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On channels with positive quantum zero-error capacity having vanishing *n*-shot capacity

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Abstract We show that unbounded number of channel uses may be necessary for perfect transmission of quantum information. For any n, we explicitly construct low-dimensional quantum channels (input dimension 4, Choi rank 2 or 4) whose quantum zero-error capacity is positive, but the corresponding n-shot capacity is zero. We give estimates for quantum zero-error capacity of such channels as a function of n and show that these channels can be chosen in any small vicinity (in the cb-norm) of a classical–quantum channel. Mathematically, this property means appearance of an ideal (noiseless) subchannel only in sufficiently large tensor power of a channel. Our approach (using special continuous deformation of a maximal commutative $*$-subalgebra of M_4) also gives low-dimensional examples of the superactivation of 1-shot quantum zero-error capacity. Finally, we consider multi-dimensional construction which increases the estimate for quantum zero-error capacity of channels having vanishing n-shot capacity.

Keywords Pseudo-diagonal quantum channel · Error-correcting code · Noncommutative graph · Maximal commutative $*$-algebra

1 Introduction

It is well known that the rate of information transmission over classical and quantum communication channels can be increased by simultaneous use of many copies of a

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channel. It is this fact that implies necessity of regularization in definitions of different capacities of a channel [1,2].

In this paper, we show that zero-error transmission of quantum information over a quantum channel may require unbounded number of channel uses. We prove by explicit construction that for any given n, there is a channel Φ_n such that

$$\bar{Q}_0\left(\Phi_n^{\otimes n}\right) = 0, \quad \text{but} \quad Q_0(\Phi_n) > 0, \tag{1}$$

where $\bar{Q}_0$ and Q_0 are, respectively, the 1-shot and the asymptotic quantum zero-error capacities defined in Sect. 2.

This effect is closely related to the recently discovered phenomenon of superactivation of quantum channel capacities [3–6]. Indeed, (1) is equivalent to existence of $m > n$ such that

$$\bar{Q}_0(\Phi_n) = \bar{Q}_0\left(\Phi_n^{\otimes 2}\right) = \ldots = \bar{Q}_0\left(\Phi_n^{\otimes(m-1)}\right) = 0, \quad \text{but} \quad \bar{Q}_0\left(\Phi_n^{\otimes m}\right) > 0. \tag{2}$$

Mathematically, (2) means that all the channels $\Phi_n, \Phi_n^{\otimes 2}, \ldots, \Phi_n^{\otimes(m-1)}$ have no ideal (noiseless) subchannels but the channel $\Phi_n^{\otimes m}$ has.

We show how for any given n to explicitly construct a pseudo-diagonal quantum channel Φ_n with the input dimension $d_A = 4$ and the Choi rank $d_E \geq 2$ satisfying (2) by determining its noncommutative graph. We also obtain the estimate for m as a function of n, which gives the lower bound for $Q_0(\Phi_n)$ in (1). This shows that

$$\sup_{\Phi}\left\{Q_0(\Phi) \mid \bar{Q}_0(\Phi^{\otimes n}) = 0\right\} \geq \frac{2\ln(3/2)}{\pi n} \quad \forall n. \tag{3}$$

It is also observed that a channel Φ_n satisfying (1) and (2) can be obtained by arbitrarily small deformation (in the cb-norm) of a classical–quantum channel with $d_A = d_E = 4$.

The main problem in finding the channel Φ_n is to show nonexistence of error-correcting codes for the channel $\Phi_n^{\otimes n}$ (provided the existence of such codes is proved for $\Phi_n^{\otimes m}$). We solve this problem by using special continuous deformation of a maximal commutative $*$-subalgebra of 4×4 matrices as the noncommutative graph of Φ_n and by noting that the Knill–Laflamme error-correcting conditions are violated for any maximal commutative $*$-subalgebra with the positive dimension-independent gap (Lemma 3).

Our construction also gives low-dimensional examples of the superactivation of 1-shot quantum zero-error capacity. In particular, it gives an example of symmetric superactivation with $d_A = 4$, $d_E = 2$ (simplifying the example in [7]) and shows that such superactivation is possible for two channels with $d_A = d_E = 4$ if one of them is arbitrarily close (in the cb-norm) to a classical–quantum channel.

In the last section, we consider multi-dimensional generalization of our basic construction. It gives examples of channels which amplify the lower bound in (3) by the factor $\frac{\pi}{2\ln 2} \approx 2.26$. Unfortunately, we did not managed to show that the value in the left side of (3) is $+\infty$ (as it is reasonable to conjecture). Estimation of this value remains an open question.

It should mention that the necessity of unbounded number of channel uses to see quantum ε-error capacity has been recently shown in [8].[1]

2 Preliminaries

Let $\mathcal{H}$ be a finite-dimensional Hilbert space, $\mathfrak{B}(\mathcal{H})$—the algebra of all linear operators in $\mathcal{H}$, $\mathfrak{S}(\mathcal{H})$—the closed convex subset of $\mathfrak{B}(\mathcal{H})$ consisting of positive operators with unit trace called *quantum states* [1,2]. We can identify $\mathcal{H}$ and $\mathfrak{B}(\mathcal{H})$, respectively, with the unitary space $\mathbb{C}^n$ and the algebra $\mathfrak{M}_n$ of all $n \times n$ matrices, where $n = \dim \mathcal{H}$.

Denote by I_n and Id_n the unit operator in the space $\mathbb{C}^n$ and the identity transformation of the algebra $\mathfrak{M}_n$ correspondingly.

For any matrix $A \in \mathfrak{M}_n$ denote by Υ_A the operator of Schur multiplication by A in $\mathfrak{M}_n$ (also called the Hadamard multiplication). Its *cb*-norm will be denoted $\|\Upsilon_A\|_{\mathrm{cb}}$. It coincides with the operator norm of Υ_A and is also called the Schur (or Hadamard) multiplier norm of A (see [9,10] and the references therein).

Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel, i.e., a completely positive trace-preserving linear map [1,2]. Stinespring's theorem implies the existence of a Hilbert space $\mathcal{H}_E$ and of an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \mathrm{Tr}_{\mathcal{H}_E} V\rho V^*, \quad \rho \in \mathfrak{S}(\mathcal{H}_A).$$

The minimal dimension of $\mathcal{H}_E$ is called Choi rank of Φ and denoted d_E.

The quantum channel

$$\mathfrak{S}(\mathcal{H}_A) \ni \rho \mapsto \widehat{\Phi}(\rho) = \mathrm{Tr}_{\mathcal{H}_B} V\rho V^* \in \mathfrak{S}(\mathcal{H}_E)$$

is called *complementary* to the channel Φ [1,11]. The complementary channel is defined uniquely up to isometrical equivalence [11, the Appendix].

The 1-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel Φ is defined as $\sup_{\mathcal{H} \in q_0(\Phi)} \log_2 \dim \mathcal{H}$, where $q_0(\Phi)$ is the set of all subspaces $\mathcal{H}_0$ of $\mathcal{H}_A$ on which the channel Φ is perfectly reversible (in the sense that there is a channel Θ such that $\Theta(\Phi(\rho)) = \rho$ for all states ρ supported by $\mathcal{H}_0$). Any subspace $\mathcal{H}_0 \in q_0(\Phi)$ is called *error-correcting code* for the channel Φ [1,13].

The (asymptotic) quantum zero-error capacity is defined by regularization: $Q_0(\Phi) = \sup_n n^{-1} \bar{Q}_0(\Phi^{\otimes n})$ [6,12,13].

