BIRATIONAL RIGIDITY OF FANO VARIETIES AND FIELD EXTENSIONS

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The modern study of the birational properties of Fano varieties started with the works of Iskovskikh; see the surveys [Isk01, Che05] and the many references there. A key concept that emerged in this area is birational rigidity.

Let X be a Fano variety with $\mathbb{Q}$-factorial, terminal singularities and Picard number 1. Roughly speaking, X is called birationally rigid if X can not be written in terms of Fano varieties in any other way. The precise definition is the following.

Definition 1. A *Mori fiber space* is a projective morphism $f : X \to Y$ such that

(1) X is $\mathbb{Q}$-factorial with terminal singularities,
(2) the relative Picard number of X/Y is 1, and
(3) $-K_X$ is f-ample.

Let X be a Fano variety with $\mathbb{Q}$-factorial, terminal singularities and Picard number 1 defined over a field k. X is called *birationally rigid* if X is not birational to the total space of any Mori fiber space, save the trivial one $f : X \to \operatorname{Spec} k$.

For our purposes, it is better to separate this condition into 2 parts:

(4) Let X' be a Fano variety with $\mathbb{Q}$-factorial, terminal singularities and Picard number 1 over k that is birational to X. Then X' is isomorphic to X.
(5) The only rational map $g : X \dashrightarrow Y$ with rationally connected general fibers is the constant map $X \dashrightarrow$ (point).

In characteristic 0, [BCHM06] implies that a map $g : X \dashrightarrow Y$ as in (5) leads to a Mori fiber space $g' : X' \to Y'$ with X' birational to X but possibly Y' not birational to Y. Thus in characteristic 0 the two versions are equivalent but in positive characteristic this is not known.

The aim of this note is to settle some foundational questions about the behavior of birational rigidity in extensions of algebraically closed fields. In all cases when the birational rigidity of a variety has been proved, the untwisting of birational maps is done by clear geometric constructions that are independent of the algebraically closed field of definition. Our first result shows that, over algebraically closed fields, any untwisting behaves similarly.

Theorem 2. *Let k be an algebraically closed field and $K \supset k$ an algebraically closed overfield. Let X_k be a Fano variety with Picard number 1 defined over k. Assume that*

(1) *either* $\operatorname{char} k = 0$ *and X_k has $\mathbb{Q}$-factorial, terminal singularities*
(2) *or that X_k is smooth.*

Then X_k is birationally rigid iff X_K is birationally rigid.

Remark 3. I don't know how $\mathbb{Q}$-factoriality behaves in flat families in positive characteristic. This is the main reason why smoothness is assumed. For instance, a cone over an elliptic curve is $\mathbb{Q}$-factorial over $\bar{\mathbb{F}}_p$, but not $\mathbb{Q}$-factorial over any other algebraically closed field. This example has log canonical singularities, but I do not know how to exclude this phenomenon for Fano varieties with terminal singularities. (There is also the slight problem that the definition of terminal may not be clear in positive characteristic.)

There is a very interesting problem related to Theorem 2.

Question 4. *Let X be a Fano variety over a field k such that X is birationally rigid over the algebraic closure $\bar{k}$. Is X birationally rigid over k?*

In the terminology of Cheltsov, this asks if the notions of birational rigidity and universal birational rigidity coincide or not.

I think that this is very unlikely but I do not have a counter example. If G denotes the Galois group $\mathrm{Gal}(\bar{k}/k)$, then the k-forms of $X_{\bar{k}}$ are classified by $H^1(G, \mathrm{Aut}(X_{\bar{k}}))$ and two such forms are birational if they have the same image in $H^1(G, \mathrm{Bir}(X_{\bar{k}}))$. For an arbitrary variety the map $H^1(G, \mathrm{Aut}(X_{\bar{k}})) \to H^1(G, \mathrm{Bir}(X_{\bar{k}}))$ is not injective. It is quite interesting that for many birationally rigid varieties, the group $\mathrm{Bir}(X_{\bar{k}})$ is a split extension

$$1 \to \Gamma \to \mathrm{Bir}(X_{\bar{k}}) \to \mathrm{Aut}(X_{\bar{k}}) \to 1$$

for a subgroup Γ generated by the "obvious" birational self-maps. For all such examples, $H^1(G, \mathrm{Aut}(X_{\bar{k}})) \to H^1(G, \mathrm{Bir}(X_{\bar{k}}))$ is an injection. See [Mel04, Che08, Shr08] for several relevant examples.

