A GEOMETRIC APPROACH TO THE TWO-DIMENSIONAL JACOBIAN CONJECTURE

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1. INTRODUCTION

Suppose $f(x, y), g(x, y)$ are two polynomials with complex coefficients. The classical Jacobian Conjecture (due to Keller) asserts the following.

Conjecture. (Jacobian Conjecture in dimension two) If the Jacobian of the pair (f, g) is a non-zero constant, then the map $(x, y) \mapsto (f(x, y), g(x, y))$ is invertible. Note that the opposite is clearly true, because the Jacobian of any polynomial map is a polynomial, and, when the map is invertible, it must have no zeroes, so it is a constant.

The Jacobian Conjecture and its generalizations received considerable attention in the past, see [1]. It is notorious for its subtlety, having produced a substantial number of wrong "proofs", by respectable mathematicians.

From the point of view of a birational geometer, the most natural approach to the two-dimensional Jacobian Conjecture is the following. Suppose a counterexample exists. It gives a rational map from P^2 to P^2. After a sequence of blowups of points, we can get a surface X with two maps: $\pi : X \to P^2$ (projection onto the origin P^2) and $\varphi : X \to P^2$ (the lift of an original rational map).

Note that X contains a Zariski open subset isomorphic to A^2 and its complement, $\pi^*((\infty))$, is a tree of smooth rational curves. We will call these curves exceptional, or curves at infinity. The structure of this tree is easy to understand inductively, as it is built from a single curve (∞) on P^2 by a sequence of two operations: blowing up a point on one of the curves of blowing up a point of intersection of two curves. However, a non-inductive description is probably impossible, which is the first difficulty in this approach. Another difficulty comes from the fact that the exceptional curves on X may behave very differently with respect to the map φ. More precisely, there are four types of curves E.

type 1) $\varphi(E) = (\infty)$

type 2) $\varphi(E)$ is a point on (∞)

type 3) $\varphi(E)$ is a curve, different from (∞)

type 4) $\varphi(E)$ is a point not on (∞)

From a first glance, the situation appears almost hopeless. The goal of this paper is to show that it really is not that bad. In particular, for a given graph of curves, one can essentially always tell which curves are of which type, and there is a fairly restrictive family of graphs that can potentially appear in a counterexample to the JC.

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2. Preliminary observations

We change the notation slightly from the Introduction.

Suppose $X = P^2$, $Y = P^2$ and $\varphi_X^Y : X --- > Y$ is a rational map. Suppose further that on an open subset $A^2 \subset P^2 = X$ the map φ is defined, unramified, and $\varphi(A^2) \subseteq A^2 \subset P^2 = Y$. By a sequence of blowups at smooth points, we get a surface X_1 with a birational map $\pi : X_1 \to X$ and a generically finite map $\varphi = \varphi_{X_1}^Y : X_1 \to Y$ such that $\varphi_{X_1}^Y = \varphi_X^Y \circ \pi$.

The blowups that lead to X_1 can be done outside of $A^2 \subset X$. So $X_1 = A^2 \cup (\cup E_i)$, where E_i are rational curves. The following observation is straightforward.

Proposition 1. *1) The curves E_i form a tree.*

2) One of E_i is $\pi^{-1}(\infty)$, all others are mapped to points by π.

3) The classes of E_i form a basis in the Picard group of X_1.

The structure of X_1 is largely determined by the graph of intersections of E_i. The vertices of this graph correspond to E_i-s and are usually labeled by $(-E_i^2)$. The edges correspond to the points of intersections of two different E_i-s. The graph is a tree.

This graph is not so easy to deal with because blowing up a point changes the self-intersections of the curves passing through it. Inspired by the Minimal Model Program, we consider a different labeling of this graph. We consider the augmented canonical class of X_1, $\bar{K}_{X_1} = K_{X_1} + \sum_i E_i$. It can be uniquely written as a linear combination of E_i, $\bar{K}_{X_1} = \sum_i a_i E_i$. We label the vertices of the intersection graph by these numbers a_i.

