Normalization of a Poisson algebra is Poisson

D. Kaledin

To Vassily Alexeevich Iskovskikh, on his 70-th birthday

Introduction

It is well-known that functions on a smooth symplectic manifold M are equipped with a canonical skew-linear operation called the Poisson bracket. The bracket is compatible with multiplication in a certain precise way. Formalizing this structure, one obtains the notion of a Poisson algebra (see Definition 1.1).

The definition of a Poisson algebra is quite general; among other things, it involves no assumption of smoothness. Recently there appeared good reasons to study Poisson algebras in full generality. In particular, they seem to be quite useful in the study of the so-called symplectic singularities initiated by A. Beauville [B].

However, while non-trivial Poisson structures on smooth manifolds have been under close scrutiny for fifty years or more, the general theory is much less developed. It seems that even the simplest facts are not known, or at least, not easy to find in the existing literature.

The goal of the present note is to prove one of these simple facts – namely, we prove that under some natural assumptions, the integral closure of a Poisson algebra is again Poisson (Theorem 1.5). The exposition is essentially self-contained. We need a couple of preliminary lemmas which are definitely not new, but not quite standard, either. For the convenience of the reader, we have taken the liberty of re-proving them from scratch.

1 Statements and definitions.

Fix once and for all a base field k of characteristic $\operatorname{\mathsf{char}} k = 0$.

Definition 1.1. A *Poisson algebra* over the field k is a commutative algebra A over k equipped with an additional skew-linear operation $\{-,-\} : A \otimes A \to A$ such that

$$(1.1) \quad \{a, bc\} = \{a, b\}c + \{a, c\}b \quad , \quad 0 = \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\},$$

for all $a, b, c \in A$. An ideal $I \subset A$ is called a *Poisson ideal* if $\{i, a\} \in I$ for any $i \in I$, $a \in A$.

Additionally, we will always assume that a Poisson algebra A has a unit element $1 \in A$ such that $\{1, a\} = 0$ for every $a \in A$.

Definition 1.2. A *Poisson scheme* over k is a scheme X over k equipped with a skew-linear bracket in the structure sheaf $\mathcal{O}_X$ satisfying (1.1).

Lemma 1.3. *Let A be a Poisson algebra.*

(i) *For any multiplicative system $S \subset A$, the localization $A[S^{-1}]$ carries a canonical Poisson algebra structure.*

(ii) *Any associated prime ideal $\mathfrak{p} \subset A$ is a Poisson ideal.*

(iii) *The radical $J \subset A$ of the algebra A is a Poisson ideal.*

Proof. For (i), set

$$\{a_1 s_1^{-1}, a_2 s_2^{-1}\} = \{a_1, a_2\}(s_1 s_2)^{-1} - \{a_1, s_2\} a_2 \left(s_1 s_2^2\right)^{-1} \\ - \{s_1, a_2\} a_1 \left(s_1^2 s_2\right)^{-1} + a_1 a_2 \{s_1, s_2\} \left(s_1^2 s_2^2\right)^{-1}.$$

For (ii), note that $\mathfrak{p} \subset A$ is the kernel of the canonical Poisson map from the Poisson algebra A to the fraction field $A_{\mathfrak{p}}$. For (iii), note that J is the intersection of all the associated primes. □

Lemma 1.3 (i), in particular, means that the spectrum of a Poisson algebra is a Poisson scheme. We also note the following geometric corollary.

Corollary 1.4. *Let X be a Poisson scheme over k. Then the reduction X_{red} of the scheme X is a Poisson scheme, and so is every irreducible component X_0 of the reduction X_{red}.* □

Our main result is the following.