Note also that the question (4) is not equivalent to the above Galois cohomology problem. Given X_k, in birational geometry we are also interested in birational equivalences $X_k \sim X'_k$ where X'_k has Picard number 1 over k but higher Picard number over $\bar{k}$.

In order to study birational maps of products of Fano varieties, [Che08] introduced a variant of question (4). (In the terminology of [Che08], one asks for varieties for which $\mathrm{Bir}(X)$ "universally untwists maximal centers.") Our second result characterizes such varieties. For this we need to define the dimension of $\mathrm{Bir}(X)$.

Definition 5. Let X be a projective variety over an algebraically closed field k. A birational map $\phi : X \dashrightarrow X$ can be identified with the closure of its graph $\mathrm{graph}(\phi) \subset X \times X$. This construction realizes $\mathrm{Bir}(X)$ as an open subscheme of $\mathrm{Hilb}(X \times X)$ or of $\mathrm{Chow}(X \times X)$. Let $\mathrm{graph} : \mathrm{Bir}(X) \to \mathrm{Hilb}(X \times X)$ denote this injection. In general, $\mathrm{graph}\bigl(\mathrm{Bir}(X)\bigr)$ is an at most countable union of finite type subschemes.

We can now define the *dimension* of $\mathrm{Bir}(X)$ as the supremum of the dimensions of all irreducible subsets of $\mathrm{graph}\bigl(\mathrm{Bir}(X)\bigr)$.

This representation, however, is not particularly unique. Let us call a subset $Z \subset \mathrm{Bir}(X)$ *constructible* if $\mathrm{graph}(Z)$ is constructible as a subset of $\mathrm{Hilb}(X \times X)$ (that is, a finite union of locally closed subvarieties). The notion of a constructible subset is independent of the birational model of X and the constructible structure is compatible with the group multiplication in $\mathrm{Bir}(X)$. The dimension of a constructible subset is also well defined.

We can also define the dimension of $\mathrm{Bir}(X)$ as the supremum of the dimensions of all constructible subsets of $\mathrm{Bir}(X)$.

Theorem 6. *Let k be an algebraically closed field of characteristic 0. Let X be a Fano variety with $\mathbb{Q}$-factorial, terminal singularities and Picard number 1 defined over k. The following are equivalent:*

(1) *X is birationally rigid and* $\dim \operatorname{Bir}(X) = 0$.
(2) *X is birationally rigid and* $\operatorname{Bir}(X_K) = \operatorname{Bir} X$ *for every overfield* $K \supset k$.
(3) *X_K is birationally rigid and* $\operatorname{Bir}(X_K) = \operatorname{Bir} X$ *for every overfield* $K \supset k$.

If k is uncountable, these are further equivalent to

(4) *X is birationally rigid and* $\operatorname{Bir}(X)$ *is countable.*

7 (Comments on pliability)**.** The works of Corti suggest that instead of birational rigidity, one should consider varieties which are birational to only finitely many different Mori fiber spaces up to square equivalence [CR00, CM04]. These are called *varieties with finite pliability.*

The proof of Theorem 2 also shows that this notion is independent of the choice of an algebraically closed field of defintion.

The proof of Theorem 6 implies that if X has finite pliability and $\operatorname{Bir}(X)$ is zero dimensional then any Mori fiber space birational to $X \times U$ is of the form $X' \times U$ where X' is a Mori fiber space birational to X.

