Superadditivity of the convex closure of the output entropy of a quantum channel

M. E. Shirokov

One of the main recent achievements of quantum information theory [1] is the proof of the equivalence of several (sub-, super-) additivity conjectures for finite-dimensional quantum channels and systems [2] (see the survey in [3]). The main purpose of this paper is to give a generalization of this result to the infinite-dimensional case.

Let $\mathscr{H}$ and $\mathscr{H}'$ be separable Hilbert spaces. A quantum channel is a linear completely positive trace-preserving map $\Phi\colon \mathfrak{T}(\mathscr{H}) \mapsto \mathfrak{T}(\mathscr{H}')$, where $\mathfrak{T}(\mathscr{H})$ is the ideal of all trace class operators on $\mathscr{H}$. In particular, Φ generates an affine map of the convex set $\mathfrak{S}(\mathscr{H})$ of states (that is, density operators) on the space $\mathscr{H}$ to the set $\mathfrak{S}(\mathscr{H}')$ of states on the space $\mathscr{H}'$ [1].

Important characteristics of a quantum channel are the output entropy $H_\Phi(\rho) = H(\Phi(\rho))$, a lower-semicontinuous concave function on the input state space with values in $[0,+\infty]$, and its convex closure (see [4]), denoted by $\widehat{H}_\Phi(\rho)$ and called the $\widehat{H}$-function of the channel Φ. It was shown in [5] that the convex closure of the output entropy of an arbitrary quantum channel Φ is given by the expression

$$\widehat{H}_\Phi(\rho) = \inf_\mu \int_{\mathfrak{S}(\mathscr{H})} H_\Phi(\sigma)\,\mu(d\sigma), \tag{1}$$

where the infimum is taken over all probability measures on $\mathfrak{S}(\mathscr{H})$ with barycentre ρ, and that this infimum is always achieved at some measure supported on the set of pure states.

A continuity condition for the $\widehat{H}$-function of an infinite-dimensional channel Φ was obtained in [5] (Proposition 7). This condition is equivalent to the condition

$$\Bigl\{\lim_{n\to+\infty} H_\Phi(\rho_n) = H_\Phi(\rho_0) < +\infty\Bigr\} \implies \Bigl\{\lim_{n\to+\infty} \widehat{H}_\Phi(\rho_n) = \widehat{H}_\Phi(\rho_0)\Bigr\}. \tag{2}$$

The *superadditivity property of the $\widehat{H}$-function* for channels Φ and Ψ is the validity of the inequality

$$\widehat{H}_{\Phi\otimes\Psi}(\omega) \geqslant \widehat{H}_\Phi(\omega^{\mathscr{H}}) + \widehat{H}_\Psi(\omega^{\mathscr{K}}) \tag{3}$$

for all $\omega \in \mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$, where $\omega^{\mathscr{H}} = \operatorname{Tr}_{\mathscr{K}}\omega$ and $\omega^{\mathscr{K}} = \operatorname{Tr}_{\mathscr{H}}\omega$. This property implies the *additivity of the minimal output entropy* for the channels Φ and Ψ (see [1], [3]). If Φ and Ψ are partial traces, then the superadditivity of the $\widehat{H}$-function means the *superadditivity of the Entanglement of Formation* (EoF), which is an important characteristic of the state of a bipartite quantum system indicating the degree of entanglement of this state (see [1], [5]).

For finite-dimensional channels Φ and Ψ the superadditivity of the $\widehat{H}$-function is equivalent to the *additivity of the Holevo capacity* of these channels *with arbitrary constraints* [6]. One of the obstacles to proving an analogous assertion for infinite-dimensional channels is the existence of *superentangled states*—pure states of a bipartite system having partial

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traces with infinite entropy (see Remark 4 in [7]). Namely, the existence of such states prevented, until now, proving the superadditivity of the $\widehat{H}$-function (and even the additivity of the minimal output entropy) for several classes of infinite-dimensional channels for which the additivity of the Holevo capacity with arbitrary constraints was derived in [7] from the corresponding finite-dimensional results [6], [8].

The problem of superentangled states can be partially solved by the following assertion based on the continuity condition (2) and on some other results in [5].

Lemma. *Let* $\Phi\colon \mathfrak{S}(\mathscr{H}) \mapsto \mathfrak{S}(\mathscr{H}')$ *and* $\Psi\colon \mathfrak{S}(\mathscr{K}) \mapsto \mathfrak{S}(\mathscr{K}')$ *be arbitrary quantum channels. The inequality* (3) *holds for all* ω *in* $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ *if for an arbitrary finite-rank state* ω_0 *in* $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ *with finite output entropy* $H_{\Phi\otimes\Psi}(\omega_0)$ *there exists a sequence* $\{\omega_n\}$ *of states in* $\mathfrak{S}(\mathscr{H}\otimes\mathscr{K})$ *such that the following properties are satisfied*:

(i) $\lim_{n\to+\infty}\omega_n = \omega_0$ *and* $\lim_{n\to+\infty} H_{\Phi\otimes\Psi}(\omega_n) = H_{\Phi\otimes\Psi}(\omega_0)$;

(ii) *the inequality* (3) *holds with* $\omega=\omega_n$ *for all* $n\in\mathbb{N}$.

Using this lemma, Proposition 7 and Theorem 2 in [7], as well as Theorem 1 in [9], we obtain the following infinite-dimensional version of the results in [9], [6], [8].

Proposition. *Let* Ψ *be an arbitrary channel. Superadditivity of the* $\widehat{H}$*-function holds in the following cases*: (i) Φ *is a noiseless channel*; (ii) Φ *is an entanglement-breaking channel* (*see* [3]); (iii) Φ *is a channel complementary* (*see* [9]) *to an entanglement-breaking channel*; (iv) Φ *is a direct sum mixture* (*see* [6]) *of a noiseless channel and a channel* Φ_0 *such that superadditivity of the* $\widehat{H}$*-function holds for the channels* Φ_0 *and* Ψ.

The above lemma makes it possible to prove the following assertion.

Theorem 1. *If superadditivity of the* $\widehat{H}$*-function holds for all finite-dimensional quantum channels, then this property holds also for all infinite-dimensional quantum channels.*

Corollary 1. *If superadditivity of* EoF *holds for all states of a finite-dimensional bipartite quantum system, then this property holds also for all states of an infinite-dimensional bipartite quantum system.*

Corollary 2. *If additivity of the minimal output entropy holds for all finite-dimensional quantum channels, then this property holds also for all infinite-dimensional quantum channels.*

Theorem 1 and Theorem 3 in [7] provide the following infinite-dimensional generalization of Shor's theorem [2].

Theorem 2. *The following properties are equivalent*: (i) *additivity of the Holevo capacity holds for all infinite-dimensional quantum channels with arbitrary constraints*; (ii) *additivity of the minimal output entropy holds for all infinite-dimensional quantum channels*; (iii) *superadditivity of* EoF *holds for all states of an arbitrary infinite-dimensional bipartite quantum system.*

The last property in Theorem 2 is equivalent to the superadditivity of the $\widehat{H}$-function for all infinite-dimensional quantum channels.

All the above assertions are proved in [10].

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M. E. Shirokov
Steklov Mathematical Institute,
Russian Academy of Sciences
E-mail: msh@mi.ras.ru

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