FACTORIALITY OF COMPLETE INTERSECTIONS IN $\mathbb{P}^5$.

DIMITRA KOSTA

Abstract. Let X be a complete intersection of two hypersurfaces F_n and F_k in $\mathbb{P}^5$ of degree n and k respectively with $n \geq k$, such that the singularities of X are nodal and F_k is smooth. We prove that if the threefold X has at most $(n+k-2)(n-1)-1$ singular points, then it is factorial.

1. Introduction

In this paper we shall extend to the complete intersection setting a recent theorem of Cheltsov [4], in which he obtained a sharp bound for the number of nodes a threefold hypersurface can have and still be factorial.

Suppose that X is the complete intersection of two hypersurfaces F_n and F_k in $\mathbb{P}^5$ of degree n and k respectively with $n \geq k$, such that X is a nodal threefold. We will prove the following.

Theorem 1.1. *Suppose that F_k is smooth. Then the threefold X is $\mathbb{Q}$-factorial, when*

$$|Sing(X)| \leq (n+k-2)(n-1)-1 \ .$$

The next example of a non-factorial nodal complete intersection threefold suggests that the number of nodes, that a hypersurface can have while being factorial, should be strictly less than $(n+k-2)^2$.

Example 1.2. Let X be the complete intersection in $\mathbb{P}^5$ of two smooth hypersurfaces

$$F = x_3 f_1(x_0,x_1,x_2,x_3,x_4,x_5) + x_4 f_2(x_0,x_1,x_2,x_3,x_4,x_5) + x_5 f_3(x_0,x_1,x_2,x_3,x_4,x_5) = 0$$
$$G = x_3 g_1(x_0,x_1,x_2,x_3,x_4,x_5) + x_4 g_2(x_0,x_1,x_2,x_3,x_4,x_5) + x_5 g_3(x_0,x_1,x_2,x_3,x_4,x_5) = 0$$

where f_1, f_2, f_3 are general hypersurfaces of degree $n-1$ and g_1, g_2, g_3 general hypersurfaces of degree $k-1$. Then the singular locus $\mathrm{Sing}(X)$, which is given by the vanishing of the polynomials

$$x_3 = x_4 = x_5 = f_1 g_2 - f_2 g_1 = f_1 g_3 - f_3 g_1 = 0 \ ,$$

consists of exactly $(n+k-2)^2$ nodal points and the threefold X is not factorial.

Therefore, we can expect the following stated in [3] to be true.

Conjecture 1.3. *Suppose that F_k is smooth. Then the threefold X is $\mathbb{Q}$-factorial, when*

$$|Sing(X)| \leq (n+k-2)(n+k-2)-1 \ .$$

The assumption of Theorem 1.1 about the smoothness of F_k is essential, as Example 28 in [3] suggests.

In the case of a nodal threefold hypersurface in $\mathbb{P}^4$, namely when $k=1$, several attempts where made towards proving Theorem 1.1, as one can see in [5] and [12]. However, a complete proof for $k=1$ was given in [4].

2. Preliminaries

Let Σ be a finite subset in $\mathbb{P}^N$. The points of Σ impose independent linear conditions on homogeneous forms in $\mathbb{P}^N$ of degree ξ, if for every point P of the set Σ there is a homogeneous form on $\mathbb{P}^N$ of degree ξ that vanishes at every point of the set $\Sigma \backslash P$ and does not vanish at the point P.

The following result, which relates the notion of $\mathbb{Q}$-factoriality with that of independent linear conditions, is due to [6] and was stated in the present form in [3].

I would like to thank Ivan Cheltsov for suggesting the problem to me and for useful comments.

Theorem 2.1. *The threefold X is $\mathbb{Q}$-factorial in the case when its singular points impose independent linear conditions on the sections of $H^0(\mathcal{O}_{\mathbb{P}^5}(2n+k-6)|_G)$.*

The following result was proved in [11] and follows from a result of J.Edmonds [9].

Theorem 2.2. *The points of Σ impose independent linear conditions on homogeneous forms of degree $\xi \geq 2$ if at most $\xi k+1$ points of Σ lie in a k-dimensional linear subspace of $\mathbb{P}^N$.*

By [1] and [7] we also know the following.