With this labeling we now describe what happens when a point is blown up, in any of the intermediate steps in getting from X to X_1.

Proposition 2. *When a point is blown up, going from X_1' to X_1'', one of the following two operations is performed to the graph of exceptional curves:*

1) A new vertex is added to the graph, connected to one of the vertices. It is labeled a_i+1, where a_i is the label of the vertex it is connected to.

2) A new vertex is introduced on the edge connecting two vertices, "breaking" the edge into two edges. The new vertex gets labeled with $a_i + a_j$, where a_i and a_j are the labels of the two vertices it is connected to.

Proof. The first case corresponds to blowing up a point on one of the curves. The second case corresponds to blowing up an intersection of two curves. The augmented canonical class calculations are straightforward and are left to the reader.

Notice that once a vertex is created, its label never changes, which is in sharp contrast with the traditional labeling.

The following observation is true for any X_1, unrelated to the map φ. It is easily proven by induction on the number of exceptional curves, using the above proposition.

Proposition 3. *For any two adjacent vertices E_i, E_j of the graph of X_1, $gcd(a_i, a_j) = 1$. In particular, no two adjacent vertices have even labels.*

The following example serves two purposes. It shows how the graph of X_1 is constructed from the graph of $X = P^2$, and we will use it to compare our labeling with the traditional self-intersection labeling.
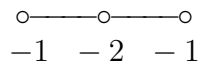
Example. We start with $X = P^2$, its graph is the following.

$$\circ$$
$$-2$$

Blowing up a point, we get
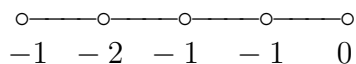
$$\circ\!\!-\!\!-\!\!\circ$$
$$-1 \quad -2$$
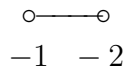
Blowing up another point, on the pullback of (∞), we get

$$\circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\circ$$
$$-1 \quad -2 \quad -1$$

Blowing up a point on a newly blown up curve, we get

$$\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ$$
$$-1 \quad -2 \quad -1 \quad 0$$
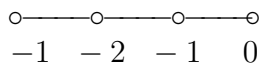
Then we blow up the intersection of the last two curves and get

$$\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ$$
$$-1 \quad -2 \quad -1 \quad -1 \quad 0$$

Blowing up another intersection point, we get

$$\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ$$
$$-1 \quad -2 \quad -1 \quad -2 \quad -1 \quad 0$$
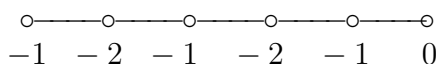
Blowing up another point, we get the following graph

$$\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ$$
$$0 \quad -1 \quad -2 \quad -1 \quad -2 \quad -1 \quad 0$$
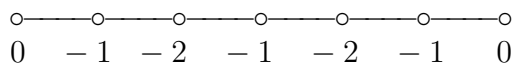
Finally, blowing up four more points (in any order) we get the following:

```
1○                              ○1
 |                               |
 ○──○──○──○──○──○○
 | 0  −1  −2  −1  −2  −1  0 |
1○                              ○1
```
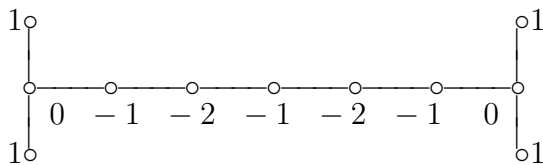
For most of the exceptional curves, one can easily recover their self-intersection from the graph, using the adjunction formula:

$(K_{X_1} + E_i)E_i = -2$, so $\bar{K}_{X_1} \cdot E_i = -2 + \#(E_j \text{ adjacent to } E_i)$

Thus, if $\bar{K}_{X_1} = \sum a_i E_i$, we have

$$a_i E_i^2 + \sum_{E_j \text{ adj. } E_i} a_j = -2 + \#(E_j \text{ adjacent to } E_i)$$

So if $a_i \neq 0$, E_i^2 can be easily calculated.