It is well known that a channel Φ is perfectly reversible on a subspace $\mathcal{H}_0$ if and only if the restriction of the complementary channel $\widehat{\Phi}$ to the subset $\mathfrak{S}(\mathcal{H}_0)$ is completely depolarizing, i.e., $\widehat{\Phi}(\rho_1) = \widehat{\Phi}(\rho_2)$ for all states ρ_1 and ρ_2 supported by $\mathcal{H}_0$ [1, Ch.10]. It follows that the 1-shot quantum zero-error capacity of a channel Φ is completely determined by the set $\mathcal{G}(\Phi) \doteq \widehat{\Phi}^*(\mathfrak{B}(\mathcal{H}_E))$ called *noncommutative graph* of Φ [13]. In particular, the Knill–Laflamme error-correcting condition (cf. [14]) implies the following lemma.

[1] It is surprising that this result and the preliminary arXiv version of the present paper appeared simultaneously.

Lemma 1 *A set* $\{\varphi_k\}_{k=1}^d$ *of unit orthogonal vectors in* $\mathcal{H}_A$ *is a basis of error-correcting code for a channel* $\Phi:\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$ *if and only if*

$$\langle\varphi_l|A|\varphi_k\rangle=0 \quad and \quad \langle\varphi_l|A|\varphi_l\rangle=\langle\varphi_k|A|\varphi_k\rangle \quad \forall A\in\mathfrak{L},\ \ \forall k\neq l, \tag{4}$$

where $\mathfrak{L}$ *is any subset of* $\mathfrak{B}(\mathcal{H}_A)$ *such that* $\mathrm{lin}\mathfrak{L}=\mathcal{G}(\Phi)$.

This lemma shows that $\bar{Q}_0(\Phi)\geq\log_2 d$ if and only if there exists a set $\{\varphi_k\}_{k=1}^d$ of unit vectors in $\mathcal{H}_A$ satisfying condition (4).

Remark 1 Since a subspace $\mathfrak{L}$ of the algebra $\mathfrak{M}_n$ of $n\times n$ matrices is a noncommutative graph of a particular channel if and only if

$$\mathfrak{L} \text{ is symmetric } (\mathfrak{L}=\mathfrak{L}^*) \text{ and contains the unit matrix} \tag{5}$$

(see Lemma 2 in [6] or Proposition 2 in [15]), Lemma 1 shows that one can "construct" a channel Φ with $\dim\mathcal{H}_A=n$ having positive (correspondingly, zero) 1-shot quantum zero-error capacity by taking a subspace $\mathfrak{L}\subset\mathfrak{M}_n$ satisfying (5) for which the following condition is valid (correspondingly, not valid)

$$\exists\varphi,\psi\in[\mathbb{C}^n]_1 \text{ such that } \langle\psi|A|\varphi\rangle=0 \text{ and } \langle\varphi|A|\varphi\rangle=\langle\psi|A|\psi\rangle \quad \forall A\in\mathfrak{L}, \tag{6}$$

where $[\mathbb{C}^n]_1$ is the unit sphere of $\mathbb{C}^n$. $\square$

We will use the following two notions.

Definition 1 [1] A channel $\Phi:\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$ is called *classical–quantum* if it has the representation

$$\Phi(\rho)=\sum_k\langle k|\rho|k\rangle\sigma_k,$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_A$ and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$.

Definition 2 [16] A channel $\Phi:\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$ is called *pseudo-diagonal* if it has the representation

$$\Phi(\rho)=\sum_{i,j}c_{ij}\langle\psi_i|\rho|\psi_j\rangle|i\rangle\langle j|,$$

where $\{c_{ij}\}$ is a Gram matrix of a collection of unit vectors, $\{|\psi_i\rangle\}$ is a collection of vectors in $\mathcal{H}_A$ such that $\sum_i|\psi_i\rangle\langle\psi_i|=I_{\mathcal{H}_A}$ – the unit operator in $\mathcal{H}_A$ and $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H}_B$.

Pseudo-diagonal channels are complementary to entanglement-breaking channels and vice versa [11,16].

3 Basic example

For any given $\theta \in \mathrm{T} \doteq (-\pi, \pi]$, consider the subspace

$$\mathfrak{L}_\theta = \left\{ M = \begin{bmatrix} a & b & \gamma c & d \\ b & a & d & \bar{\gamma} c \\ \bar{\gamma} c & d & a & b \\ d & \gamma c & b & a \end{bmatrix}, \quad a, b, c, d \in \mathbb{C}, \quad \gamma = \exp\left(\tfrac{\mathrm{i}}{2}\theta\right) \right\} \tag{7}$$

of $\mathfrak{M}_4$. This subspace satisfies condition (5) and has the following property

$$A = W_4^* A W_4 \quad \forall A \in \mathfrak{L}_\theta, \quad \text{where} \quad W_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

Remark 2 The subspace $\mathfrak{L}_\theta$ can be considered as a deformation of the maximal commutative $*$-algebra $\mathfrak{L}_0$. To clarify the form of this deformation, note that the family of subspaces $\mathfrak{L}_\theta$ is unitary equivalent to the family of the subspaces

$$\mathfrak{L}_\theta^s = \left\{ M = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} + \tfrac{1}{4}(d + c - b - a)\, T_\theta, \quad a, b, c, d \in \mathbb{C} \right\} \tag{9}$$

where

$$T_\theta = \begin{bmatrix} u & 0 & 0 & v \\ 0 & u & v & 0 \\ 0 & -v & -u & 0 \\ -v & 0 & 0 & -u \end{bmatrix}, \quad u = 1 - \Re\gamma, \quad v = \mathrm{i}\Im\gamma.$$

Indeed, by representing the matrix M in (7) as $M = A + cB$, where

$$A = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \bar{\tau} \\ \bar{\tau} & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \end{bmatrix}, \quad \tau = \gamma - 1,$$

is easy to see that $S^{-1} B S = T_\theta$ and

$$S^{-1} A S = \begin{bmatrix} \tilde{a} & 0 & 0 & 0 \\ 0 & \tilde{b} & 0 & 0 \\ 0 & 0 & \tilde{c} & 0 \\ 0 & 0 & 0 & \tilde{d} \end{bmatrix}, \quad \text{where} \quad S = \tfrac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

$\tilde{a} = a - b - c + d,\ \ \tilde{b} = a + b - c - d,\ \ \tilde{c} = a - b + c - d,\ \ \tilde{d} = a + b + c + d.$

Denote by $\widehat{\mathfrak{L}}_\theta$ the set of all channels whose noncommutative graph coincides with $\mathfrak{L}_\theta$. Since $\dim \mathfrak{L}_\theta = 4$, Proposition 2 in [15] shows the set $\widehat{\mathfrak{L}}_\theta$ contains infinitely many different channels with $d_A \doteq \dim \mathcal{H}_A = 4$ and $d_E \geq 2$. It is essential that one can choose families $\{\Phi_\theta\} \subset \{\widehat{\mathfrak{L}}_\theta\}$ continuous with respect to θ.

Lemma 2 1) *There is a family $\{\Phi^1_\theta\}$ of pseudo-diagonal channels (see Def. 2) with $d_E = 2$ such that $\Phi^1_\theta \in \widehat{\mathfrak{L}}_\theta$ for each θ.*

2) *There is a family $\{\Phi^2_\theta\}$ of pseudo-diagonal channels with $d_E = 4$ such that $\Phi^2_\theta \in \widehat{\mathfrak{L}}_\theta$ for each θ and Φ^2_0 is a classical–quantum channel (see Def. 1).*

The families $\{\Phi^1_\theta\}$ and $\{\Phi^2_\theta\}$ can be chosen continuous in the following sense:

$$\Phi^k_\theta(\rho) = \mathrm{Tr}_{\mathcal{H}^k_E} V^k_\theta \rho \left[V^k_\theta\right]^*, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \quad k = 1, 2, \tag{10}$$

where V^1_θ, V^2_θ are continuous families of isometries, $\mathcal{H}^1_E = \mathbb{C}^2$, $\mathcal{H}^2_E = \mathbb{C}^4$.[2]

Lemma 2 is proved in the Appendix, where representations (10) are constructed explicitly by using the unitary equivalence of $\mathfrak{L}_\theta$ and $\mathfrak{L}^s_\theta$.

