On the notion of entanglement in Hilbert spaces

A. S. Kholevo [Holevo], M. E. Shirokov, and R. F. Werner

A density operator (state) on a tensor product $\mathcal{H} \otimes \mathcal{K}$ of Hilbert spaces is *separable* if it is in the convex closure of the subset of all tensor product states. Non-separable states are called *entangled*. These concepts are of great importance in quantum information theory, but they have been studied in depth only in the finite-dimensional context [1]. In this note we give a general integral representation for separable states and provide the first example of separable states that are not countably decomposable. We also prove a structure theorem for quantum communication channels that are entanglement-breaking, generalizing the finite-dimensional result of [2]. In the finite-dimensional case such channels can be characterized as having a Stinespring–Kraus representation (3) with operators V_j of rank 1. The above example implies the existence of infinite-dimensional entanglement-breaking channels having no such representation.

In what follows, $\mathcal{H}, \mathcal{K}, \dots$ are separable Hilbert spaces, $\mathfrak{T}(\mathcal{H})$ is the Banach space of trace-class operators and $\mathfrak{S}(\mathcal{H})$ is the convex subset of all density operators on $\mathcal{H}$. For brevity we shall also call them *states*, having in mind that a density operator ρ uniquely determines a normal state on the algebra $\mathfrak{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. Equipped with the trace-norm topology, $\mathfrak{S}(\mathcal{H})$ is a complete separable metric space. If π is a Borel probability measure on $\mathfrak{S}(\mathcal{H})$, then the Bochner integral

$$\bar{\rho}(\pi) = \int_{\mathfrak{S}(\mathcal{H})} \sigma\, \pi(d\sigma) \tag{1}$$

defines a state called the *barycenter* of π.

The following lemma, which strengthens the Choquet decomposition for the case of closed convex subsets of $\mathfrak{S}(\mathcal{H})$, is proved using the compactness criterion for subsets of probability measures on $\mathfrak{S}(\mathcal{H})$ [3]. We denote by $\overline{\mathrm{co}}\mathcal{A}$ the convex closure of a set $\mathcal{A}$ [4].

Lemma. *Let $\mathcal{A}$ be a closed subset of $\mathfrak{S}(\mathcal{H})$. Then $\overline{\mathrm{co}}\mathcal{A}$ coincides with the set of barycenters of all Borel probability measures supported by $\mathcal{A}$.*

Definition 1. A state on $\mathcal{H} \otimes \mathcal{K}$ is called *separable* if it is in the convex closure of the subset of all tensor product states $\rho \otimes \sigma$, where $\rho \in \mathfrak{S}(\mathcal{H})$ and $\sigma \in \mathfrak{S}(\mathcal{K})$. A state is called *entangled* if it is not separable.

In this definition one can replace the set of all product states by the set of all products of pure states (extreme points of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$). The subset $\mathfrak{P}(\mathcal{H} \otimes \mathcal{K})$ of pure states is closed in the trace-norm topology. The lemma then implies that a state ρ is separable if and only if there is a Borel probability measure μ on $\mathfrak{P}(\mathcal{H}) \times \mathfrak{P}(\mathcal{K})$ such that

$$\rho = \int_{\mathfrak{P}(\mathcal{H})} \int_{\mathfrak{P}(\mathcal{K})} |\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi|\, \mu(d\varphi d\psi). \tag{2}$$

In the finite-dimensional case application of Carathéodory's theorem reduces this to the familiar definition of a separable state as a finite convex combination of pure product states [1]. In general, we call the state *countably decomposable* if it is possible to find a representation (2) with purely atomic measure μ.

A *channel* is a positive trace-preserving linear map Φ from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ such that the dual map $\Phi^* : \mathfrak{B}(\mathcal{H}') \mapsto \mathfrak{B}(\mathcal{H})$ (which exists since Φ is bounded) is completely positive. An arbitrary channel admits a (non-unique) Stinespring–Kraus representation

$$\Phi(\rho) = \sum_j V_j \rho V_j^*, \tag{3}$$

where the V_j are bounded operators from $\mathcal{H}$ to $\mathcal{H}'$ such that $\sum_j V_j^* V_j = I$.

This work was carried out with the partial support of the A. von Humboldt Foundation and the grant NSh no. 1758.2003.1.

AMS 2000 *Mathematics Subject Classification.* 47A67, 47B10, 94A40.