8 (Proof of Theorem 2)**.** Assume first that X_K is birationally rigid. Let $X'_k \to Y_k$ be a Mori fiber space whose total space is k-birational to X_k. Then $X'_K \to Y_K$ is a Mori fiber space whose total space is K-birational to X_K, thus X'_K is K-isomorphic to X_K.

$\operatorname{Isom}(X_k, X'_k)$ is a k-variety which has a K-point. Thus it also has a k-point and so X'_k is k-isomorphic to X_k. Thus X_k is birationally rigid.

The converse follows from the next, more general, result.

Proposition 9. *Let k be an algebraically closed field, U a k-variety and $X_U \to U$ a flat family of Fano varieties with terminal singularities. If* $\operatorname{char} k \neq 0$ *then assume in addition that $X_U \to U$ is smooth. Assume that the set*

$$R(U) := \{u \in U(k) : X_u \text{ is } \mathbb{Q}\text{-factorial, birationally rigid and } \rho(X_u) = 1\}$$

is Zariski dense. Then the geometric generic fiber X_K of $X_U \to U$ is birationally rigid.

Proof. It is easy to see that the Picard number of X_K is also 1. Let $\pi_K : X'_K \to S_K$ be a Mori fiber space and $\phi_K : X_K \dashrightarrow X'_K$ a birational map.

Possibly after replacing U by a generically finite ramified cover, we may assume that the above varieties and maps are defined over U. Thus we have

$$X_U \overset{\phi_U}{\dashrightarrow} X'_U \overset{\pi_U}{\to} S_U.$$

If $S_U \to U$ has positive dimensional fibers, then for almost all $u \in R(U)$, the fiber X_u is birational to a nontrivial Mori fiber space; a contradiction. Thus we may assume that $S_U = U$ and the Picard number of X'_K is also 1.

Assume for the moment that the Picard number of X'_u is also 1 for general $u \in R(U)$. Then $X_u \cong X'_u$ since X_u is birationally rigid.

Let us now consider the scheme $\operatorname{Isom}(X_U, X'_U)$ parametrizing isomorphisms between the fibers of $X_U \to U$ and $X'_U \to U$. Since all isomorphisms preserve the ample anti-canonical class, $\operatorname{Isom}(X_U, X'_U)$ is a scheme of finite type over U (cf. [Kol96, I.1.10]). By assumption $\operatorname{Isom}(X_U, X'_U) \to U$ has nonempty fibers over the dense subset $R(U) \subset U$. Therefore $\operatorname{Isom}\bigl(X_U, X'_U\bigr)$ dominates U and so the geometric generic fiber X_K of $X_U \to U$ is isomorphic to the geometric generic fiber X^*_K of $X^*_U \to U$, as required.

We have seen that the Picard number of the geometric generic fiber of $X'_U \to U$ also 1. In characteristic 0, this implies that the Picard number of X'_u is also 1 for general $u \in U(k)$. (If $X'_U \to U$ is smooth and $k = \mathbb{C}$, then $\operatorname{Pic}(X'_u) = H^2(X'_u, \mathbb{Z})$ implies this. The general singular case is treated in [KM92, 12.1.7].) In positive characteristic, the topological arguments of [KM92, 12.1.7] do not apply and I do not know if the rank of the Picard group is a constructible function for families of Fano varieties.

In our case, the following auxiliary argument does the trick.

Let $Z \to X_U \times_U X'_U$ be the normalization of the closure of the graph of ϕ_U with projections $p : Z \to X_U$ and $p' : Z \to X'_U$. Let $E \subset Z$ (resp. $E' \subset Z$) be the exceptional divisors of p (resp. p'). Then

$$\begin{aligned} \rho(Z_K) &= \rho(X_K) + \#\{\text{irreducible components of } E_K\}, \quad \text{and} \\ &= \rho(X'_K) + \#\{\text{irreducible components of } E'_K\}. \end{aligned}$$