Theorem 1 *Let Φ_θ be a channel in $\widehat{\mathfrak{L}}_\theta$ and $n \in \mathbb{N}$ be arbitrary.*

A) $\bar{Q}_0(\Phi_\theta) > 0$ *if and only if* $\theta = \pi$ *and* $\bar{Q}_0(\Phi_\pi) = 1$.

B) *If $\theta_1 + \ldots + \theta_n = \pi (\mathrm{mod}\, 2\pi)$, then $\bar{Q}_0(\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}) > 0$ and there exist 2^n mutually orthogonal 2-D error-correcting codes for the channel $\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}$. For each binary n-tuple $(x_1, \ldots x_n)$, the corresponding error-correcting code is spanned by the images of the vectors*

$$|\varphi\rangle = \tfrac{1}{\sqrt{2}} [\, |1 \ldots 1\rangle + \mathrm{i}\, |2 \ldots 2\rangle\,], \quad |\psi\rangle = \tfrac{1}{\sqrt{2}} [\, |3 \ldots 3\rangle + \mathrm{i}\, |4 \ldots 4\rangle\,], \tag{11}$$

under the unitary transformation $U_{x_1} \otimes \ldots \otimes U_{x_n}$, where $\{|1\rangle, \ldots, |4\rangle\}$ is the canonical basis in $\mathbb{C}^4$, $U_0 = I_4$ and $U_1 = W_4$ (defined in (8)).

C) *If $|\theta_1| + \ldots + |\theta_n| \leq 2 \ln(3/2)$, then $\bar{Q}_0(\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}) = 0$.*

Remark 3 It is easy to show that $\bar{Q}_0(\Phi^{\otimes n}_\theta) = \bar{Q}_0(\Phi^{\otimes n}_{-\theta})$ and that the set of all θ such that $\bar{Q}_0(\Phi^{\otimes n}_\theta) = 0$ is open. Hence, for each n, there is $\varepsilon_n > 0$ such that $\bar{Q}_0(\Phi^{\otimes n}_\theta) = 0$ if $|\theta| < \varepsilon_n$ and $\bar{Q}_0(\Phi^{\otimes n}_{\pm\varepsilon_n}) > 0$. Theorem 1 shows that $\varepsilon_1 = \pi$ and $2\ln(3/2)/n < \varepsilon_n \leq \pi/n$ for $n > 1$. Since assertion C is proved by using quite coarse estimates, one can conjecture that $\varepsilon_n = \pi/n$ for $n > 1$. There exist some arguments confirming validity of this conjecture for $n = 2$.

Remark 4 Assertion B of Theorem 1 can be strengthened as follows:

B') *If $\theta_1 + \ldots + \theta_n = \pi (\mathrm{mod}\, 2\pi)$, then there exist 2^n mutually orthogonal projectors $P_{\bar{x}}$ of rank 2 indexed by a binary n-tuple $\bar{x} = (x_1, \ldots x_n)$ such that*

$$P_{\bar{x}} A P_{\bar{x}} = \lambda(A) P_{\bar{x}} \quad \forall A \in \mathfrak{L}_{\theta_1} \otimes \ldots \otimes \mathfrak{L}_{\theta_n},$$

[2] This implies continuity of these families in the *cb*-norm [17].

where $\lambda(A) \in \mathbb{C}$ *does not depend on* $\bar{x}$. $P_{\bar{x}}$ *is the projector on the subspace* $U_{x_1} \otimes \ldots \otimes U_{x_n}(\mathcal{H}_0)$, *where* $\mathcal{H}_0$ *is the linear hull of vectors* (11).

This follows from the proof of Theorem 1 presented below.

Theorem 1 implies the main result of this paper.

Corollary 1 *Let* n *be arbitrary and* m *be a natural number such that* $\theta_* = \pi/m \leq 2\ln(3/2)/n$. *Then*

$$\bar{Q}_0\left(\Phi_{\theta_*}^{\otimes n}\right) = 0 \quad but \quad \bar{Q}_0\left(\Phi_{\theta_*}^{\otimes m}\right) \geq 1 \quad and\ hence \quad Q_0(\Phi_{\theta_*}) \geq 1/m. \tag{12}$$

There exist 2^m *mutually orthogonal 2-D error-correcting codes for the channel* $\Phi_{\theta_*}^{\otimes m}$.

Relation (12) means that it is not possible to transmit any quantum information with no errors by using $\leq n$ copies of the channel Φ_{θ_*}, but such transmission is possible if the number of copies is $\geq m$.

Remark 5 In (12), one can take $\Phi_{\theta_*} = \Phi_{\theta_*}^1$—a channel from the family described in the first part of Lemma 2. So, Corollary 1 shows that for any n, there exists a channel Φ_n with $d_A = 4$ and $d_E = 2$ such that $\bar{Q}_0(\Phi_n^{\otimes n}) = 0$ and

$$Q_0(\Phi_n) \geq \left(\left[\frac{\pi n}{2\ln(3/2)}\right] + 1\right)^{-1} = \frac{2\ln(3/2)}{\pi n} + o(1/n), \quad n \to +\infty,$$

where $[x]$ is the integer part of x.

It is natural to ask about the maximal value of quantum zero-error capacity of a channel with given input dimension having vanishing n-shot capacity, i.e., about the value

$$S_d(n) \doteq \sup_{\Phi\,:\,d_A=d} \left\{Q_0(\Phi) \mid \bar{Q}_0(\Phi^{\otimes n}) = 0\right\}, \tag{13}$$

where the supremum is over all quantum channels with $d_A \doteq \dim \mathcal{H}_A = d$. We may also consider the value

$$S_*(n) \doteq \sup_d S_d(n) = \lim_{d\to+\infty} S_d(n) \leq +\infty. \tag{14}$$

The sequences $\{S_d(n)\}_n$ and $\{S_*(n)\}_n$ are nonincreasing, and the first of them is bounded by $\log_2 d$. Theorem 2 in [7] shows that

$$S_{2d}(1) \geq \frac{\log_2 d}{2} \quad \text{and hence} \quad S_*(1) = +\infty.$$

It seems reasonable to conjecture that $S_*(n) = +\infty$ for all n. A possible way to prove this conjecture is discussed at the end of Sect. 4.

It follows from the superadditivity of quantum zero-error capacity that

$$S_{d^k}(n) \geq kS_d(nk) \quad \text{and hence} \quad S_*(n) \geq kS_*(nk) \quad \text{for any } k, n. \tag{15}$$

These relations show that the assumption $S_*(n_0) < +\infty$ for some n_0 implies

$$S_d(n) = O(1/n) \text{ for each } d \quad \text{and} \quad S_*(n) = O(1/n) \text{ if } n \geq n_0.$$

By Corollary 1, we have

$$S_4(n) \geq \left(\left[\frac{\pi n}{2\ln(3/2)}\right] + 1\right)^{-1} = \frac{2\ln(3/2)}{\pi n} + o(1/n), \quad \forall n. \tag{16}$$

This and (15) imply the estimation

$$S_{4^k}(n) \geq k\,\frac{2\ln(3/2)}{\pi kn} + o(1/(kn)) = \frac{2\ln(3/2)}{\pi n} + o(1/(kn)),$$

which shows that

$$S_*(n) \geq \frac{2\ln(3/2)}{\pi n} \quad \forall n. \tag{17}$$

In Sect. 4, we will improve these lower bounds by considering the multi-dimensional generalization of the above construction.

Remark 6 Since the parameter θ_* in Corollary 1 can be taken arbitrarily close to zero, the second part of Lemma 2 shows that the channel Φ_{θ_*}, for which $\bar{Q}_0(\Phi_{\theta_*}^{\otimes n}) = 0$ and $Q_0(\Phi_{\theta_*}) > 0$, can be chosen in any small vicinity (in the *cb*-norm) of the classical–quantum channel Φ_0^2.

