakly converges to the measure $\mu \circ \Phi^{-1}$ (this can be verified directly by using the definition of weak convergence and noting that strong convergence for sequences of quantum operations is equivalent to uniform convergence on compact subsets of $\mathfrak{S}(\mathcal{H}_A)$; see the proof of Lemma 1 in [13]).

Let $\{P_n\}$ be an increasing sequence of finite-rank projectors in $\mathfrak{B}(\mathcal{H}_B)$ strongly converging to I_B. Consider the sequence of quantum operations $\Phi_n = \Pi_n \circ \Phi$, where $\Pi_n[\sigma] = P_n[\sigma]P_n$. Then

$$\widehat{\Phi}_n[\rho] = \mathrm{Tr}_{\mathcal{H}_B} P_n \otimes I_{\mathcal{H}_E} V \rho V^*, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \tag{38}$$

where V is the isometry from the Stinespring representation (2) for the channel Φ.

The sequences $\{\Phi_n\}$ and $\{\widehat{\Phi}_n\}$ strongly converge to the channels Φ and $\widehat{\Phi}$ correspondingly. Let $\rho = \sum_k \lambda_k |k\rangle\langle k|$ and $|\varphi_\rho\rangle = \sum_k \sqrt{\lambda_k}|k\rangle \otimes |k\rangle$. Since $H(\rho) < +\infty$ and $S(\Phi_n[\rho]) < +\infty$, the triangle inequality implies $S(\widehat{\Phi}_n[\rho]) < +\infty$. So, we have

$$\begin{aligned} I(\rho, \Phi_n) &= H\big(\Phi_n \otimes \mathrm{Id}_R[|\varphi_\rho\rangle\langle\varphi_\rho|] \,\|\, \Phi_n[\rho] \otimes \varrho\big) \\ &= -S(\widehat{\Phi}_n[\rho]) + S(\Phi_n[\rho]) + a_n \\ &= -\chi_{\widehat{\Phi}_n}(\mu) + \chi_{\Phi_n}(\mu) + a_n, \end{aligned} \tag{39}$$

where

$$a_n = -\sum_k \mathrm{Tr}(\Phi_n[|k\rangle\langle k|])\lambda_k \log \lambda_k, \tag{40}$$

and the last equality is obtained by using (37) and coincidence of the functions $\rho \mapsto S(\Phi[\rho])$ and $\rho \mapsto S(\widehat{\Phi}[\rho])$ on the set of pure states.

Since the function $\Phi \mapsto I(\rho, \Phi)$ is lower semicontinuous (by lower semicontinuity of the relative entropy) and $I(\rho, \Phi_n) \le I(\rho, \Phi)$ for all n by monotonicity of the relative entropy under the action of the quantum operation $\Pi_n \otimes \mathrm{Id}_R$, we have

$$\lim_{n\to+\infty} I(\rho, \Phi_n) = I(\rho, \Phi). \tag{41}$$

We will also prove that

$$\lim_{n\to+\infty} \chi_{\Phi_n}(\mu) = \chi_\Phi(\mu) \qquad \text{and} \qquad \lim_{n\to+\infty} \chi_{\widehat{\Phi}_n}(\mu) = \chi_{\widehat{\Phi}}(\mu). \tag{42}$$

The first relation in (42) follows from the lower semicontinuity of the function $\Phi \mapsto \chi_\Phi(\mu)$ established before and from the inequality $\chi_{\Phi_n}(\mu) \le \chi_\Phi(\mu)$ valid for all n by monotonicity of the χ-quantity under the action of the quantum operation Π_n.

To prove the second relation in (42), note that (38) implies $\widehat{\Phi}_n[\rho] \le \widehat{\Phi}[\rho]$ for any state $\rho \in \mathfrak{S}(\mathcal{H}_A)$. Hence, Lemma 2 in [13] shows that

$$\chi_{\widehat{\Phi}_n}(\mu) \le \chi_{\widehat{\Phi}}(\mu) + f(\operatorname{Tr}\widehat{\Phi}_n[\rho]), \tag{43}$$

where $f(x) = -2x\log x - (1-x)\log(1-x)$, for any measure $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ with finite support and with barycenter ρ. Let μ be an arbitrary measure in $\mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ with barycenter ρ, and let $\{\mu_k\}$ be the sequence of measures with finite support with the same barycenter constructed in the proof of Lemma 1 in [14] which weakly converges to the measure μ. Validity of inequality (43) for the measure μ is derived from its validity for all measures μ_k by using lower semicontinuity of the function $\mu \mapsto \chi_{\widehat{\Phi}_n}(\mu)$ and the inequality $\chi_{\widehat{\Phi}}(\mu_k) \le \chi_{\widehat{\Phi}}(\mu)$, which is valid for all k by the construction of the sequence $\{\mu_k\}$ and convexity of the relative entropy.

