The additivity problem for constrained quantum channels

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One of the central problems of quantum information theory [1] is the additivity conjecture for the channel capacity of quantum communication channels [2]–[4]. Recently Shor showed [5] that the validity of this conjecture for all quantum channels was equivalent to a whole series of (super)additivity properties for other important characteristics such as minimal output entropy and entanglement of formation. The present paper introduces the new property of strong additivity (additivity for channels with input constraints) with several equivalent formulations, establishes that it is satisfied for some nontrivial classes of channels and notes that the validity of the additivity conjecture for all channels implies strong additivity.

Let $\mathcal{H}$, $\mathcal{H}'$ be finite-dimensional unitary spaces. By a quantum channel we mean a completely positive trace-preserving linear map $\Phi\colon \mathfrak{B}(\mathcal{H}) \mapsto \mathfrak{B}(\mathcal{H}')$, where $\mathfrak{B}(\mathcal{H})$ is the algebra of all operators on $\mathcal{H}$. In particular, Φ generates an affine map of the convex set $\mathfrak{S}(\mathcal{H})$ of states (density operators) on the space $\mathcal{H}$ into the set $\mathfrak{S}(\mathcal{H}')$ of states on $\mathcal{H}'$ [1].

The quantum analogue of Shannon's theorem, established in [4], gives the following expression for the classical channel capacity:

$$C(\Phi) = \lim_{n\to\infty} n^{-1}\overline{C}(\Phi^{\otimes n}),$$

where $\Phi^{\otimes n} = \Phi \otimes \cdots \otimes \Phi$ is the n-th tensor power of the channel Φ, and

$$\overline{C}(\Phi) = \max_{\{\pi_i,\rho_i\}} \biggl\{ H\biggl(\sum_i \pi_i \Phi(\rho_i)\biggr) - \sum_i \pi_i H(\Phi(\rho_i)) \biggr\}. \tag{1}$$

Here $H(\rho) = -\operatorname{Tr}\rho\log\rho$ is the von Neumann entropy, and the maximum is taken over arbitrary input ensembles $\{\pi_i,\rho_i\}$, representing a finite collection of states $\{\rho_i\} \subset \mathfrak{S}(\mathcal{H})$ with corresponding probabilities $\{\pi_i\}$. The additivity conjecture states that

$$\overline{C}(\Phi\otimes\Psi) = \overline{C}(\Phi) + \overline{C}(\Psi), \tag{2}$$

where $\Phi\otimes\Psi$ is the tensor product of the channels Φ, Ψ. Its validity for a fixed channel Φ and an arbitrary channel Ψ implies, in particular, that $\overline{C}(\Phi) = C(\Phi)$. For a survey of results in this connection, see [3], [5].

Denoting the expression in curly brackets in (1) by $\chi_\Phi(\{\pi_i,\rho_i\})$, we introduce the function

$$\chi_\Phi(\rho) = \max_{\rho_{\rm av}=\rho} \chi_\Phi(\{\pi_i,\rho_i\}) = H(\Phi(\rho)) - \widehat{H}_\Phi(\rho), \tag{3}$$

where $\rho_{\rm av} = \sum_i \pi_i\rho_i$ is the mean of the ensemble $\{\pi_i,\rho_i\}$ and $\widehat{H}_\Phi(\rho) = \min_{\rho_{\rm av}=\rho}\sum_i \pi_i H(\Phi(\rho_i))$ is the convex closure [6] of the output entropy of the channel $H(\Phi(\rho))$. The function $\chi_\Phi(\rho)$, just like $H(\Phi(\rho))$, is a concave continuous function on $\mathfrak{S}(\mathcal{H})$. We also introduce the function

$$\nu_H(\Phi, A) = \min_{\rho\in\mathfrak{S}(\mathcal{H})} [H(\Phi(\rho)) + \operatorname{Tr} A\rho], \tag{4}$$

which is adjoint to the minimal output entropy [6].

Let $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ be a closed subset of states on the input channel. We consider the quantity

$$\overline{C}(\Phi,\mathcal{A}) = \max_{\rho\in\mathcal{A}} \chi_\Phi(\rho) = \max_{\rho_{\rm av}\in\mathcal{A}} \chi_\Phi(\{\pi_i,\rho_i\}). \tag{5}$$

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In applications $\mathcal{A}$ is usually defined by the linear inequality $\operatorname{Tr} A\rho \leqslant \alpha$, where A is a positive operator (for example, the energy), and $\alpha \geqslant 0$.