Theorem 1 also gives examples of the superactivation of 1-shot quantum zero-error capacity.

Corollary 2 *If $\theta \neq 0, \pi$, then the following superactivation property*

$$\bar{Q}_0(\Phi_\theta) = \bar{Q}_0(\Phi_{\pi-\theta}) = 0 \quad \textit{and} \quad \bar{Q}_0(\Phi_\theta \otimes \Phi_{\pi-\theta}) > 0$$

holds for any channels $\Phi_\theta \in \widehat{\mathfrak{L}}_\theta$ and $\Phi_{\pi-\theta} \in \widehat{\mathfrak{L}}_{\pi-\theta}$. For any $\theta \in \mathrm{T}$, there exist 4 mutually orthogonal 2-D error-correcting codes for the channel $\Phi_\theta \otimes \Phi_{\pi-\theta}$, one of them is spanned by the vectors

$$|\varphi\rangle = \tfrac{1}{\sqrt{2}}\,[\,|11\rangle + \mathrm{i}\,|22\rangle\,], \quad |\psi\rangle = \tfrac{1}{\sqrt{2}}\,[\,|33\rangle + \mathrm{i}\,|44\rangle\,], \tag{18}$$

others are the images of this subspace under the unitary transformations $I_4 \otimes W_4$, $W_4 \otimes I_4$ and $W_4 \otimes W_4$ (the operator W_4 is defined in (8)).

Remark 7 Corollary 2 shows that the channel $\Phi^1_{\pi/2}$ (taken from the first part of Lemma 2) is an example of the symmetric superactivation of 1-shot quantum zero-error capacity *with Choi rank* 2.[3]

By taking the family $\{\Phi^2_\theta\}$ from the second part of Lemma 2 and tending θ to zero, we see from Corollary 2 that *the superactivation of 1-shot quantum zero-error capacity may hold for two channels with* $d_A = d_E = 4$ *if one of them is arbitrarily close (in the cb-norm) to a classical–quantum channel.*

Note that the entangled subspace spanned by the vectors (18) is an error-correcting code for the channel $\Phi^2_0 \otimes \Phi^2_\pi$ (and hence for the channel $\Phi^2_0 \otimes \mathrm{Id}_4$) despite the fact that Φ^2_0 is a classical–quantum channel.

Proof of Theorem 1 A) It is easy to verify that the subspace $\mathfrak{L}_\pi$ satisfies condition (6) with the vectors $|\varphi\rangle = [1, \mathrm{i}, 0, 0]^\top$, $|\psi\rangle = [0, 0, 1, \mathrm{i}]^\top$.

To prove that $\bar{Q}_0(\Phi_\theta) = 0$ for all $\theta \neq \pi$, it suffices to show that condition (6) is not valid for the subspace $\mathfrak{L}^s_\theta$ defined in (9) if $\theta \neq \pi$ (i.e.,$\gamma \neq \mathrm{i}$).

Assume the existence of unit vectors $|\varphi\rangle = [x_1, x_2, x_3, x_4]^\top$ and $|\psi\rangle = [y_1, y_2, y_3, y_4]^\top$ in $\mathbb{C}^4$ such that

$$\langle\psi|M|\varphi\rangle = 0 \quad \text{and} \quad \langle\psi|M|\psi\rangle = \langle\varphi|M|\varphi\rangle \quad \text{for all } M \in \mathfrak{L}^s_\theta. \tag{19}$$

Since condition (19) is invariant under the rotation

$$|\varphi\rangle \mapsto p|\varphi\rangle - q|\psi\rangle, \quad |\psi\rangle \mapsto \bar{q}|\varphi\rangle + \bar{p}|\psi\rangle, \quad |p|^2 + |q|^2 = 1,$$

we may consider that $y_1 = 0$.

By taking successively ($a = -1, b = c = d = 0$), ($b = -1, a = c = d = 0$), ($c = 1, a = b = d = 0$) and ($d = 1, a = b = c = 0$), we obtain from (19) the following equations

$$\begin{aligned} \bar{y}_1 x_1 = \bar{y}_2 x_2 = -\bar{y}_3 x_3 = -\bar{y}_4 x_4 &= \tfrac{1}{4}\langle\psi|T_\theta|\varphi\rangle, \\ |x_1|^2 - |y_1|^2 = |x_2|^2 - |y_2|^2 = |y_3|^2 - |x_3|^2 &= |y_4|^2 - |x_4|^2 \\ &= \tfrac{1}{4}\left(\langle\varphi|T_\theta|\varphi\rangle - \langle\psi|T_\theta|\psi\rangle\right). \end{aligned}$$

Since $y_1 = 0$ and $\|\varphi\| = \|\psi\| = 1$, the above equations imply

$$y_1 = y_2 = x_3 = x_4 = 0$$

and

$$|x_1|^2 = |x_2|^2 = |y_3|^2 = |y_4|^2 = \tfrac{1}{4}\left(\langle\varphi|T_\theta|\varphi\rangle - \langle\psi|T_\theta|\psi\rangle\right) = 1/2. \tag{20}$$

So, $|\varphi\rangle = [x_1, x_2, 0, 0]^\top$ and $|\psi\rangle = [0, 0, y_3, y_4]^\top$, where $[x_1, x_2]^\top$ and $[y_3, y_4]^\top$ are unit vectors in $\mathbb{C}^2$. It follows from (20) that

[3] This strengthens the result in [7], where a similar example with Choi rank 3 and the same input dimension was constructed.

$$2=\left\langle \begin{matrix} x_1 \\ x_2 \end{matrix} \middle| \begin{matrix} u & 0 \\ 0 & u \end{matrix} \middle| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\rangle - \left\langle \begin{matrix} y_3 \\ y_4 \end{matrix} \middle| \begin{matrix} -u & 0 \\ 0 & -u \end{matrix} \middle| \begin{matrix} y_3 \\ y_4 \end{matrix} \right\rangle = 2u,$$

which can be valid only if $\gamma = \mathrm{i}$, i.e., $\theta = \pi$.

The above arguments also show that $\bar{Q}_0(\Phi_\pi) = 1$, since the assumption $\bar{Q}_0(\Phi_\pi) > 1$ implies, by Lemma 1, the existence of orthogonal unit vectors ϕ_1, ϕ_2, ϕ_3 such that condition (19) with $\varphi = \phi_i$, $\psi = \phi_j$ is valid for all $i \neq j$.

B) Let $M_1 \in \mathfrak{L}_{\theta_1}, \ldots, M_n \in \mathfrak{L}_{\theta_n}$ be arbitrary and $X = M_1 \otimes \ldots \otimes M_n$. To prove that the linear hull $\mathcal{H}_0$ of vectors (11) is an error-correcting code for the channel $\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}$, it suffices, by Lemma 1, to show that

$$\langle\psi|X|\varphi\rangle = 0 \quad \text{and} \quad \langle\psi|X|\psi\rangle = \langle\varphi|X|\varphi\rangle. \tag{21}$$