However, when $a_i = 0$, it is not that easy. One can see in the above example, the left curve with $a_i = 0$ has self-intersection (-3), while the right one has self-intersection (-4), despite the symmetry of the graph. One can remedy this situation by keeping track of the strict pullback of infinity. We do not need it in this paper, and the details are left to an ambitious reader.

Note that the subgraph of vertices with negative labels is connected. It is separated from the "positive" vertices by the "zero" vertices. Moreover, the "zero" vertices are only connected to vertices with labels (-1) or 1.

Now we are going to make use of the map φ. The main idea is to use the adjunction formula for φ to get a formula for $\bar{K}_{X_1}$.

Recall from the Introduction the four types of curves E_i. For every curve of type 1 or 3 denote by d_i the degree of its image on $Y = P^2$, by f_i the degree of the map onto its image and by r_i the ramification index. Denote by L the class of the line on $Y = P^2$.

Proposition 4. *There exist integers b_i for the curves E_i of types 2 and 4 such that*

$$\bar{K}_{X_1} = \varphi^*(-2L) + \sum_{type(E_i)=3} r_i E_i + \sum_{type(E_i)=2 or 4} b_i E_i$$

Proof. Consider the differential form ω on $Y = P^2$ that has the pole of order 3 at (∞) and no other poles or zeroes. Because φ is unramified on the $A^2 \subset X$, there is a differential form on X_1, such that its divisor of zeroes and poles is $\varphi^*(-3L) + \sum_i c_i E_i$, where c_i can be calculated locally at a general point of each E_i.

Notice that for the curves E_i of types 1 and 3, $c_i = r_i - 1$, and

$$\varphi^*(L) = \sum_{type(E_i)=1} r_i E_i + \sum_{type(E_i)=2} e_i E_i$$

for some e_i. Thus,

$$\bar{K}_{X_1} = K_{X_1} + \sum E_i = \varphi^*(-3L) + \sum_{type(E_i)=1 or 3} r_i E_i + \sum_{type(E_i)=2 or 4} (c_i+1) E_i =$$

$$= \varphi^*(-2L) + \sum_{type(E_i)=3} + \sum_{type(E_i)=2 or 4} b_i E_i$$

Q.E.D.

Note that because E_i are independent in the Picard group of X_1, the above representation of $\bar{K}_{X_1}$ is unique and must match with the labeling of the graph of E_i. As a corollary, we have the following observation.

Proposition 5. *1) Any curve of type 1 has a negative even label.*
2) Any curve of type 3 has a positive label.

Proof. Note that $\varphi^*(-2L)$ only involves curves of type 1 and 2. Q.E.D.

Additionally, the union of curves of type 1 and 2 must be connected, as a specialization (set-theoretically) of a pullback of a generic L on $Y = P^2$. This means that the corresponding subgraph is connected.

Every curve of type 3 must intersect with one of the curves of type 1 or 2, while the curves of type 4 do not intersect with curves of type 1 or 2. (This follows from the projection formula of the intersection theory: if E is a curve on X_1, $E \cdot \varphi^*(L) = (\varphi_* E) \cdot L$.)

On the other hand, a type 3 curve can not intersect a type 1 curve, because negative and positive labels are never adjacent. Because the

graph of the exceptional curves on X_1 is a tree, no two curves of type 3 intersect with each other. Putting this all together, we must have the following. The tree of curves on X_1 has a connected subtree containing all curves of type 1 and 2. Some of the vertices of this subtree may have one or more curves of type 3 connected to them. Then some of these type 3 curves may have trees of type 4 curves connected to them. Additionally, no two curves of type 1 are adjacent, and the subtree of curves of type 1 and 2 contains the connected subtree of curves with negative labels.