Inequality (43) and lower semicontinuity of the function $\Phi \mapsto \chi_\Phi(\mu)$ imply the second relation in (42).

Since the sequence $\{a_n\}$ defined in (40) obviously tends to $H(\rho)$, relations (39), (41), and (42) imply (36). $\triangle$

APPENDIX

Gaussian Classical-Quantum Channels

The main applications of infinite-dimensional quantum information theory are related to Bosonic systems; for a detailed description of them, we refer the reader to [1, Ch.12]. Let $\mathcal{H}_A$ be the irreducible representation space of the canonical commutation relations (CCR)

$$W_A(z_A)W_A(z'_A) = \exp\Bigl(-\frac{i}{2}z_A^\top \Delta_A z'_A\Bigr)W_A(z'_A + z_A) \tag{44}$$

with a coordinate symplectic space (Z_A, Δ_A) and the Weyl system $W_A(z_A) = \exp(iR_A \cdot z_A)$, $z_A \in Z_A$. Here R_A is the row vector of canonical variables in $\mathcal{H}_A$, and Δ_A is the nondegenerate skew-symmetric commutation matrix of components of R_A. The Gaussian channel $\Phi\colon \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$, with a similar description for $\mathcal{H}_B$, is defined via the action of its dual Φ^* on the Weyl operators:

$$\Phi^*[W_B(z_B)] = W_A(Kz_B)\exp\Bigl[il^\top z_B - \frac{1}{2}z_B^\top \alpha z_B\Bigr], \tag{45}$$

where K is the matrix of a linear operator $Z_B \to Z_A$, $l \in Z_B$, and α is a real symmetric matrix satisfying

$$\alpha \ge \pm\frac{i}{2}\left(\Delta_B - K^\top \Delta_A K\right).$$

Proposition 5. *Let $\Phi_{K,l,\alpha}$ be the Gaussian channel with parameters K, l, and α.*

1. *The channel $\Phi_{K,l,\alpha}$ is c-q if and only if*

$$K^\top \Delta_A K = 0.$$

In this case it is of discrete type if and only if $K = 0$, i.e., the channel is completely depolarizing;

2. *If $\operatorname{rank} K = \dim Z_A$, then the channel $\Phi_{K,l,\alpha}$ has no c-q subchannels of discrete type.*

Proof. 1. Since the family $\{W_B(z_B)\}_{z_B \in Z_B}$ generates $\mathfrak{B}(\mathcal{H}_B)$, all the operators $\Phi^*_{K,l,\alpha}(A)$, $A \in \mathfrak{B}(\mathcal{H}_B)$, commute if and only if operators (45) (i.e., $W_A(Kz_B)$) commute for all z_B. By (44),

$$W_A(Kz_B)W_A(Kz'_B) = \exp\left(-iz_B^\top K^\top \Delta_A K z'_B\right) W_A(Kz'_B)W_A(Kz_B);$$

hence the first assertion follows. Assuming the discrete representation (32), we have that the operators $W_A(Kz_B) = \exp(iR_A \cdot Kz_B)$ all have pure point spectrum, which is possible only if $Kz_B \equiv 0$, since the canonical observables R_A are known to have a Lebesgue spectrum.

2. Assume that there is a subspace $\mathcal{H}_0 \subset \mathcal{H}_A$ such that

$$\Phi_{K,l,\alpha}[\rho] = \sum_k \langle k|\rho|k\rangle \sigma_k$$

for all $\rho \in \mathfrak{S}(\mathcal{H}_0)$, where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_0$. Then

$$PW_A(Kz_B)P \exp\left[il^\top z_B - \frac{1}{2} z_B^\top \alpha z_B\right] = P\Phi^*_{K,l,\alpha}[W_B(z_B)]P = \sum_k [\operatorname{Tr} W_B(z_B)\sigma_k] |k\rangle\langle k|,$$

where $P = \sum\limits_k |k\rangle\langle k|$ is the projector onto $\mathcal{H}_0$. It follows that $\langle k|W_A(Kz_B)|j\rangle = 0$ for all $k \neq j$. But this cannot be valid, since $\{Kz_B \mid z_B \in Z_B\} = Z_A$ and hence the family $\{W_A(Kz_B)\}_{z_B \in Z_B}$ of Weyl operators acts irreducibly on $\mathcal{H}_A$. △

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