We have

$$\begin{aligned} 2\langle\psi|X|\varphi\rangle &= \langle 3\ldots 3|X|1\ldots 1\rangle + \mathrm{i}\langle 3\ldots 3|X|2\ldots 2\rangle - \mathrm{i}\langle 4\ldots 4|X|1\ldots 1\rangle \\ &\quad + \langle 4\ldots 4|X|2\ldots 2\rangle = c_1\ldots c_n(\bar{\gamma}_1\ldots\bar{\gamma}_n + \gamma_1\ldots\gamma_n) \\ &\quad + d_1\ldots d_n(\mathrm{i}-\mathrm{i}) = 0, \end{aligned}$$

since $\gamma_1 \ldots \gamma_n = \pm\mathrm{i}$,

$$\begin{aligned} 2\langle\varphi|X|\varphi\rangle &= \langle 1\ldots 1|X|1\ldots 1\rangle + \mathrm{i}\langle 1\ldots 1|X|2\ldots 2\rangle - \mathrm{i}\langle 2\ldots 2|X|1\ldots 1\rangle \\ &\quad + \langle 2\ldots 2|X|2\ldots 2\rangle = a_1\ldots a_n(1+1) + b_1\ldots b_n(\mathrm{i}-\mathrm{i}) = 2a_1\ldots a_n \end{aligned}$$

and

$$\begin{aligned} 2\langle\psi|X|\psi\rangle &= \langle 3\ldots 3|X|3\ldots 3\rangle + \mathrm{i}\langle 3\ldots 3|X|4\ldots 4\rangle - \mathrm{i}\langle 4\ldots 4|X|3\ldots 3\rangle \\ &\quad + \langle 4\ldots 4|X|4\ldots 4\rangle = a_1\ldots a_n(1+1) + b_1\ldots b_n(\mathrm{i}-\mathrm{i}) = 2a_1\ldots a_n. \end{aligned}$$

So, the both equalities in (21) are valid.

To prove that the subspace $U_{\bar{x}}(\mathcal{H}_0)$, where $U_{\bar{x}} = U_{x_1} \otimes \ldots \otimes U_{x_n}$, is an error-correcting code for the channel $\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}$, it suffices to note that (8) implies that $U_{\bar{x}}^* A U_{\bar{x}} = A$ for all $A \in \mathfrak{L}_{\theta_1} \otimes \ldots \otimes \mathfrak{L}_{\theta_n}$.

C) To show that $\bar{Q}_0\left(\bigotimes_{k=1}^n \Phi_{\theta_k}\right) = 0$ if $\sum_{k=1}^n |\theta_k| \leq 2\ln(3/2)$ note that $\mathfrak{L}_\theta = \Upsilon_{D(\theta)}(\mathfrak{L}_0)$ and $\bigotimes_{k=1}^n \mathfrak{L}_{\theta_k} = \bigotimes_{k=1}^n \Upsilon_{D(\theta_k)}\left(\mathfrak{L}_0^{\otimes n}\right)$, where $\Upsilon_{D(\theta)}$ is the Schur multiplication by the matrix

$$D(\theta) = \begin{bmatrix} 1 & 1 & \gamma & 1 \\ 1 & 1 & 1 & \bar{\gamma} \\ \bar{\gamma} & 1 & 1 & 1 \\ 1 & \gamma & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \bar{\tau} \\ \bar{\tau} & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \end{bmatrix}, \quad \tau = \gamma - 1. \tag{22}$$

By using (22) and Theorem 8.7 in [9] [4], it is easy to show that

$$x_k \doteq \|\Upsilon_{D(\theta_k)} - \mathrm{Id}_4\|_{\mathrm{cb}} \le |\tau_k| = |1-\gamma_k| = \left|1-\exp\left(\tfrac{\mathrm{i}}{2}\theta_k\right)\right| \le \tfrac{1}{2}|\theta_k|. \tag{23}$$

Let $\Delta_n \doteq \|\bigotimes_{k=1}^{n}\Upsilon_{D(\theta_k)} - \mathrm{Id}_{4^n}\|_{\mathrm{cb}}$. By using multiplicativity of the *cb*-norm and (23), we obtain

$$\Delta_n \le x_n\prod_{k=1}^{n-1}(1+x_k) + \Delta_{n-1} \le \prod_{k=1}^{n}(1+x_k) - 1 \le \prod_{k=1}^{n}\left(1+\tfrac{1}{2}|\theta_k|\right) - 1. \tag{24}$$

Assume that $\bar{Q}_0\left(\bigotimes_{k=1}^{n}\Phi_{\theta_k}\right) > 0$. Then, Lemma 1 implies existence of unit vectors φ and ψ in $\mathcal{H}_A^{\otimes n} = \mathbb{C}^{4^n}$ such that

$$\langle\psi|\Psi(A)|\varphi\rangle = 0 \quad \text{and} \quad \langle\varphi|\Psi(A)|\varphi\rangle = \langle\psi|\Psi(A)|\psi\rangle \quad \forall A \in \mathfrak{L}_0^{\otimes n},$$

where $\Psi = \bigotimes_{k=1}^{n}\Upsilon_{D(\theta_k)}$. Hence, for any A in the unit ball of $\mathfrak{L}_0^{\otimes n}$, we have

$$|\langle\psi|A|\varphi\rangle| \le \Delta_n \quad \text{and} \quad |\langle\varphi|A|\varphi\rangle - \langle\psi|A|\psi\rangle| \le 2\Delta_n.$$

By using (24) and the inequality $x \ge \ln(1+x)$, it is easy to see that the assumption $\sum_{k=1}^{n}|\theta_k| \le 2\ln(3/2)$ implies $\Delta_n \le 1/2$. So, the above relations can not be valid by the below Lemma 3, since $\mathfrak{L}_0^{\otimes n}$ is a maximal commutative $*$-subalgebra of $\mathfrak{M}_{4^n}$. □

Lemma 3 *Let $\mathfrak{A}$ be a maximal commutative $*$-subalgebra of $\mathfrak{M}_n$. Then,*

$$\textit{either} \quad 2\sup_{A\in\mathfrak{A}_1}|\langle\psi|A|\varphi\rangle| > 1 \quad \textit{or} \quad \sup_{A\in\mathfrak{A}_1}|\langle|\varphi|A|\varphi\rangle - \langle\psi|A|\psi\rangle| > 1$$

for any two unit vectors φ and ψ in $\mathbb{C}^n$, where $\mathfrak{A}_1$ is the unit ball of $\mathfrak{A}$.

Proof Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be the coordinates of φ and ψ in the basis in which the algebra $\mathfrak{A}$ consists of diagonal matrices. Then,

$$\sup_{A\in\mathfrak{A}_1}|\langle\psi|A|\varphi\rangle| = \sum_{i=1}^{n}|x_i||y_i|, \quad \sup_{A\in\mathfrak{A}_1}|\langle|\varphi|A|\varphi\rangle - \langle\psi|A|\psi\rangle| = \sum_{i=1}^{n}\left||x_i|^2 - |y_i|^2\right|.$$

Let $d_i = |y_i| - |x_i|$. Assume that

$$2\sum_{i=1}^{n}|x_i||y_i| \le 1 \quad \text{and} \quad \sum_{i=1}^{n}\left||x_i|^2 - |y_i|^2\right| \le 1.$$

[4] This theorem states that $\|\Upsilon_A\|_{\mathrm{cb}} \le 1$ if and only if $A = [\langle\varphi_i|\psi_j\rangle]$ for some collections $\{\varphi_i\}$ and $\{\psi_j\}$ of vectors from the unit ball of some Hilbert space.