Proposition 6. $\pi^{-1}(\infty)$ *is of type 1 or 2.*

Proof. One can prove it using the above description of the graph of exceptional curves, but there is also the following direct geometric argument. The pullbacks of lines on $X = P^2$ form a family of rational curves C on X_1 that intersect $\pi^{-1}(\infty)$ at one point while being inside A^2 elsewhere. Consider $\varphi(C)$ for a generic C. If $\pi^{-1}(\infty)$ is of type 3 or 4 then $\varphi(C) \subseteq A^2 \subset Y$. The curve C is proper and A^2 is affine, so $\varphi(C)$ is a point, which is impossible. Q.E.D.

Until now, the variety X_1 was an arbitrary resolution at infinity of the original rational map. But we can put an additional restriction on it, to avoid unnecessary blowups.

Definition 7. *A curve E_i on X_1 is called* **final** *if there is a sequence of blowups from X to X_1 such that E_i is blown up last.*

Note that there may be more than one final curve, and $\pi^{-1}(\infty)$ is never final. In what follows, E_i is one of the exceptional curves on X_1.

Proposition 8. *Suppose that when X_1 was created, E_i was created after all of its neighbors in the graph (i.e. all adjacent vertices). Then E_i is a final curve.*

Proof. Instead of creating E_i at its due time we can change the order of blowups and create it at the last step of the process, without changing anything else. Q.E.D.

Proposition 9. *Suppose $a_i = a(E_i) \geq 2$ and it is the maximum (not necessarily a strict maximum) label among all its neighbors. Then E_i is final.*

Proof. We will prove that E_i was created after all its neighbors. First of all, no neighbor of E_i can be a blowup of a point on E_i, because its label would have been $a_i + 1$. If it were a blowup of a point of intersection of E_i and some E_j, then before the blowup we had E_i and E_j were adjacent before the blowup. Negative curves are never

adjacent to the positive curves and zero curves are only adjacent to curves with labels 1 of -1. Thus, $a_j \geq 1$. So the label of the new curve is $a_i + a_j \geq a_i + 1 > a_i$. Q.E.D.

Note that no two curves with the same label $a_i \geq 2$ can be adjacent, by Proposition 3. So every local maximum $a_i \geq 2$ is a strict maximum.

Proposition 10. *If $a_i = 1$, then E_i is final if and only if it either has only one neighbor, with label 0, or exactly two neighbors, with labels 1 and 0.*

Proof. A curve with label 1 can be created either by a blowup of a point on a curve with label 0 or by a blowup of an intersection of a curve with label 0 and a curve with label 1. Once created, it will be final if and only if no other curve is blown up as its neighbor. The rest is easy and is left to the reader.

The above two propositions allow us to easily spot the final curves in the positive part of the graph of curves. Our interest in the final curves stems from the following. If one of the final curves on X_1 is of type 2 or 4, then it can be contracted, using the $\varphi-$relative MMP to get another X_1, with two maps to X and Y and a smaller Picard number.

Definition 11. *We call X_1 **minimal** if all of its final curves are of type 1 or 3.*

Proposition 12. *If a counterexample to JC exists, it can be obtained using a minimal X_1.*

Proof. Just look at the X_1 with smallest possible Picard number. If it is not minimal, it can be created in such a way so that some curve of type 2 or 4 is blown-up last. Using MMP relative to φ, it can be blown down, maintaining the morphisms, and creating a counterexample to JC with smaller Picard number. Q.E.D.

From now on, X_1 will always be minimal.

Proposition 13. *Suppose E is a curve of type 3 on X_1. Suppose E_0 is the curve of type 2 it is adjacent to. Then the tree on the other side of E is a line $E - E_1 - ... - E_k$, where $E_1, ...E_k$ are of type 4. Additionally, the order of creation is the following: $E_0, E_k, E_{k-1}, ..., E_1, E$.*

Proof. The label of E is positive and all curves $E, E_1, ...E_k$ must be created after the curve E_0. The last one created must be of type 3, for any possible order of creation, from which the result follows. Q.E.D.