Since $\rho(X_K) = \rho(X'_K) = 1$, we obtain that E_K and E'_K have the same number of irreducible components. Similarly, for general $u \in U(k)$, Z_u is the graph of a birational map from X_u to X'_u and

$$\begin{aligned} \rho(X_u) - \rho(X'_u) &= \#\{\text{irred. comps. of } E_u\} - \#\{\text{irred. comps. of } E'_u\} \\ &= \#\{\text{irred. comps. of } E_K\} - \#\{\text{irred. comps. of } E'_K\} \\ &= 0. \end{aligned}$$

Applying this to $u \in R(U)$ we obtain that the Picard number of X'_u is also 1 for general $u \in R(U)$, as required. □

10 (Proof of Theorem 6)**.** For any projective variety X, $\operatorname{Aut}(X)$ is a scheme with countably many irreducible components and the identity component $\operatorname{Aut}^0(X)$ is a (finite dimensional) algebraic group. For Fano varieties, $\operatorname{Aut}(X)$ respects the anticanonical polarization, hence it acts faithfully on some projective embedding. In particular, $\operatorname{Aut}(X)$ is a linear algebraic group. Thus, if $\dim \operatorname{Aut}(X) > 0$ then X is ruled (cf. [Ros56] or [Kol96, IV.1.17.5]). A ruled variety is not birationally rigid (except for $X = \mathbb{P}^1$). In particular, a birationally rigid variety of dimension ≥ 2 has a finite automorphism group. $\mathbb{P}^1$ does not satisfy any of the conditions of (6.1–4). Hence from now on we may assume that $\operatorname{Aut}(X)$ is finite.

Over an uncountable algebraically closed field, a scheme which is a countable union of finite type subschemes has countably many points iff it is zero dimensional. Thus (1) and (4) are equivalent.

Let us prove next that (1) and (2) are equivalent. Let $U \subset \operatorname{Bir}(X)$ be a positive dimensional irreducible component with generic point z_g. Then z_g corresponds to a birational self-map ϕ_Z of X defined over $k(Z)$ which is not in $\operatorname{Bir}(X)$. Conversely, if $\phi_K \in \operatorname{Bir}(X_K)$ is not in $\operatorname{Bir}(X)$ then the closure of the corresponding point in $\operatorname{Hilb}(X \times X)$ gives a positive dimensional component of $\operatorname{Bir}(X)$.

It is clear that (3) $\Rightarrow$ (2).

Finally let us prove that (1) implies (3),

Let X'_K be a Mori fiber space birational to X_K. We may assume that everything is defined over a k-variety U. Thus we have

$$X_U \overset{\phi_U}{\dashrightarrow} X'_U \overset{\pi_U}{\to} S_U.$$

If $S_U \to U$ has positive dimensional fibers, then for almost all $u \in U(k)$, the fiber X_u is birational to a nontrivial Fano fiber space; a contradiction. Thus we may assume that $S_U = U$.

Here we are not allowed to replace U with an arbitrary ramified cover, hence we can not assert that X'_u has Picard number 1 for general $u \in U(k)$.

There is, however, a quasi-finite Galois cover $W \to U$ such that the whole Picard group of the geometric generic fiber of X'_W is defined over $k(W)$. Let us run the MMP on $X'_W \to W$ to end up with $\tau : X'_W \dashrightarrow X^*_W$ and a Mori fiber space $X^*_W \to S^*_W$. (This much of the MMP is known in any dimension by [BCHM06].) As before we see that $S^*_W \cong W$ and the fibers of $X^*_W \to W$ have Picard number 1. The fibers of $X^*_W \to W$ over k-points are thus isomorphic to X_k and so, arguing with $\mathrm{Isom}\big(X_k \times W, X^*_W\big)$ as above, we obtain that $X^*_W \to W$ is an étale locally trivial X-bundle over a dense open subset of W. After a possible further covering of W, we may assume that $X^*_W \cong X \times W$.

Note that the Galois group $G = \mathrm{Gal}(W/U)$ need not act trivially on the Picard group, thus we use the ordinary MMP, not the G-equivariant MMP. In particular, the G-action on X'_W does not give a regular G-action on X^*_W, only a birational G-action.