Theorem 2.3. *Let $\pi : Y \to \mathbb{P}^2$ be a blow up of distinct points $P_1, ..., P_\delta$ on $\mathbb{P}^2$. Then the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(\xi)) - \sum_{i=1}^{\delta} E_i|$ is base-point-free for all $\delta \leq \max(m(\xi+3-m)-1, m^2)$, where $E_i = \pi^{-1}(P_i)$, $\xi \geq 3$, and $m = \lfloor \frac{\xi+3}{2} \rfloor$, if at most $k(\xi+3-k)-2$ points of the set $P_1, P_2, \ldots, P_\delta$ lie on a possibly reducible curve of degree $1 \leq k \leq m$.*

What is next is an application, as stated in [12], of the modern Cayley-Bacharach theorem (see [10] or [8]).

Theorem 2.4. *Let Σ be a subset of a zero-dimensional complete intersection of the hypersurfaces $X_1, X_2, ..., X_N$ in $\mathbb{P}^N$ of degrees $d_1, ..., d_N$ respectively. Then the points of Σ impose dependent linear conditions on homogeneous forms of degree $\sum_{i=1}^{N} deg(X_i) - N - 1$ if and only if the equality $|\Sigma| = \prod_{i=1}^{N} d_i$ holds.*

Again due to [4] we have the following.

Theorem 2.5. *Let $\Lambda \subseteq \Sigma$ be a subset, let $\phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^m$ be a general projection and let*

$$\mathcal{M} \subset |\mathcal{O}_{\mathbb{P}^n}(t)|$$

be a linear subsystem that contains all hypersurfaces of degree t that pass through Λ. Suppose that

- *the inequality $|\Lambda| \geq (n+k-2)t+1$ holds,*
- *the set $\phi(\Lambda)$ is contained in an irreducible reduced curve of degree t,*

where $r > m \geq 2$. Then $\mathcal{M}$ has no base curves and either $m = 2$ or $t > n+k-2$.

Finally, next is one of our basic tools, a proof of which can be found in [2].

Theorem 2.6. *Let Σ be a finite subset in $\mathbb{P}^N$ that is a disjoint union of finite subsets Λ and Δ, and P be a point in Σ. Suppose that there is a hypersurface in $\mathbb{P}^N$ of degree $\alpha \geq 1$ that contains all points of the set $\Lambda \backslash P$ and does not contain P, and for every point Q in the set Δ there is a hypersurface in $\mathbb{P}^N$ of degree $\beta \geq 1$ that contains all points of the set $\Sigma \backslash Q$ and does not contain the point Q. Then there is a hypersurface in $\mathbb{P}^N$ of degree γ that contains the set $\Sigma \backslash P$ and does not contain the point P, where γ is a natural number such that $\gamma \geq max(\alpha, \beta)$.*

3. Proof of Theorem 1.1

Let us consider the complete intersection X of two hypersurfaces F_n and F_k in $\mathbb{P}^5$ of degrees n and k respectively, with $n \geq k$, such that X is a nodal threefold. Suppose, furthermore, that F_k is smooth and X has at most $(n+k-2)(n-1)-1$ singular points. We denote now by $\Sigma \subset \mathbb{P}^5$ the set of singular points of X.

Definition 3.1. We say that the points of a subset $\Gamma \subset \mathbb{P}^r$ have property $\star$ if at most $t(n+k-2)$ points of the set Γ lie on a curve in $\mathbb{P}^r$ of degree $t \in \mathbb{N}$.

For a proof of the following we refer the reader to [3].

Lemma 3.2. *The points of the set $\Sigma \subset \mathbb{P}^5$ have property $\star$.*

According to Theorem 2.1, for any point $P \in \Sigma$ we need to prove that there is a hypersurface of degree $2n+k-6$, that passes through all the points of the set $\Sigma \backslash P$, but not through the point P.

Remark 3.3. As we mentioned, the claim of Theorem 1.1 is true, when $k = 1$ and thus we need only consider the case $k \geq 2$. Furthermore, taking into account the following Lemma, we can assume that $n \geq 5$.