Since $\sum_{i=1}^{n}|x_i|^2=\sum_{i=1}^{n}|y_i|^2=1$, the first of these inequalities implies

$$\left|\sum_{i=1}^{n} d_i|x_i|\right|\geq 1/2 \quad \text{and} \quad \left|\sum_{i=1}^{n} d_i|y_i|\right|\geq 1/2.$$

Hence,

$$\sum_{i=1}^{n}\left||x_i|^2-|y_i|^2\right|=\sum_{i=1}^{n}|d_i|[|x_i|+|y_i|]>\left|\sum_{i=1}^{n} d_i|x_i|\right|+\left|\sum_{i=1}^{n} d_i|y_i|\right|\geq 1,$$

where the strict inequality follows from the existence of negative and positive numbers in the set $\{d_i\}_{i=1}^{n}$. This contradicts to the above assumption. $\square$

4 Multi-dimensional generalization

Note that

$$\mathfrak{L}_0=\mathfrak{A}_2^{\otimes 2}, \quad \text{where} \quad \mathfrak{A}_2=\left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix},\ a,b\in\mathbb{C}\right\},$$

and that $\mathfrak{L}_\theta$ is the image of $\mathfrak{L}_0$ under the Schur multiplication by matrix (22). So, the above construction can be generalized by considering the corresponding deformation of the maximal commutative $*$-subalgebra $\mathfrak{L}_0^p=\mathfrak{A}_2^{\otimes p}$ of $\mathfrak{M}_{2^p}$ for $p>2$. The algebra $\mathfrak{L}_0^p$ can be described recursively as follows:

$$\mathfrak{L}_0^p=\left\{\begin{bmatrix} A & B \\ B & A \end{bmatrix},\ A,B\in\mathfrak{L}_0^{p-1}\right\}, \quad \mathfrak{L}_0^1=\mathfrak{A}_2.$$

Let $p>2$ and $\theta\in\mathrm{T}\doteq(-\pi,\pi]$ be arbitrary, $\gamma=\exp\left(\frac{\mathrm{i}}{2}\theta\right)$. Let $D(\theta)$ be the $2^p\times 2^p$ matrix described as $2^{p-1}\times 2^{p-1}$ matrix $[A_{ij}]$ consisting of the blocks

$$A_{ii}=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \ \forall i, \quad A_{ij}=\begin{bmatrix} \gamma & 1 \\ 1 & \bar{\gamma} \end{bmatrix} \ \text{if} \ i<j \ \text{and} \ A_{ij}=\begin{bmatrix} \bar{\gamma} & 1 \\ 1 & \gamma \end{bmatrix} \ \text{if} \ i>j.$$

Consider the 2^p-D subspace $\mathfrak{L}_\theta^p=\Upsilon_{D(\theta)}\left(\mathfrak{L}_0^p\right)$ of $\mathfrak{M}_{2^p}$ (where $\Upsilon_{D(\theta)}$ is the Schur multiplication by the matrix $D(\theta)$, see Sect. 2). This subspace satisfies condition (5) and has the following property

$$A=W_{2^p}^{*}AW_{2^p} \quad \forall A\in\mathfrak{L}_\theta^p, \tag{25}$$

where W_{2^p} is the $2^p\times 2^p$ matrix having “1” on the main skew-diagonal and “0” on the other places. To prove (25), it suffices to show that it holds for the algebra $\mathfrak{L}_0^p=\mathfrak{A}_2^{\otimes p}$ (by using $W_{2^p}=W_2^{\otimes p}$) and to note that the map $\Upsilon_{D(\theta)}$ commutes with the transformation $A\mapsto W_{2^p}^{*}AW_{2^p}$.

Denote by $\widehat{\mathfrak{L}}^p_\theta$ the set of all quantum channels whose noncommutative graph coincides with $\mathfrak{L}^p_\theta$. By Proposition 2 in [15], the set $\widehat{\mathfrak{L}}^p_\theta$ contains pseudo-diagonal channels with $d_A = 2^p$ and d_E such that $d_E^2 \geq 2^p$.

Theorem 2 *Let $p > 1$ and $n > 1$ be given natural numbers, Φ_θ be an arbitrary channel in $\widehat{\mathfrak{L}}^p_\theta$ and $\delta_p = \frac{1}{2^{p-1}} \sum_{k=1}^{2^{p-1}} \left|\cot\left(\frac{(2k-1)\pi}{2^p}\right)\right| > 0.$*

A) *$\bar{Q}_0(\Phi_\theta^{\otimes n}) = 0$ if $|\theta| \leq \theta_n$, where θ_n is the minimal positive solution of the equation*

$$2(1-\cos(\theta/2)) + \delta_p \sin(\theta/2) = n^{-1}\ln(3/2).$$

B) *If $\theta = \pm\pi/n$, then $\bar{Q}_0(\Phi_\theta^{\otimes n}) \geq p-1$ and there exist 2^n mutually orthogonal 2^{p-1}-D error-correcting codes for the channel $\Phi_\theta^{\otimes n}$. For each binary n-tuple $(x_1,\dots x_n)$, the corresponding error-correcting code is spanned by the images of the vectors*

$$|\varphi_k\rangle = \tfrac{1}{\sqrt{2}}\left[\,|2k-1\dots 2k-1\rangle + \mathrm{i}\,|2k\dots 2k\rangle\,\right], \quad k = \overline{1, 2^{p-1}}, \tag{26}$$

under the unitary transformation $U_{x_1} \otimes \dots \otimes U_{x_n}$, where $\{|k\rangle\}$ is the canonical basis in $\mathbb{C}^{2^p}$, $U_0 = I_{2^p}$ and $U_1 = W_{2^p}$ (defined in (25)).

Remark 8 The constant δ_p is the Schur multiplier norm of the skew-symmetric $2^{p-1} \times 2^{p-1}$ matrix having "1" everywhere below the main diagonal. So, the sequence $\{\delta_p\}$ is nondecreasing. It is easy to see that $\delta_2 = 1$, $\delta_3 = \sqrt{2}$, $\delta_4 \approx 1.84$ and that $\delta_p = \left(\frac{2\ln 2}{\pi}\right) p + o(p)$ for large p [10].

Note also that $\theta_n = 2\ln(3/2)\left(n\delta_p\right)^{-1} + o(1/n)$ for large n.

Remark 9 Assertion B of Theorem 2 can be strengthened as follows:

B') *If $\theta = \pm\pi/n$ then there exist 2^n mutually orthogonal projectors $P_{\bar{x}}$ of rank 2^{p-1} indexed by a binary n-tuple $\bar{x} = (x_1,\dots x_n)$ such that*

$$P_{\bar{x}} A P_{\bar{x}} = \lambda(A) P_{\bar{x}} \quad \forall A \in [\mathfrak{L}^p_\theta]^{\otimes n},$$

where $\lambda(A) \in \mathbb{C}$ does not depend on $\bar{x}$. $P_{\bar{x}}$ is the projector on the subspace $U_{x_1} \otimes \dots \otimes U_{x_n}(\mathcal{H}_0)$, where $\mathcal{H}_0$ is the linear hull of vectors (26).

This follows from the proof of Theorem 2 presented below.

Proof of Theorem 2 A) Note that $\Upsilon_{D(\theta)} - \mathrm{Id}_{2^p}$ is the Schur multiplication by the matrix

$$-T \otimes \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} + S \otimes \begin{bmatrix} \bar{v} & 0 \\ 0 & v \end{bmatrix},$$

where T is the $2^{p-1} \times 2^{p-1}$ matrix having "0" on the main diagonal and "1" on the other places, S is the $2^{p-1} \times 2^{p-1}$ skew-symmetric matrix having "1" everywhere below the main diagonal, $u = 1 - \Re\gamma = 1 - \cos(\theta/2)$, $v = \mathrm{i}\Im\gamma = \mathrm{i}\sin(\theta/2)$.

In [10], it is shown that $\|\Upsilon_S\|_{\mathrm{cb}} = 2^{1-p}\|S\|_1 = \delta_p$. Since $\|\Upsilon_T\|_{\mathrm{cb}} \le 2$ and $\|\Upsilon_{A\otimes B}\|_{\mathrm{cb}} = \|\Upsilon_A \otimes \Upsilon_B\|_{\mathrm{cb}} = \|\Upsilon_A\|_{\mathrm{cb}}\|\Upsilon_B\|_{\mathrm{cb}}$, we have

$$x \doteq \|\Upsilon_{D(\theta)} - \mathrm{Id}_{2^p}\|_{\mathrm{cb}} \le u\|\Upsilon_T\|_{\mathrm{cb}} + |v|\|\Upsilon_S\|_{\mathrm{cb}} = 2(1-\cos(\theta/2)) + \delta_p|\sin(\theta/2)|$$

and hence $x \le n^{-1}\ln(3/2) \le \sqrt[n]{3/2} - 1$ if $|\theta| \le \theta_n$.