DOI 10.1070/RM2005v060n02ABEH000830

Definition 2. A channel Φ is called *entanglement-breaking* if for an arbitrary Hilbert space $\mathcal{K}$ and an arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ the state $(\Phi \otimes \mathrm{Id}_{\mathcal{K}})(\omega)$, where $\mathrm{Id}_{\mathcal{K}}$ is the identity channel on $\mathfrak{S}(\mathcal{K})$, is separable.

Theorem 1. *A channel Φ is entanglement-breaking if and only if there is a complete separable metric space $\mathcal{X}$, a Borel $\mathfrak{S}(\mathcal{H}')$-valued function $x \mapsto \rho'(x)$, and a resolution of the identity (a positive operator-valued Borel measure) $M(dx)$ on $\mathcal{X}$ such that*

$$\Phi(\rho) = \int_{\mathcal{X}} \rho'(x)\, \mu_\rho(dx), \tag{4}$$

where $\mu_\rho(B) = \operatorname{Tr} \rho M(B)$ for all Borel sets $B \subseteq \mathcal{X}$.

The proof uses a generalization of the well-known correspondence between completely positive maps and states in $\mathcal{H}' \otimes \mathcal{K}$ (see [5]).

If there is a representation (4) with purely atomic measure $M(dx)$, then we call the channel Φ *countably decomposable*. This is easily seen to be equivalent to the channel having a representation (3) with operators V_j of rank 1. On the other hand, the channel is countably decomposable if and only if the states in $(\Phi \otimes \mathrm{Id}_{\mathcal{K}})(\mathfrak{S}(\mathcal{H} \otimes \mathcal{K}))$ are countably decomposable. This reduces the question of the existence of entanglement-breaking channels which have no representation (3) with operators of rank 1 to the question of the existence of a separable state which is not countably decomposable. We give a construction of such states below.

Let $\mathbb{T}$ be the one-dimensional torus parametrized as the interval $[0, 2\pi)$ with addition mod 2π, and let $\mathcal{H} = L^2(\mathbb{T})$ with the normalized Lebesgue measure $\frac{dx}{2\pi}$. We consider the unitary representation $x \to V_x$ of $\mathbb{T}$, where $(V_u\psi)(x) = \psi(x-u)$.

Theorem 2. *For arbitrary state vectors $|\varphi_j\rangle \in \mathcal{H}_j \simeq L^2(\mathbb{T})$, $j = 1, 2$, with non-vanishing Fourier coefficients the separable state*

$$\rho_{12} = \int_0^{2\pi} V_x^{(1)} |\varphi_1\rangle\langle\varphi_1| V_x^{(1)*} \otimes V_x^{(2)} |\varphi_2\rangle\langle\varphi_2| V_x^{(2)*} \, \frac{dx}{2\pi}$$

on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is not countably decomposable.

Corollary 1. *There exists an entanglement-breaking channel which has no representation* (3) *with operators of rank* 1.

The proof is by explicit construction using the example of Theorem 2 and the correspondence between channels and the states on the tensor product of the spaces. We conjecture that the subset of states that are not countably decomposable is dense in the set of all separable states. The following result is a step in this direction.

Corollary 2. *There is a separable state that is not countably decomposable in an arbitrary neighborhood of an arbitrary pure product state.*

Bibliography

[1] R. F. Werner, "Quantum information theory–an invitation", Quantum Information. An Introduction to Basic Theoretical Concepts and Experiments, Chapt. 2, (Springer Tracts Modern Phys., vol. 173) Springer-Verlag, Berlin, 2001; e-print quant-ph/0101061.

[2] M. Horodecki, P. W. Shor, and M. B. Ruskai, *Rev. Math. Phys.* **15** (2003), 629–641; e-print quant-ph/0302031.

[3] A. S. Holevo and M. E. Shirokov, *Teor. Veroyatnost. i Primenen.* **50**:1 (2005); English transl. in *Theory Probab. Appl.* **50** (2005); e-print quant-ph/0408176.

[4] A. D. Ioffe and V. M. Tikhomirov, *Theory of extremal problems*, Nauka, Moscow 1974; English transl., North-Holland, Amsterdam–New York 1979.

[5] M.-D. Choi, *Linear Algebra Appl.* **10** (1975), 285–290.

Institute of Mathematical Physics, TUB, Braunschweig, Germany;
Steklov Institute of Mathematics, Russian Academy of Sciences
E-mail: holevo@mi.ras.ru, msh@mi.ras.ru

Presented by A. V. Bulinskii
Received 15/MAR/05