Theorem 1.5. *Let A_0 be an excellent Noetherian domain over k, and let A be its integral closure in its fraction field.*

(i) *Every derivation ξ of the algebra A_0 extends to a derivation of the algebra A.*

(ii) *Every Poisson bracket $\{-,-\}$ on the algebra A_0 extends to a Poisson bracket on the algebra A.*

Note that both derivations and Poisson brackets extend naturally and uniquely to the fraction field $\mathsf{Frac}\, A_0 = \mathsf{Frac}\, A$. The point is that both preserve the integral closure $A \subset \mathsf{Frac}\, A$. The first claim is well-known; nevertheless, we will prove it, because it is needed in the proof of (ii).

The geometric corollary (in fact, an equivalent geometric formulation) of Theorem 1.5 is the following.

Corollary 1.6. *Let X_0 be an excellent Noetherian integral scheme over k, and let X be its normalization. Then every vector field ξ on X_0 and every Poisson scheme structure on X_0 extend to X.*

2 Discrete valuation rings.

To prove Theorem 1.5, we first study the situation in codimension 1. In this section, assume given an excellent local Noetherian algebra A_0 over k of dimension 1. Let K_0 be its residue field. Let A be the integral closure of the algebra A_0. Since A_0 is excellent, A is finite over A_0; therefore it is a semilocal ring with a finite number l of maximal ideals $\mathfrak{m}_i \subset A$, $1 \leq i \leq l$. For every such $\mathfrak{m}_i$, the localization $A_{\mathfrak{m}_i}$ is a normal local ring of dimension 1, thus a discrete valuation ring whose residue field $K_i = A/\mathfrak{m}_i$ is a finite extension of the residue field K_0. Denote the valuation on $A_{\mathfrak{m}_i}$ by v_i, and fix uniformizing elements $\pi_i \in A_{\mathfrak{m}_i}$, $v_i(\pi_i) = 1$. The ring A is regular and coincides with the intersection

$$A = \bigcap_{1 \leq i \leq l} A_{\mathfrak{m}_i} \subset \mathsf{Frac}(A) \tag{2.1}$$

in the fraction field $\mathsf{Frac}(A) = \mathsf{Frac}(A_0)$.

Lemma 2.1. *In the assumptions above, for any i, $1 \leq i \leq l$, there exists a single element $x \in A_{\mathfrak{m}_i}$ generating $A_{\mathfrak{m}_i}$ over A_0.*

Proof. By the Primitive Element Theorem, the residue field K_i is generated over K_0 by a single element, say $\overline{x}$. Let $P(x)$ be the minimal polynomial for $\overline{x}$ over K_0. Lift $\overline{x}$ to an element $x \in A\mathfrak{m}_i$ and consider

$$y = P(x) \in A.$$

By definition, we have $y = 0 \mod \pi_i$, so that $v_i(y) > 0$. If $v_i(y) = 1$, we are done: x and y generate $A_{\mathfrak{m}_i}$ over A_0, and $y = P(x)$. If not, replace y with

$$y' = P(x + \pi_i).$$

By the binomial formula, we have

$$y' = P'(\overline{x})\pi_i \mod \pi_i^2.$$

Since the polynomial P is minimal, its derivative P' satisfies $P'(\overline{x}) \neq 0$. Therefore $v_i(y') = 1$, and we are done: $A_{\mathfrak{m}_i}$ is generated over A_0 by $x + \pi_i$. □

Lemma 2.2. *Every derivation $\xi_0 : A_0 \to A_0$ of the algebra A_0 extends to a derivation of the algebra A.*

Proof. Consider the formal power series algebra $B = A[[t]]$ in one indeterminate t. Since A is finite over A_0, its fraction field

$$\mathsf{Frac}(B) \subset \mathsf{Frac}(A)((t)) = \mathsf{Frac}(A_0)((t))$$

coincides with $\mathsf{Frac}(A_0[[t]])$. Moreover, B a regular local algebra, in particular, it is integrally closed (see, for example, [AC, Prop. 14]). Therefore it is the integral closure of the power series algebra $B_0 = A_0[[t]]$. By functoriality, every automorphism of the algebra B_0 extends to an automorphism of its integral closure B. Consider the automorphism $\sigma_0 : B_0 \to B_0$ given by

$$\sigma_0(t) = t \qquad \sigma_0(a) = \exp(t\xi)(a) \text{ for } a \in A_0 \subset B_0.$$