Lemma 3.4. *The threefold X is $\mathbb{Q}$-factorial, when*

$$|Sing(X)| \leq (n+k-2)(n-1)-1 \text{ and } k \leq n \leq 4 \ .$$

Proof. Indeed, we consider the projection

$$\psi : \mathbb{P}^5 \dashrightarrow \Pi \cong \mathbb{P}^2 \ ,$$

from a general plane Γ of $\mathbb{P}^5$ to another general plane $\Pi \cong \mathbb{P}^2$, that sends the set Σ to $\psi(\Sigma) = \Sigma'$. Choose a point $P \in \Sigma$ and put $P' = \psi(P)$. We have the following cases.

- If $2 = n \geq k = 2$, then $|\Sigma| \leq 1$ and the result holds according to Theorem 2.1.
- If $3 = n \geq k = 2$, then $|\Sigma| \leq 5$ and it imposes independent linear conditions on forms of degree 2.
- If $3 = n \geq k = 3$, then $|\Sigma| \leq 7$ and it imposes independent linear conditions on forms of degree 3.
- If $4 = n \geq k = 2$, then $|\Sigma| \leq 11$ and at most $4t$ points lie on a curve in $\mathbb{P}^5$ of degree t. So, the 11 points of Σ impose independent linear conditions on forms of degree 4.
- If $4 = n \geq k = 3$, then $|\Sigma| \leq 14$ and at most $5t$ points lie on a curve in $\mathbb{P}^5$ of degree t.

  If the points of $\Sigma' \subset \Pi$ satisfy property $\star$, then the set $\Sigma' \backslash P'$ satisfies the requirements of Theorem 2.3 for $\xi = 5$ and this implies that the set Σ imposes independent linear conditions on forms of degree 5.

  Suppose on the contrary that the points Σ' do not satisfy Theorem 2.3 for $\xi = 5$. In this case there is a curve C_2 of degree 2 in Π that passes through at least 11 points of Σ'. If we take the cone over C_2 with vertex Γ, we obtain a hypersurface f_2 in $\mathbb{P}^5$. Denote by Λ_2 the points of Σ that lie on f_2. From Theorem 2.4 it follows that the points of Λ_2 impose independent linear conditions on homogeneous forms of degree $5(2-1)-1 = 4$, since Λ_2 is a subset of the complete intersection of hypersurfaces of degree 2 in $\mathbb{P}^5$. The set $|\Sigma \backslash \Lambda_2| \leq 3$ imposes independent linear conditions on forms of degree 2 and, by applying Theorem 2.6 to the two disjoint sets Λ_2 and $\Sigma \backslash \Lambda_2$, we get that the points of Σ impose independent linear conditions on forms of degree 5.
- $4 = n \geq k = 4$. Then $|\Sigma| \leq 17$ and at most $6t$ points lie on a curve $C_t \in \mathbb{P}^5$ of degree t.

  If the points of $\Sigma' \subset \Pi$ satisfy property $\star$, then the set $\Sigma' \backslash P'$ satisfies the requirements of Theorem 2.3 for $\xi = 6$ and this implies that the set Σ imposes independent linear conditions on forms of degree 6.

  Suppose on the contrary that the points Σ' do not satisfy Theorem 2.3 for $\xi = 6$. In this case there is a curve C_2 of degree 2 in Π that passes through at least 13 points of Σ'. If we take the cone over C_2 with vertex Γ, we obtain a hypersurface f_2 in $\mathbb{P}^5$. Denote by Λ_2 the points of Σ that lie on f_2. From Theorem 2.4 it follows that the points of Λ_2 impose independent linear conditions on homogeneous forms of degree $5(2-1)-1 = 4$, since Λ_2 is a subset of the complete intersection of hypersurfaces of degree 2 in $\mathbb{P}^5$. The set $|\Sigma \backslash \Lambda_2| \leq 4$ imposes independent linear conditions on forms of degree 2 and, by applying Theorem 2.6 to the two disjoint sets Λ_2 and $\Sigma \backslash \Lambda_2$, we get that the points of Σ impose independent linear conditions on forms of degree 6.