Assume that $\bar{Q}_0(\Phi_\theta^{\otimes n}) > 0$ for some $\theta \in [-\theta_n, \theta_n]$. By repeating the arguments from the proof of part C of Theorem 1, we obtain

$$|\langle\psi|A|\varphi\rangle| \le \Delta_n \quad \text{and} \quad |\langle\varphi|A|\varphi\rangle - \langle\psi|A|\psi\rangle| \le 2\Delta_n \tag{27}$$

for some unit vectors $\varphi, \psi \in \mathbb{C}^{2pn}$ and all A in the unit ball of $[\mathfrak{L}_0^p]^{\otimes n}$, where

$$\Delta_n \doteq \left\|\Upsilon_{D(\theta)}^{\otimes n} - \mathrm{Id}_{2^{pn}}\right\|_{\mathrm{cb}} \le (x+1)^n - 1 \le 1/2.$$

Since $[\mathfrak{L}_0^p]^{\otimes n}$ is a maximal commutative $*$-subalgebra of $\mathfrak{M}_{2^{pn}}$, Lemma 3 shows that (27) cannot be valid.

B) Let $\theta = \pm\pi/n$. To prove that the linear hull $\mathcal{H}_0$ of vectors (26) is an error-correcting code for the channel $\Phi_\theta^{\otimes n}$, it suffices, by Lemma 1, to show that

$$\langle\varphi_l|M_1 \otimes \ldots \otimes M_n|\varphi_k\rangle = 0 \quad \forall M_1, \ldots, M_n \in \mathfrak{L}_\theta^p, \ \forall k, l$$

and that

$$\langle\varphi_l|M_1 \otimes \ldots \otimes M_n|\varphi_l\rangle = \langle\varphi_k|M_1 \otimes \ldots \otimes M_n|\varphi_k\rangle \quad \forall M_1, \ldots, M_n \in \mathfrak{L}_\theta^p, \ \forall k, l.$$

Since any matrix in $\mathfrak{L}_\theta^p$ can be described as $2^{p-1} \times 2^{p-1}$ matrix $[A_{ij}]$ consisting of the blocks

$$A_{ii} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \forall i \quad \text{and} \quad A_{ij} = \begin{bmatrix} \bar{\gamma}_{ij}c_{ij} & d_{ij} \\ d_{ij} & \gamma_{ij}c_{ij} \end{bmatrix} \ \forall i \ne j,$$

where $\gamma_{ij} = \exp\left(\mathrm{i}s_{ij}\theta/2\right)$, $s_{ij} = \mathrm{sgn}(i-j)$ and a, b, c_{ij}, d_{ij} are some complex numbers, the above relations are proved by the same way as in the proof of part B of Theorem 1 (by using $\gamma_{ij}^n + \bar{\gamma}_{ij}^n = 0$).

To prove that the subspace $U_{\bar{x}}(\mathcal{H}_0)$, where $U_{\bar{x}} = U_{x_1} \otimes \ldots \otimes U_{x_n}$, is an error-correcting code for the channel $\Phi_\theta^{\otimes n}$, it suffices to note that (25) implies that $U_{\bar{x}}^* A U_{\bar{x}} = A$ for all $A \in [\mathfrak{L}_\theta^p]^{\otimes n}$. □

Corollary 3 *Let n be arbitrary and m be a natural number such that $\theta_* = \pi/m \le \theta_n$. Then*

$$\bar{Q}_0\left(\Phi_{\theta_*}^{\otimes n}\right) = 0 \quad \text{but} \quad \bar{Q}_0\left(\Phi_{\theta_*}^{\otimes m}\right) \ge p-1 \quad \text{and hence} \quad Q_0\left(\Phi_{\theta_*}\right) \ge (p-1)/m.$$

There exist 2^m mutually orthogonal 2^{p-1}-D error-correcting codes for the channel $\Phi_{\theta_*}^{\otimes m}$.

Remark 10 Corollary 3 (with Proposition 2 in [15] and Remark 8) shows that for any n, there exists a channel Φ_n with $d_A = 2^p$ and arbitrary d_E satisfying the inequality $d_E^2 \geq 2^p$ such that

$$\bar{Q}_0\left(\Phi_n^{\otimes n}\right) = 0 \quad \text{and} \quad Q_0(\Phi_n) \geq \frac{p-1}{[\pi/\theta_n]+1} = \frac{2\ln(3/2)(p-1)}{\pi n \delta_p} + o(1/n),$$

where $[x]$ is the integer part of x, and hence, we have the following lower bounds for the values $S_d(n)$ and $S_*(n)$ (introduced in 13,14)

$$S_{2^p} \geq \frac{2\ln(3/2)(p-1)}{\pi n \delta_p} + o(1/n) \quad \text{and} \quad S_*(n) \geq \frac{2\ln(3/2)(p-1)}{\pi n \delta_p}$$

(the later inequality is obtained from the former by using relation (15)).

Since $\delta_2 = 1$, the above lower bounds with $p = 2$ coincide with (16,17).

Since $\delta_3 = \sqrt{2}$, Remark 10 with $p = 3$ shows that for any n, there exists a channel Φ_n with $d_A = 8$ and $d_E = 3$ such that

$$\bar{Q}_0\left(\Phi_n^{\otimes n}\right) = 0 \quad \text{and} \quad Q_0(\Phi_n) \geq \sqrt{2} \times \frac{2\ln(3/2)}{\pi n} + o(1/n).$$

Hence,

$$S_8(n) \geq \sqrt{2} \times \frac{2\ln(3/2)}{\pi n} + o(1/n).$$

Comparing this estimation with (16), we see that the increasing input dimension d_A from 4 to 8 gives the amplification factor $\sqrt{2}$ for the quantum zero-error capacity of a channel having vanishing n-shot capacity (more precisely, for the lower bound of this capacity).

In general, Remark 10 shows that our construction with the input dimension $d_A = 2^p$ amplifies lower bound (17) for $S_*(n)$ by the factor $\Lambda_p = \delta_p^{-1}(p-1)$. By Remark 8, the nondecreasing sequence Λ_p has a finite limit:

$$\lim_{p\to+\infty} \Lambda_p = \Lambda_* \doteq \frac{\pi}{2\ln 2} \approx 2.26.$$

Hence, $\Lambda_* \approx 2.26$ is the maximal amplification factor for $S_*(n)$ which can be obtained by increasing input dimension. So, we have

$$S_*(n) \geq \Lambda_* \frac{2\ln(3/2)}{\pi n} = \frac{\log_2(3/2)}{n} \qquad \forall n.$$

Unfortunately, we have not managed to show the existence of a channel with *arbitrary* quantum zero-error capacity and vanishing n-shot capacity, i.e., to prove the conjecture $S_*(n) = +\infty$ for all n. This can be explained as follows.

According to Theorem 2, if the input dimension of the channel Φ_θ increases as 2^p, then the dimension of error-correcting code for the channel $\Phi_\theta^{\otimes m}$, $\theta = \pi/m$, increases as 2^{p-1}. But simultaneously, the norm of the map $\Upsilon_{D(\theta)} - \mathrm{Id}_{2^p}$ characterizing deformation of a maximal commutative $*$-subalgebra increases as $\delta_p \sin(\theta/2) \sim p\theta/2$ for large p and small θ, so, to guarantee vanishing of the n-shot capacity of Φ_θ by using Lemma 3, we have to decrease the value of θ as $O(1/p)$. Since $\theta = \pi/m$, we see that $\bar{Q}_0(\Phi_\theta^{\otimes m})$ and m have the same increasing rate $O(p)$, which does not allow to obtain large values of $Q_0(\Phi_\theta)$.