Extend it to an automorphism $\sigma : B \to B$ of the algebra B. Setting

$$\xi(a) = \frac{\partial}{\partial t}\sigma(a) \mod t$$

gives a derivation $\xi : A \to A = B/tB$ extending the given derivation ξ_0. $\square$

Remark 2.3. Apparently, this result was first proved by A. Seidenberg [S] back in 1966 (moreover, he did not need the assumption of excellence). However, it seems that this is not universally known. In particular, and I am grateful to M. Lehn and D. van Straten for bringing this to my attentuon, the result also appears as Lemma 2.33 on page 36 of SGA7.2 (with essentially the same proof as here).

Lemma 2.4. *Assume that the algebra A_0 is equipped with a Poisson bracket $\{-,-\}$. Then this bracket extends uniquely to the algebra A.*

Proof. Extend the bracket to the fraction field $\mathsf{Frac}\, A$. We have to prove that $\{f, g\} \in A$ for every $f, g \in A$. By (2.1), it suffices to prove it for each of the $A_{\mathfrak{m}_i}$ instead of A. Let $x \in A_{\mathfrak{m}_i}$ be the generator provided by Lemma 2.1. It suffices to prove that $\{x, x\} \in A_{\mathfrak{m}_i}$ and $\{x, f\} \in A_{\mathfrak{m}_i}$ for every $f \in A_0$. But $\{x, x\} = 0$ tautologically, and $\{x, f\} \in A_{\mathfrak{m}_i}$ by Lemma 2.2 (define $\xi_0 : A_0 \to A_0$ by $\xi_0(a) = \{a, f\}$, and note that any derivation $\xi : A \to A$ preserves all the localizations $A_{\mathfrak{m}_i} \subset \mathsf{Frac}(A)$). $\square$

3 Proof of the Theorem.

We can now prove Theorem 1.5. It is more convenient to approach it in the geometric form of the Corollary 1.6. Thus, let X_0 be a Noetherian integral scheme, and let X be its normalization. By Lemma 2.2 and Lemma 2.4, Corollary 1.6 holds for the open complement $U \subset X$ to a subscheme $Z \subset X$ of codimension $\mathsf{codim}\, Z \geq 2$. Therefore we have a derivation and/or a Poisson bracket on the structure sheaf $\mathcal{O}_U$. This induces a derivation and/or a Poisson bracket on the sheaf $j_*\mathcal{O}_U$, where $j : U \hookrightarrow X$ is the embedding. Since $\mathsf{codim}\, Z \geq 2$, and X is normal, we have $\mathcal{O}_X \cong j_*\mathcal{O}_U$. $\square$

Acknowledgements. It is a great pleasure to dedicate the paper to Vassily Alexeevich Iskovskikh; I hope to be able to do the same in ten years. I would like to thank E. Amerik, R. Bezrukavnikov, A. Kuznetsov, M. Rovinsky and M. Verbitsky for stimulating discussions. I am extremely gratefull to M. Lehn for re-visiting this paper some time after it was first posted to the web, indicating some serious gaps in the argument, and helping me to fix them. The reference to Seidenberg's paper was communicated to me by an anonymous referee in "Functional analysis and Applications" (who rejected the paper since one new result per four pages is not enough for that journal).

References

[AC] N. Bourbaki, *Algèbgre Commutative, Ch. 5*, Hermann, Paris, 1964.

[B] A. Beauville, *Symplectic singularities*, Invent. Math. **139** (2000), 541–549.

[S] A. Seidenberg, *Derivations and integral closure*, Pacific J. of Math. **116** (1966), 167–173.

Steklov Math Institute
Moscow, USSR
E-mail: kaledin@mccme.ru