As we saw above, for $3 \leq n \leq 5$ the points of Σ impose independent linear conditions on forms of degree $2n+k-6$, and thus, by Theorem 2.1, the threefold X is $\mathbb{Q}$-factorial. □

Lemma 3.5. *Suppose that all the singularities of X lie on a plane $\Pi \subset \mathbb{P}^5$. Then for any point $P \in \Sigma$ there is hypersurface of degree $(2n+k-6)$ that contains $\Sigma \backslash P$, but does not contain the point P.*

Proof. By Remark 3.3, we can see that $\xi = 2n+k-6 \geq 6$. Also, we have

$$|\Sigma \backslash P| \leq \max \left\{ \lfloor \frac{2n+k-3}{2} \rfloor (2n+k-3-\lfloor \frac{2n+k-3}{2} \rfloor) - 1, \lfloor \frac{2n+k-3}{2} \rfloor^2 \right\} \ ,$$

for $k \geq 2$ and $n \geq 5$. In order to show that at most $t(2n+k-3-t)-2$ points of Σ lie on a curve of degree t in Π, it is enough to show that

$$t(2n+k-3-t)-2 \geq t(n+k-2) \Longleftrightarrow t(n-t-1) \geq 2, \text{ for all } t \leq \frac{2n+k-3}{2} \ .$$

For $t = 1$ the inequality holds, since $n \geq 5$, and we can assume that $t \geq 2$. It remains to show that $t < n-1$. Suppose on the contrary that $t \geq n-1$. The quantity $t(2n+k-3-t)-2$ rises for all $n-1 \leq t \leq \lfloor \frac{2n+k-3}{2} \rfloor$ and we have

$$|\Sigma \backslash P| \leq (n-1)(n+k-2) - 2 \leq t(2n+k-3-t) - 2 \ .$$

Therefore we see that the requirement of Theorem 2.3, that at most $t(2n+k-3-t)-2$ points of Σ lie on a curve of degree t in Π is satisfied by the set $\Sigma \backslash P$ for all $t \leq \frac{2n+k-3}{2}$. So there is a hypersurface of degree $(2n+k-6)$ that contains $\Sigma \backslash P$, but does not contain point P. □

Taking into account Theorem 2.5, we can reduce to the case Σ is a finite set in $\mathbb{P}^3$, such that at most $(n+k-2)t$ of its points are contained in a curve in $\mathbb{P}^3$ of degree $t \in \mathbb{N}$. Now fix a general plane $\Pi \in \mathbb{P}^3$ and let

$$\phi : \mathbb{P}^3 \dashrightarrow \Pi \cong \mathbb{P}^2$$

be a projection from a sufficiently general point $O \in \mathbb{P}^3$. Denote by $\Sigma' = \phi(\Sigma)$ and $P' = \phi(P)$.

Lemma 3.6. *Suppose that the points of $\Sigma' \subseteq \Pi$ have the property $\star$. Then there is a hypersurface of degree $2n+k-6$ that contains $\Sigma \backslash P$ and does not contain P.*

Proof. The points of the set Σ' satisfy the requirements of Theorem 2.3, following the proof of Lemma 3.5. Thus, there is a curve C in Π of degree $2n+k-6$, that passes through all the points of the the set $\Sigma' \backslash P'$, but not through the point P'. By taking the cone in $\mathbb{P}^3$ over the curve C with vertex O, we obtain the required hypersurface. □

We may assume then, that the points of the set $\Sigma' \subseteq \Pi$ do not have property $\star$. Then there is a subset $\Lambda_r^1 \subseteq \Sigma$ with $|\Lambda_r^1| > r(n+k-2)$, but after projection the points

$$\phi(\Lambda_r^1) \subseteq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

are contained in a curve $C_r \subseteq \Pi$ of degree r. Moreover, we may assume that r is the smallest natural number, such that at least $(n+k-2)r+1$ points of Σ' lie on a curve of degree r, which implies that the curve C_r is irreducible and reduced.