Thus, the main obstacle for proving the conjecture $S_*(n) = +\infty$ consists in the unavoidable growth of the norm of the map $\Upsilon_{D(\theta)} - \mathrm{Id}_{2^p}$ as $p \to +\infty$ (for fixed θ).

First, there was a hope to solve this problem by using a freedom in choice of the deformation map $\Upsilon_{D(\theta)}$. Indeed, instead of the matrix $D(\theta)$ introduced before the definition of $\mathfrak{L}_\theta^p$, one can use the matrix $D(\theta, S) = [A_{ij}]$ consisting of the blocks

$$A_{ii} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \forall i, \quad A_{ij} = \begin{bmatrix} \gamma & 1 \\ 1 & \bar{\gamma} \end{bmatrix} \quad \text{if} \quad s_{ij} = -1 \quad \text{and} \quad A_{ij} = \begin{bmatrix} \bar{\gamma} & 1 \\ 1 & \gamma \end{bmatrix} \quad \text{if} \quad s_{ij} = 1,$$

where $S = [s_{ij}]$ is *any* skew-symmetric $2^{p-1} \times 2^{p-1}$ matrix such that $s_{ij} = \pm 1$ for all $i \neq j$. For the corresponding subspace $\mathfrak{L}_{\theta,S}^p = \Upsilon_{D(\theta,S)}\left(\mathfrak{L}_0^p\right)$, the main assertions of Theorem 2 are valid (excepting the assertion about 2^m error-correcting codes) with the constant δ_p replaced by the norm $\|\Upsilon_S\|_{\mathrm{cb}}$ (in our construction $S = S_*$ is the matrix having "1" everywhere below the main diagonal and $\delta_p = \|\Upsilon_{S_*}\|_{\mathrm{cb}}$). But the further analysis (based on the results from [10]) has shown that

$$\|\Upsilon_S\|_{\mathrm{cb}} \geq \delta_p = \|\Upsilon_{S_*}\|_{\mathrm{cb}}$$

and hence

$$\|\Upsilon_{D(\theta,S)} - \mathrm{Id}_{2^p}\|_{\mathrm{cb}} \geq \|\Upsilon_{D(\theta,S_*)} - \mathrm{Id}_{2^p}\|_{\mathrm{cb}}$$

for any skew-symmetric $2^{p-1} \times 2^{p-1}$ matrix S such that $s_{ij} = \pm 1$ for all $i \neq j$. So, by using the above modification, we cannot increase the lower bound for $Q_0(\Phi_\theta)$. The useless of some other modifications of the map $\Upsilon_{D(\theta)}$ was also shown.

It is interesting to note that the norm growth of the map $\Upsilon_{D(\theta)} - \mathrm{Id}_{2^p}$ is a *cost of the symmetry requirement* for the subspace $\mathfrak{L}_\theta^p$. Indeed, if we omit this requirement, then we would use the matrix $\widetilde{D}(\theta) = [A_{ij}]$ consisting of the blocks

$$A_{ii} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \forall i \quad \text{and} \quad A_{ij} = \begin{bmatrix} \gamma & 1 \\ 1 & \bar{\gamma} \end{bmatrix} \forall i \neq j,$$

for which $\|\Upsilon_{\widetilde{D}(\theta)} - \mathrm{Id}_{2^p}\|_{\mathrm{cb}} \leq 2|\gamma - 1| \leq \theta$ for all p.

It seems that the above obstacle is technical and can be overcome (within the same construction of a channel) by finding a way to prove the equality $\bar{Q}_0(\Phi^{\otimes n}) = 0$ not using estimations of the distance between the unit balls of $[\mathfrak{L}_\theta^p]^{\otimes n}$ and of $[\mathfrak{L}_0^p]^{\otimes n}$. Anyway the question concerning the value

$$S_*(n) \doteq \sup_{\Phi} \left\{ Q_0(\Phi) \mid \bar{Q}_0(\Phi^{\otimes n}) = 0 \right\}$$

remains open.

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Appendix: Proof of Lemma 2

Show first that for each θ, one can construct basis $\{A_i^\theta\}_{i=1}^4$ of $\mathfrak{L}_\theta$ consisting of positive operators with $\sum_{i=1}^4 A_i^\theta = I_4$ such that:

(1) the function $\theta \mapsto A_i^\theta$ is continuous for $i = \overline{1,4}$;
(2) $\{A_i^0\}_{i=1}^4$ consists of mutually orthogonal 1-rank projectors.

Recall that $\mathfrak{L}_\theta$ is unitary equivalent to the subspace $\mathfrak{L}_\theta^s$ defined by (9).

Denote by $\|T_\theta\|$ the operator norm of the matrix T_θ involved in (9). Note that the function $\theta \mapsto T_\theta$ is continuous, $T_0 = 0$ and $\|T_\theta\| \leq \|T_\pi\| = 2$. Let

$$\tilde{A}_1^\theta = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} - \tfrac{1}{4}(\alpha - \beta)\, T_\theta, \quad \tilde{A}_2^\theta = \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} - \tfrac{1}{4}(\alpha - \beta)\, T_\theta,$$
$$\tilde{A}_3^\theta = \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} + \tfrac{1}{4}(\alpha - \beta)\, T_\theta, \quad \tilde{A}_4^\theta = \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} + \tfrac{1}{4}(\alpha - \beta)\, T_\theta$$

be operators in $\mathfrak{L}_\theta^s$, where $\beta = \min\left\{\frac{3}{16}, \frac{1}{4}\|T_\theta\|\right\}$ and $\alpha = 1 - 3\beta$. It is easy to verify that $\tilde{A}_i^\theta \geq 0$ for all i and $\sum_{i=1}^4 \tilde{A}_i^\theta = I_4$. Then $\{A_i^\theta = S\tilde{A}_i^\theta S^{-1}\}_{i=1}^4$, where S is the unitary matrix defined after (9), is a required basis of $\mathfrak{L}_\theta$.

Let $m \geq 2$ and $\{|\psi_i\rangle\}_{i=1}^4$ be a collection of unit vectors in $\mathbb{C}^m$ such that $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^4$ is a linearly independent subset of $\mathfrak{M}_m$. It is easy to show (see the proof of Corollary 1 in [15]) that $\mathfrak{L}_\theta$ is a noncommutative graph of the pseudo-diagonal channel

$$\Phi_\theta(\rho) = \mathrm{Tr}_{\mathbb{C}^m} V_\theta \rho V_\theta^*,$$

where

$$V_\theta : |\varphi\rangle \mapsto \sum_{i=1}^4 \left[A_i^\theta\right]^{1/2} |\varphi\rangle \otimes |i\rangle \otimes |\psi_i\rangle$$

is an isometry from $\mathcal{H}_A = \mathbb{C}^4$ into $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^m$ ($\{|i\rangle\}$ is the canonical basis in $\mathbb{C}^4$). By property 1 of the basis $\{A_i^\theta\}_{i=1}^4$, the function $\theta \mapsto V_\theta$ is continuous.

The first part of Lemma 2 follows from this construction with $m = 2$.

To prove the second part assume that $m = 4$ and $|\psi_i\rangle = |i\rangle$, $i = \overline{1,4}$. Property 2 of the basis $\{A_i^\theta\}_{i=1}^4$ implies

$$V_0|\varphi\rangle = \sum_{i=1}^4 \langle e_i|\varphi\rangle|e_i\rangle \otimes |i\rangle \otimes |i\rangle,$$

where $\{|e_i\rangle\}_{i=1}^4$ is an orthonormal basis in $\mathbb{C}^4$. Hence, $\Phi_0(\rho) = \sum_{i=1}^4 \langle e_i|\rho|e_i\rangle\sigma_i$, $\sigma_i = |e_i \otimes i\rangle\langle e_i \otimes i|$, is a classical–quantum channel.

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