3. Other varieties and further analysis

We start with the theorem that shows that type 3 curves must exist in a counterexample to the JC.

Theorem 14. *Suppose X_1 and φ provide a counterexample to the JC. Then X_1 contains a curve of type 3, where φ is ramified.*

Proof. Consider a generic line L on the target variety $Y = P^2$. The curve $C = \varphi^{-1}(L)$ is smooth and irreducible ("Bertini's theorem"). Moreover, we can assume that for all but finitely many lines L' that only intersect L "at infinity", $C' = \varphi^{-1}(L')$ is smooth and irreducible. We can also assume that L does not pass through the images of the exceptional curves of types 2 and 4, so C does not intersect these curves on X. Suppose that the genus of C is g, the map $H = \varphi_{|C} C \to L$ has degree n and the number of points of C "at infinity" is k. (There is a special point ∞ on L, the only one not lying in A^2. The number k is the number of points of C mapped to it, in a set-theoretic sense.) Because the map φ is only ramified at the exceptional curves of X, the map H could only by ramified at these k points at infinity. By Hurwitz formula, we have

$$2g - 2 = -2n + r,$$

where r is the total ramification at infinity. We have $g \geq 0$, $n \geq 1$ and $r \leq n - k$. So

$$-2 \leq 2g - 2 \leq -2n + n - k = -n - k \leq -2$$

Thus all the inequalities above are equalities, $g = 0$, $n = 1$, and $k = 1$.

The curve $\pi(C)$ is now a rational curve on P^2, and its part on A^2 is smooth, while its intersection with the infinity consists of one point. By the celebrated Abhyankar-Moh-Suzuki Theorem (see [1]) there is a polynomial automorphism of A^2 that maps it to the standard line $y = 0$. Precomposing the map φ with this automorphism, we now assume that the line $y = 0$ is mapped by φ to a line on A^2. By a linear transformation in the target variety we assume that it is mapped to the line $y = 0$ there.

The map φ restricted to this line is $1 - to - 1$ so it is of the form $\varphi(x, 0) = (ax + b, 0)$. By another linear transformation we can assume that it is $\varphi(x, 0) = (x, 0)$. Now consider the preimages by φ of the lines $y = c$ on the target variety, for generic c. By the same argument as before these curves are rational and smooth. They do not intersect $y = 0$ except "at infinity". Because these curves are also isomorphic to A^1, they are given by two polynomials $(x(t), y(t))$. Because $y(t)$ is never zero, it must be a constant. So these curves are of the form $y = d$

for some constant d that depends on c. Thus the second coordinate function of the map φ only depends on y. This is a polynomial and it must be a linear one. By a linear change of variables, we can assume that it is just y. Now the map φ has the following form:

$$\varphi(x, y) = (a(y)x + b(y), y)$$

Clearly, the polynomial $a(x)$ must be a constant. The map φ is then easily invertible, thus X_1 is not a counterexample to the JC. Q.E.D.

Now we want to make further use of the morphism $\varphi : X_1 \to Y$. First, we decompose it into a composition of two morphisms, birational and finite (Stein decomposition):

$$X_1 \longrightarrow Y_1 \longrightarrow Y$$

Hear the fist morphism is birational and denoted φ_1, and the second one is finite and denoted φ_2.

Here Y_1 is a normal surface, and one can talk about its canonical class, defined modulo numerical equivalence. By adjunction, we have:

$$K_{Y_1} = \varphi_2^* K_Y + \sum (r_i - 1) E_i,$$

where r_i is the ramification index, and E_i are dimension 1 images of the curves E_i of types 1 and 3 on X_1.