By repeating how we constructed Λ_r^1, we obtain a non-empty disjoint union of subsets

$$\Lambda = \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i \subseteq \Sigma \ ,$$

such that $|\Lambda_j^i| > j(n+k-2)$, the points of the set

$$\phi(\Lambda_j^i) \subseteq \Sigma'$$

are contained in an irreducible curve in Π of degree j, and the points of the subset

$$\phi(\Sigma \backslash \Lambda) \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

have property $\star$, where $c_j \geq 0$. Let Ξ_j^i be the base locus of the linear subsystem in $|\mathcal{O}_{\mathbb{P}^3}(j)|$ of all surfaces of degree j passing through the set Λ_j^i. Then according to Theorem 2.5, the base locus Ξ_j^i is a finite set of points and we have $c_r > 0$ and

$$|\Sigma \backslash \Lambda| < (n-1)(n+k-2) - \sum_{i=r}^{l} i(n+k-2)c_i = (n+k-2)\left(n-1-\sum_{i=r}^{l} ic_i\right) \ .$$

Corollary 3.7. *The inequality $\sum_{i=r}^{l} ic_i \leq n-2$ holds.*

Put $\Delta = \Sigma \cap (\cup_{j=r}^{l} \cup_{i=1}^{c_j} \Xi_j^i)$. Then $\Lambda \subseteq \Delta \subseteq \Sigma$.

Lemma 3.8. *The points of the set Δ impose independent linear conditions on forms of degree $2n+k-6$.*

Proof. We have the exact sequence

$$0 \longrightarrow \mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2n+k-6) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2n+k-6) \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \ ,$$

where $\mathcal{I}_\Delta$ is the ideal sheaf of the closed subscheme Δ of $\mathbb{P}^3$. Then the points of Δ impose independent linear conditions on forms of degree $2n+k-6$, if and only if

$$h^1\left(\mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2n+k-6)\right) = 0 \ .$$

We assume on the contrary that $h^1\left(\mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2n+k-6)\right) \neq 0$. Let $\mathcal{M}$ be a linear subsystem in $|\mathcal{O}_{\mathbb{P}^3}(n-2)|$ that contains all surfaces that pass through all points of the set Δ. Then the base locus of $\mathcal{M}$ is zero-dimensional, since $\sum_{i=r}^{l} ic_i \leq n-2$ and

$$\Delta \subseteq \cup_{j=r}^{l} \cup_{i=1}^{c_j} \Xi_j^i \ ,$$

but Ξ_j^i is a zero-dimensional base locus of a linear subsystem of $|\mathcal{O}_{\mathbb{P}^3}(j)|$. Let Γ be the complete intersection

$$\Gamma = M_1 \cdot M_2 \cdot M_3 \ ,$$

of three general surfaces M_1, M_2, M_3 in $\mathcal{M}$. Then Γ is zero-dimensional and Δ is closed subscheme of Γ. Let

$$\mathcal{I}_\Upsilon = \mathrm{Ann}\left(\mathcal{I}_\Delta / \mathcal{I}_\Gamma\right) \ .$$

Then

$$0 \neq h^1\left(\mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2n+k-6)\right) = h^0\left(\mathcal{I}_\Upsilon \otimes \mathcal{O}_{\mathbb{P}^3}(n-k-4)\right) - h^0\left(\mathcal{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^3}(n-k-4)\right) \ .$$

Therefore $h^0\left(\mathcal{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^3}(n-k-4)\right) \neq 0$ and there is a surface $F \in |\mathcal{I}_\Upsilon \otimes \mathcal{O}_{\mathbb{P}^3}(n-k-4)|$. We have

$$(n-k-4)(n-2)^2 = F \cdot M_2 \cdot M_3 \geq h^0(\mathcal{O}_\Upsilon) = h^0(\mathcal{O}_\Gamma) - h^0(\mathcal{O}_\Delta) = (n-2)^3 - |\Delta| \ ,$$

which implies $|\Delta| \geq (k+2)(n-2)^2$. But $|\Delta| \leq |\Sigma| < (n-1)(n+k-2)$, which is impossible since $k \geq 2$ and $n \geq 5$. □

We see that $\Delta \subsetneq \Sigma$. Put $\Gamma = \Sigma \backslash \Delta$ and $d = 2n+k-6-\sum_{i=r}^{l} ic_i$.

