Characterization of convex μ-compact sets

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The properties of compact sets in the context of convex analysis have been studied by many authors (see [1] and the references therein). It is natural to ask about possible generalizations of results proved for compact convex sets to non-compact sets. In [2] one such generalization concerning the particular class of sets called μ-compact sets is considered. In [2], [3] it is shown that for this class of sets, which includes all compact convex sets as well as some non-compact sets widely used in applications, many results of the Choquet theory [1] and the Vesterstrøm–O'Brien theory [4], [5] can be proved. In this paper we give a characterization of a convex μ-compact set in terms of properties of functions defined on this set.

In what follows, $\mathscr{A}$ is a bounded convex complete separable metrizable subset of a locally convex space, $C(\mathscr{A})$ is the set of all continuous bounded functions on $\mathscr{A}$, and $M(\mathscr{A})$ is the set of all Borel probability measures on $\mathscr{A}$ endowed with the weak-convergence topology [6]. Let $\overline{\mathrm{co}}\, f$ be the convex closure of a function f [7] (the lower envelope in the terminology of [1]).

The barycenter $\mathbf{b}\,(\mu)$ of an arbitrary measure $\mu \in M(\mathscr{A})$ is the state defined by the Pettis integral (cf. [6]):

$$\mathbf{b}(\mu) = \int_{\mathscr{A}} x\,\mu(dx) \in \mathscr{A}. \tag{1}$$

Definition. A set $\mathscr{A}$ is said to be μ-*compact* if the pre-image of any compact subset of $\mathscr{A}$ under the barycenter map (1) is a compact subset of $M(\mathscr{A})$.

Any compact set is μ-compact. Indeed, compactness of $\mathscr{A}$ implies compactness of $M(\mathscr{A})$ [6]. In [2], [3] the μ-compactness property is proved for the following non-compact closed sets:

– bounded parts of the positive cones of the Banach space l_1 and the Banach space $\mathfrak{T}(\mathscr{H})$ of trace class operators in a separable Hilbert space $\mathscr{H}$;

– a variation-bounded set of Borel measures, endowed with the weak-convergence topology, on an *arbitrary* complete separable metric space;

– a norm-bounded set of positive linear operators, endowed with the strong operator topology, on the Banach spaces l_1 and $\mathfrak{T}(\mathscr{H})$.

In particular, this implies μ-compactness of the set of all Borel probability measures, endowed with the weak-convergence topology, on an *arbitrary* complete separable metric space, of the set of quantum states, and of the set of quantum operations endowed with the strong operator topology [8].

It is essential to note that the μ-compactness property of a convex set is not purely topological but reflects a special relation between the topology and the convex structure of this set [3].

The following theorem shows that the class of convex μ-compact sets can be characterized by the continuity of the operation of convex closure with respect to monotone pointwise converging sequences of functions.

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Theorem. *The following properties are equivalent*:

(i) *the set* $\mathscr{A}$ *is* μ*-compact*;

(ii) *for an arbitrary increasing sequence* $\{f_n\}$ *of functions in* $C(\mathscr{A})$ *converging pointwise to a function* f_0 *in* $C(\mathscr{A})$, *the sequence* $\{\overline{\mathrm{co}}\, f_n\}$ *converges pointwise to the function* $\overline{\mathrm{co}}\, f_0$;

(iii) *for an arbitrary increasing sequence* $\{f_n\}$ *of lower semicontinuous functions bounded below on* $\mathscr{A}$ *and converging pointwise to a function* f_0, *the sequence* $\{\overline{\mathrm{co}}\, f_n\}$ *converges pointwise to the function* $\overline{\mathrm{co}}\, f_0$.

If these equivalent properties hold, then for an arbitrary decreasing sequence $\{f_n\}$ *of lower semicontinuous bounded functions on* $\mathscr{A}$ *converging pointwise to a lower semicontinuous bounded function* f_0, *the sequence* $\{\overline{\mathrm{co}}\, f_n\}$ *converges pointwise to the function* $\overline{\mathrm{co}}\, f_0$.

The proof of this theorem is presented in [9], while some its applications in quantum information theory are considered in [10].

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