Define $\bar{K}_[Y_1] = K_{Y_1} + \sum E_i$. Then $\bar{K}_{Y_1}$ looks very simple:

$$\bar{K}_{Y_1} = \varphi_2^*(-3L) + \sum r_i E_i = \varphi_2^*(-2L) + \sum_{type(E_i)=3} r_i E_i.$$

Additionally, we decompose φ_1 using the Log Minimal Model Program for $(X_1, \sum E_i)$ relative to φ_1 :

$$X_1 \longrightarrow X_2 \longrightarrow Y_1$$

Here the first map is φ_3, and the second one is φ_4. Note hat $(X_1, \varphi_3(\sum E_i))$ has Kawamata log-terminal singularities and $\bar{K}_{X_2} = K_{X_2} + \sum \varphi_3(E_i)$ is φ_4-nef.

Because $\bar{K}_{X_2}$ is φ_4-nef,

$$\bar{K}_{X_2} = \varphi_4^*(\bar{K}_{Y_1}) + \sum_{\varphi_4(E_i)=pt} c_i E_i, \quad \text{where} \quad c_i \leq 0$$

So

$$\bar{K}_{X_2} = \varphi_4^* \varphi_2^*(-2L) + \varphi_4^*(\sum_{type(E_i)=3} r_i E_i) + \sum_{\varphi_4(E_i)=pt} c_i E_i$$

The following calculation is very important for the paper. Consider $\bar{K}_{X_2} \cdot (\sum_{type(E_i)=3} r_i E_i)$. It equals

$$-2 \sum_{type(E_i)=3} (r_i d_i f_i) + \sum_{\varphi_4(E_j)=pt} c_j \sum_{type(E_i)=3} r_i E_i + \varphi_4^*(\sum_{type(E_i)=3} r_i E_i) \sum_{type(E_i)=3} r_i E_i$$

On the other hand,

$$\bar{K}_{X_2} \cdot E_i = (K_{X_2} + E_i)E_i + \sum_{E_j \neq E_i} E_j E_i \geq -2 + 1 = -1$$

So, $\bar{K}_{X_2} \cdot \sum r_i E_i \geq -\sum r_i$. Thus,

$$\sum r_i(-2d_i f_i) + \varphi_4^*(\sum_{E_i \subset Y_1, type(E_i)=3} r_i E_i) \cdot \sum_{E_i \subset X_2, type(E_i)=3} r_i E_i \geq -\sum r_i$$

Note that

$$\varphi_4^*(\sum r_i E_i) \cdot \sum r_i E_i = (\varphi_4^* \sum_{E_i \subset Y_1, type(E_i)=3} r_i E_i)^2$$

So we have

$$(\varphi_4^* \sum_{E_i \subset Y_1, type(E_i)=3} r_i E_i)^2 \geq \sum r_i(2d_i f_i - 1) \geq 0.$$

Pulling back to X_1, we get

$$(\sum_{E_i \subset Y_1, type(E_i)=3} r_i \varphi_1^*(E_i))^2 \geq 0$$

Proposition 15. *The curve $\pi^{-1}(\infty)$ is of type 2.*

Proof. By Proposition 6, it is of type 1 or 2. If it is of type 1, then it is not included in $\varphi_1^* \sum r_i E_i$ above. So $\varphi_1^*(\sum r_i E_i)$ above consists of curves contractible by π. So, its self-intersection is negative, contradicting he calculation above. Q.E.D.

Now we define a graph of curves on X_2, similar to that on X_1. Notice that $(X_2, \sum E_i)$ has Kawamata log terminal singularities. As a result, the curves E_i have simple normal crossings at smooth points of X_1. The only difference from X_1 is that the curves may pass through singularities of X_2. Denote this graph by Γ_{X_2}. It is obtained from Γ_{X_1} by contracting all type 4 curves and some type 2 curves. Suppose $E_i \subset X_2$ is of type 3.

Note that every curve of type 3 on X_1 intersects the union of curves of type 2 at exactly one point, and does not intersect curves of type 1. When the curves of type 2 are contracted, on Y_1, every curve of type 3 intersects the union of curves of type 1 at exactly one point.