Lemma 3.9. *The inequality $d \geq 3$ holds.*

Proof. Suppose that $d \leq 2$. Since $\sum_{i=r}^{l} ic_i \leq n-2$ due to Corollary 3.7, we have

$$2 \geq d = 2n+k-6-\sum_{i=r}^{l} ic_i \geq 2n+k-6-(n-2) = n+k-4 \geq 3 \ ,$$

which is impossible. □

For the number of points of the set Γ' we have

$$|\Gamma'| = |\Gamma| \leq |\Sigma \backslash \Lambda| \leq (n+k-2)\left(n-1-\sum_{i=r}^{l} ic_i\right) - 2 \ ,$$

and for $d = 2n+k-6-\sum_{i=r}^{l} ic_i$, since $n \geq 5$ and $k \geq 2$, we get

$$|\Gamma'| \leq (n+k-2)\left(n-1-\sum_{i=r}^{l} ic_i\right) - 2 \leq \max\left\{\lfloor\frac{d+3}{2}\rfloor\left(d+3-\lfloor\frac{d+3}{2}\rfloor\right)-1, \lfloor\frac{d+3}{2}\rfloor^2\right\} \ .$$

Lemma 3.10. *If the points of the set Γ impose dependent linear conditions on forms of degree d, then at most d points of the set Γ' lie on a line in $\Pi \cong \mathbb{P}^2$.*

Proof. Let us assume on the contrary that there is a line that contains at least $d+1$ points of Γ. Since the points of Γ satisfy property $\star$, at most $n+k-2$ of its points lie on a line, thus

$$n+k-2 \geq d+1 = 2n+k-6-\sum_{i=r}^{l} ic_i + 1 \ ,$$

which along with Corollary 3.7 implies that

$$n - 3 \leq \sum_{i=r}^{l} ic_i \leq n - 2 \ .$$

If $\sum_{i=r}^{l} ic_i = n-2$, then $|\Gamma| \leq n+k-4$ and we get a contradiction as no more than $n+k-4 < d+1$ points can lie on a line. If $\sum_{i=r}^{l} ic_i = n-3$, then $|\Gamma| \leq 2(n+k-3)$ and according to Theorem 2.2 the points of Γ impose independent linear conditions on forms of degree $d = n + k - 3$, which contradicts our assumption. By Theorem 2.5 the number of points of Γ' that can lie on a line $\Pi \cong \mathbb{P}^2$ is at most d. □

Lemma 3.11. *At most*

$$t(d + 3 - t) - 2$$

points of the set Γ' *lie on a curve in* $\Pi \cong \mathbb{P}^2$ *of degree* t*, for every* $t \leq \frac{d+3}{2}$*.*

Proof. We need to check the condition that at most $t(d + 3 - t) - 2$ points of Γ' lie on a curve of degree t only for $2 \leq t \leq \frac{d+3}{2}$, such that

$$t(d + 3 - t) - 2 < |\Gamma'| \ .$$

Because the set Γ' satisfies property $\star$, at most $(n + k - 2)t$ of its points can lie on a curve of degree t and therefore it is enough to prove that

$$t(d + 3 - t) - 2 \geq (n + k - 2)t \Longleftrightarrow t\left(n - 1 - \sum_{i=r}^{l} ic_i - t\right) \geq 2 \ , \text{ for all } 2 \leq t \leq \frac{d+3}{2} \ .$$