We consider two channels $\Phi\colon \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi\colon \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ with constraints defined by sets $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathfrak{S}(\mathcal{K})$ respectively. For the channel $\Phi \otimes \Psi$ it is natural to introduce the 'inherited' constraint defined by $\omega_{\mathrm{av}}^{\Phi} = \operatorname{Tr}_{\mathcal{K}} \omega_{\mathrm{av}} \in \mathcal{A}$ and $\omega_{\mathrm{av}}^{\Psi} = \operatorname{Tr}_{\mathcal{H}} \omega_{\mathrm{av}} \in \mathcal{B}$, where ω_{av} is the average of the input ensemble $\{\mu_i, \omega_i\}$. We denote the corresponding subset of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ by $\mathcal{A} \otimes \mathcal{B}$. The additivity conjecture for constrained channels states that

$$\overline{C}(\Phi \otimes \Psi; \mathcal{A} \otimes \mathcal{B}) = \overline{C}(\Phi; \mathcal{A}) + \overline{C}(\Psi; \mathcal{B}). \tag{6}$$

Theorem 1. *Let Φ and Ψ be arbitrary channels. The following properties are equivalent*:

(i) (6) *holds for arbitrary closed subsets $\mathcal{A}$ and $\mathcal{B}$*;

(ii) (6) *holds for subsets $\mathcal{A}$ and $\mathcal{B}$ defined respectively by the inequalities* $\operatorname{Tr} A\rho_{\mathrm{av}} \leqslant \alpha$ *and* $\operatorname{Tr} B\sigma_{\mathrm{av}} \leqslant \beta$ *for arbitrary A, α, B, β*;

(iii) $\chi_{\Phi\otimes\Psi}(\omega) \leqslant \chi_{\Phi}(\omega^{\Phi}) + \chi_{\Psi}(\omega^{\Psi}) \quad \forall\, \omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$; (7)

(iv) $\widehat{H}_{\Phi\otimes\Psi}(\omega) \geqslant \widehat{H}_{\Phi}(\omega^{\Phi}) + \widehat{H}_{\Psi}(\omega^{\Psi}) \quad \forall\, \omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$; (8)

(v) $\nu_H(\Phi \otimes \Psi, A \otimes I + I \otimes B) = \nu_H(\Phi, A) + \nu_H(\Psi, B) \ \ \forall\, A \in \mathfrak{B}_+(\mathcal{H}),\ \ \forall\, B \in \mathfrak{B}_+(\mathcal{K})$.

If one of these equivalent conditions is satisfied we shall say that *strong additivity* holds for the channels Φ and Ψ.

We consider a channel $\Phi\colon \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$, and let $E \in \mathfrak{B}(\mathcal{H})$, $0 \leqslant E \leqslant I$. Let $q \in [0;1]$ and $d \in \mathbb{N} = \{1, 2, \dots\}$. The Shor extension $\widehat{\Phi}(E, q, d)$ acts with probability $1 - q$ like the channel Φ, and with probability q makes a measurement in $\mathcal{H}$ with two outcomes $\{0, 1\}$, described by a resolution of the identity $\{I - E, E\}$. If the outcome of the measurement is 1, then $\log d$ bits of classical information are sent to the output, and otherwise a failure signal. For a precise definition see [7], [5]. The asymptotics are interesting when q tends to zero and d to infinity, in such a way that $q \log d = \lambda$ remains constant. Then with high probability $\widehat{\Phi}$ acts like Φ on the input states ρ, occasionally sending a large quantity of information proportional to the quantity $\operatorname{Tr} \rho E$. This explains the connection between the channel capacity of the extension $\widehat{\Phi}$ and that of the channel Φ with a linear constraint on the input.

Theorem 2. *Let $\Phi\colon \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ and $\Psi\colon \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')$ be arbitrary channels with a fixed constraint $\mathcal{B}$ on the input of the second channel. The following assertions are equivalent*:

(i) *the additivity* (6) *holds for the channel Φ with an arbitrary closed constraint on the input and the channel Ψ with the constraint $\mathcal{B}$*;

(ii) *the additivity property is satisfied asymptotically as $d \to \infty$ for the sequence of channels $\{\widehat{\Phi}(E, \lambda/\log d, d)\}_{d\in\mathbb{N}}$ with an arbitrary operator $0 \leqslant E \leqslant I$ and an arbitrary $\lambda > 0$, and for the channel Ψ with the constraint $\mathcal{B}$.*

Applying this theorem twice, we get the following.

Corollary. *The validity of the additivity conjecture* (2) *for all channels implies the strong additivity property* (6).

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