Thus, in (10.1) the maps are G-equivariant, the G-action is trivial on the X-factor on the left but on $X^*_W \cong X \times W$ we have a so far unknown birational G-action which commutes with projection to W.

$$X \times W = X_W \overset{\phi}{\dashrightarrow} X'_W \overset{\tau}{\dashrightarrow} X^*_W \cong X \times W. \tag{10.1}$$

A birational map $X \times W \dashrightarrow X \times W$ which commutes with projection to W corresponds to a rational map $W \dashrightarrow \mathrm{Bir}(X)$. Since $\mathrm{Bir}(X)$ is zero dimensional, any such birational map is obtained by $\mathrm{Bir}(X)$ acting on the X-factor. Thus the G-action on $X^*_W \cong X \times W$ is given by

$$(x, w) \mapsto \big(\phi(g)(x), g(w)\big)$$

where $\phi : G \to \mathrm{Bir}(X)$ is a homomorphism.

Applying this argument to $\tau \circ \phi$, we get that

$$\big(\tau \circ \phi\big)(x, w) = \big(\rho(x), w\big)$$

for some $\rho \in \mathrm{Bir}(X)$.

The G-equivariance of $\tau \circ \phi$ gives the equality

$$\big(\rho(x), g(w)\big) = \big(\phi(g)\rho(x), g(w)\big) \quad \forall g \in G.$$

Thus $\phi(g)$ is the identity and so the G-action on $X^*_W \cong X \times W$ is in fact trivial on the X-factor, hence biregular.

We can now take the quotient by G to obtain

$$\tau/G : X'_U = X'_W/G \dashrightarrow X^*_W/G \cong X \times U.$$

Note that τ does not extract any divisor, thus τ/G also does not extract any divisor. Since X'_U has relative Picard number 1 over U, we conclude that τ/G also does not contract any divisor. By a lemma of Matsusaka and Mumford (cf. [KSC04, 5.6]), this implies that τ/G is an isomorphism. □

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References

[BCHM06] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, http://www.citebase.org/abstract?id=oai:arXiv.org:math/0610203, 2006.

[Che05] Ivan Cheltsov, *Birationally rigid Fano varieties*, Uspekhi Mat. Nauk **60** (2005), no. 5(365), 71–160. MR MR2195677 (2007d:14028)

[Che08] ———, *Fano varieties with many selfmaps*, Adv. Math. **217** (2008), no. 1, 97–124. MR MR2357324

[CM04] Alessio Corti and Massimiliano Mella, *Birational geometry of terminal quartic 3-folds. I*, Amer. J. Math. **126** (2004), no. 4, 739–761. MR MR2075480 (2005d:14019)

[CR00] Alessio Corti and Miles Reid, *Foreword*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 1–20.

[Isk01] V. A. Iskovskikh, *Birational rigidity of Fano hypersurfaces in the framework of Mori theory*, Uspekhi Mat. Nauk **56** (2001), no. 2(338), 3–86. MR MR1859707 (2002g:14017)

[KM92] János Kollár and Shigefumi Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc. **5** (1992), no. 3, 533–703. MR MR1149195 (93i:14015)

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 32, Springer-Verlag, Berlin, 1996. MR MR1440180 (98c:14001)

[KSC04] János Kollár, Karen E. Smith, and Alessio Corti, *Rational and nearly rational varieties*, Cambridge Studies in Advanced Mathematics, vol. 92, Cambridge University Press, Cambridge, 2004. MR MR2062787 (2005i:14063)

[Mel04] Massimiliano Mella, *Birational geometry of quartic 3-folds. II. The importance of being $\mathbb{Q}$-factorial*, Math. Ann. **330** (2004), no. 1, 107–126. MR MR2091681 (2005h:14030)

[Ros56] Maxwell Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443. MR MR0082183 (18,514a)

[Shr08] Constantin Shramov, *Birational automorphisms of nodal quartic threefolds*, http://www.citebase.org/abstract?id=oai:arXiv.org:0803.4348, 2008.

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