Proposition 16. *For every curve E_i of type 3 on Y_1 the point above is $\varphi_1(\pi^{-1}(\infty))$.*

Proof. Suppose there is a point $y \in Y_1$ on the union of type 1 curves, which is not $\varphi_1(\pi^*(\infty))$ and which has some type 3 curves passing through it. Consider

$$\bar{K}_{Y_1} \cdot \sum_{y \in E_i, type(E_i)=3} r_i E_i$$

On the one hand,

$$\bar{K}_{Y_1} \cdot E_i \geq (K_{Y_1} + E_i)E_i \geq -2,$$

so

$$\bar{K}_{Y_1} \cdot \sum_{y \in E_i, type(E_i)=3} r_i E_i \geq -2 \sum_{y \in E_i, type(E_i)=3} r_i$$

On the other hand,

$$\bar{K}_{Y_1} \cdot E_i \geq (K_{Y_1} + E_i)E_i = -2 \sum_{y \in E_i, type(E_i)=3} r_i E_i =$$

$$= -2 \sum_{y \in E_i, type(E_i)=3} (f_i d_i) + \left(\sum_{y \in E_i, type(E_i)=3} r_i E_i \right)^2$$

(Note that curves of type 3 can only intersect at the union of curves of type 1.)

So,

$$\left(\sum_{y \in E_i, type(E_i)=3} r_i E_i \right)^2 = 2 \sum_{y \in E_i, type(E_i)=3} r_i(f_i d_i - 1) \geq 0.$$

But, pulling back to X_1,

$$\left(\sum_{y \in E_i, type(E_i)=3} \varphi_4^*(r_i E_i) \right)^2 < 0,$$

because $\displaystyle\sum_{y \in E_i, type(E_i)=3} \varphi_4^*(r_i E_i)$ does not include $\pi^{-1}(\infty)$, contradiction. Q.E.D.

Proposition 17. *On Y_1, all exceptional curves contain $\varphi_1(\pi^{-1}(\infty))$ and there are no other points of intersection.*

Proof. By the proposition above, every curve of type 3 contains $\varphi_1(\pi^{-1}(\infty))$ and this is the only point of intersection with other curves. Now consider a curve E_i of type 1. Suppose it does not contain

$\varphi_1(\pi^{-1}(\infty))$. Then it does not intersect any of the curves of type 3 on Y_1.

On Y_1 we have:

$$\bar{K}_{Y_1} \cdot E_i \geq (K_{Y_1} + E_i)E_i \geq -2$$

On the other hand,

$$\bar{K}_{Y_1} \cdot E_i = (-2\varphi_2^*(L) + \sum_{type(E_j)=3} r_j E_j) \cdot E_i = -2\varphi_2^*(L) \cdot E_i \leq -2$$

The inequalities above became equalities if and only if E_i intersects no other curves and is smooth. This would make it the only curve of type 1, which would have to intersect with some curves of type 3, contradiction. Q.E.D.

Thus, we know that every curve of type 1 on Y_1 contains $\varphi_1(\pi^{-1}(\infty))$. We now look at the graph of curves on X_1. The curves of type 2 that are mapped to $\varphi_1(\pi^{-1}(\infty))$ form a connected subgraph, containing $\pi^{-1}(\infty)$. Every curve of type 1 or 3 is attached to this subgraph. On "the other side" of curves of type 3 there may be curves of type 4, and on "the other side" of curves of type 1 there may be curves of type 2. Note that all of these "other side" curves must be created before the corresponding type 3 or type 1 curves. When mapped to Y_1, the curves of type 1 and 3 intersect at $\varphi_1(\pi^{-1}(\infty))$ and nowhere else.

We have now established that the structure of the graph of exceptional curves on a possible counterexample to the Jacobian Conjecture is far from arbitrary. There are several things one can try in order to prove the JC using this approach. In particular, one can try to unwind the map φ by postcomposing it with a suitable polynomial automorphism that blows up the point $\varphi(\pi^{-1}(\infty))$. One can also study some linear systems on X_1 and/or X_2, like $\bar{K}_{X_1} + \varphi^*(L)$.

References

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