As we saw Lemma 3.10 implies that $t \geq 2$ and we only need to show that $t < n - 1 - \sum_{i=r}^{l} ic_i$. Suppose that

$$n - 1 - \sum_{i=r}^{l} ic_i \leq t \leq \frac{d+3}{2} \ ,$$

then

$$(n - 1 - \sum_{i=r}^{l} ic_i)(n + k - 2) = (n - 1 - \sum_{i=r}^{l} ic_i)(d + 3 - (n - 1 - \sum_{i=r}^{l} ic_i)) - 2 \leq t(d + 3 - t) - 2 \ ,$$

since the quantity $t(d + 3 - t) - 2$ increases, as $t \leq \frac{d+3}{2}$ increases. But then

$$(n - 1 - \sum_{i=r}^{l} ic_i)(n + k - 2) - 2 \leq t(d + 3 - t) - 2 < |\Gamma'| \leq (n - 1 - \sum_{i=r}^{l} ic_i)(n + k - 2) - 2 \ ,$$

which is a contradiction. □

Lemma 3.12. *The points of the set* Σ *impose independent linear conditions on homogeneous forms of degree* $2n + k - 6$*.*

Proof. According to Lemma 3.9 and Lemma 3.11 all the requirements of Theorem 2.3 for $\xi = d$ are satisfied and thus, the points of Γ impose independent linear conditions on homogeneous forms of degree d. Hence, for any point Q in Γ, there is a hypersurface G_Q of degree d, such that $G_Q(\Gamma \backslash Q) = 0$ and $G_Q(Q) \neq 0$.

Furthermore, by the way the set Δ was constructed, there is a form F of degree $\sum_{i=r}^{l} ic_i$ in $\mathbb{P}^3$, that vanishes at every point of the set Δ, but does not vanish at any point of the set Γ.

Therefore, for any point $Q \in \Gamma$ we obtain a hypersurface FG_Q of degree $2n+k-6$, such that

$$FG_Q(\Sigma) = 0 \text{ and } FG_Q(Q) \neq 0 \ .$$

Also, by Lemma 3.8, for any point $R \in \Delta$ there is a hypersurface of degree $2n + k - 6$ that passes through all points of $\Delta \backslash R$, except for the point R.

By applying Theorem 2.6 to the two disjoint sets Δ and Γ, we prove the Lemma. □

References

[1] E. Bese, *On the spannedness and very ampleness of certain line bundles on the blow-ups of* $\mathbb{P}^2_{\mathbb{C}}$ *and* $\mathbb{F}_r$, Math. Ann. **262** (1983), 225-238.

[2] I. Cheltsov, *Points in projective spaces and applications*, arXiv:math.AG/0511578 v4 (2005).

[3] I. Cheltsov, *Factorial threefolds and Shokurov vanishing*, Sbornik: Mathematics **197**, no 3 (2006), 87-116.

[4] I. Cheltsov, *Factorial threefold hypersurfaces*, arxiv:math.AG/0803.3301 (2008).

[5] I. Cheltsov, J. Park, *Factorial hypersurfaces in* $\mathbb{P}^4$ *with nodes*, Geometriae Dedicata, **121** (2006), 205-219.

[6] S. Cynk, *Defect of a nodal hypersurface*, Manuscripta Mathematica **104** (2001), 325-331.

[7] E. Davis, A. Geramita, Birational morphisms to $\mathbb{P}^2$: an ideal-theoretic perspective, Math. Ann. **279** (1988), 435-448.

[8] E. Davis, A. Geramita, F. Orecchia, *Gorenstein algebras and the Cayley-Bacharach theorem*, Proc. Amer. Math. Soc. **93**, no. 4 (1985), 593-597.

[9] J. Edmonds, *Minimum partition of a matroid into independent subsets*, Journal of Research of the National Bureau of Standards- B. Mathematics and Mathematical Physics, **69B** (1965), 67-72.

[10] D. Eisenbud, M. Green, J. Harris, *Higher Castelnuovo Theory*, Asterisque **218** (1993), 187-202.

[11] D. Eisenbud, J-H. Koh, *Remarks on points in a projective space*, Commutative Algebra, Berkeley, CA, (1987), MSRI Publications **15**, Springer, New York 157-172.

[12] D. Kosta, *Factoriality condition of some nodal threefolds in* $\mathbb{P}^4$, Manuscripta Mathematica (2008), Springer, Berlin/ Heidelberg, (on line).

School of Mathematics
The University of Edinburgh
Kings Buildings, Mayfield Road
Edinburgh EH9 3JZ, UK

D.Kosta@sms.ed.ac.uk