# AN UPDATE ON SEMISIMPLE QUANTUM COHOMOLOGY AND $F$–MANIFOLDS

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*To V. A. Iskovskih for his 70th anniversary*

**Abstract.** In the first section of this note we show that the Theorem 1.8.1 of Bayer–Manin ([BaMa]) can be strengthened in the following way: *if the even quantum cohomology of a projective algebraic manifold $V$ is generically semi–simple, then $V$ has no odd cohomology and is of Hodge–Tate type.* In particular, this addressess a question in [Ci].

In the second section, we prove that *an analytic (or formal) supermanifold $M$ with a given supercommutative associative $\mathcal{O}_M$–bilinear multiplication on its tangent sheaf $\mathcal{T}_M$ is an $F$–manifold in the sense of [HeMa], iff its spectral cover as an analytic subspace of the cotangent bundle $T^*_M$ is coisotropic of maximal dimension.* This answers a question of V. Ginzburg.

Finally, we discuss these results in the context of mirror symmetry and Landau–Ginzburg models for Fano varieties.

## §0. Introduction

**0.1. Contents of the paper.** Semisimple Frobenius manifolds have many nice properties: see e. g. [Du], [Ma], [Te], [Go1], [Go2], and references therein. It is important to understand as precisely as possible, which projective algebraic manifolds $V$ have (generically) semi–simple quantum cohomology. In this case the quantum cohomology is determined by initial conditions at one point, a finite amount of numbers, and a mirror (Landau–Ginzburg model) can in many cases be described explicitly.

If $V$ has non–trivial odd cohomology, its full quantum cohomology cannot be semi–simple, but its even part is a closed Frobenius subspace, and in principle it can be semisimple. In [BaMa], Theorem 1.8.1, it was proved that if $H^{ev}_{quant}(V)$ is generically semisimple, then $h^{p,q}(V) = 0$ for $p+q \equiv 0 \bmod 2$, $p \neq q$. In the first section of this note we show that in this case $h^{p,q}(V) = 0$ for $p+q \equiv 1 \bmod 2$ as well.

Thus, the Theorem 1.8.1 of Bayer–Manin ([BaMa]) can be strengthened in the following way: *if the even quantum cohomology of a projective algebraic manifold $V$ is generically semi–simple, then $V$ has no odd cohomology and is of Hodge–Tate* *type.* In particular, for the 47 families of Fano threefolds (classified by V. A. Iskovskih, Sh. Mori and Sh. Mukai) with $b_3(V) > 0$, pure even quantum cohomology cannot be semi–simple. This answers a question discussed in [Ci], p. 826.

The second section is dedicated to a strengthening of a previously unpublished result of C. Hertling (letter dated March 09, 2005, where it was stated for the pure even case). It shows that an analytic (or formal) supermanifold $M$ with a given supercommutative associative $\mathcal{O}_M$–bilinear multiplication on its tangent sheaf $\mathcal{T}_M$ is an $F$–(super)manifold in the sense of [HeMa], iff its spectral cover as an analytic subspace of the cotangent bundle $T^*_M$ is coisotropic of maximal dimension.

This answered a question posed to Yu. Manin by V. Ginzburg.

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## §1. Semisimple quantum cohomology and Dubrovin's conjecture

**1.1. Notation.** Let $V$ be a projective manifold over $\mathbf{C}$. We denote by $H^{ev}_{quant}(V)$ its even quantum cohomology ring. As in [BaMa] and [Ba], it is a topological commutative algebra. Multiplication in it (the classical cup product plus "quantum corections") is denoted $\circ$. The space $H^{ev}(V)$ is embedded in it as a field of flat vector fields on the respective formal Frobenius manifold.

**1.2. Proposition**. *If $H^{ev}_{quant}(V)$ is generically reduced, i.e. has no nilpotents at (the local ring of) the generic point, then $H^{odd}(V) = 0$.*

**Proof.** Assume that $H^{odd}(V) \neq 0$. Let $\Delta$ be a non–zero class of an odd dimension.

First, we have $\Delta \circ \Delta = 0$. In fact, $\Delta \cup \Delta = 0$, because the cup produt is supercommutative. The quantum corrections vanish, because the correlators $\langle ... \rangle$ are also supercommutative in their arguments, so $\langle \Delta\Delta\Delta' ... \rangle = 0$. This follows from the fact that the quantum correlators come from the $S_n$–covariant maps $H^*(V)^{\otimes n} \to H^*(\overline{M}_{0,n})$, induced by algebraic correspondences (push–forwards of virtual fundamental classes). Covariance holds with respect to the action of $S_n$ on the tensor power permuting factors and introducing signs as usual in $Z_2$-graded setting. On the target it renumbers points and hence leaves the fundamental class invariant.

Now, find another (odd) class $\Delta'$ such that $g(\Delta, \Delta') = 1$, where $g$ is the Poincare form. Then we have $1 = g(\Delta, \Delta') = g(\Delta \circ \Delta', e)$ where $e$ is the identity in quantum cohomology. Hence $\Delta \circ \Delta' \in H^{ev}_{quant}(V)$ must be generically non-zero. But its square is zero because of the first remark. This contradicts the generic absence of nilpotents in $H^{ev}_{quant}(V)$.

**1.3. Theorem.** *If the even quantum cohomology of a projective algebraic manifold $V$ is generically semi–simple, then $V$ has no odd cohomology and is of Hodge–Tate type.*

**Proof.** From the generic semisimplicity and the Proposition 1.2 it follows, that $h^{p,q}(V) = 0$ for $p + q \equiv 1 \bmod 2$.

To prove that $h^{pq}(V) = 0$ for $p + q \equiv 0 \bmod 2$ and $p \neq q$, we reproduce a short reasoning from [BaMa]. It compares the Lie algebra of Euler vector fields in the semi–simple case and in the quantum cohomology case.

Firstly, in he semisimple case each Euler vector field must be of the form $E = d_0 \sum_i u_i e_i + \sum_j c_j e_j$, where $d_0$ is a constant (weight of $E$, cf. [Ma1], [Ma2]), and $(u_i)$ are (local) Dubrovin's canonical coordinates, that is, $e_i := \partial/\partial u_i$ form a complete system of pairwise orthogonal idempotents in $H^*_{quant}(V)$. Moreover, $(c_j)$ are arbitrary constants.

From this explicit description it follows directly, that if two Euler fields of non–zero weights commute, they are proportional.

On the other hand, if $h^{p,q}(V) \neq 0$ for some $p + q \equiv 0 \bmod 2$ and $p \neq q$, then $H^*_{quant}(V)$ admits two commuting and non–proportional Euler vector fields $E_1, E_2$ of weight 1. Namely, in the bihomogeneous (with respect to the $(p,q)$–grading) basis of flat vector fields $\Delta_a \in H^{p_a,q_a}(V)$, we can take

$$E_1 := \sum_a (1 - p_a) x_a \Delta_a + \sum_{p_b = q_b = 1} r_b \Delta_b,$$

$$E_2 := \sum_a (1 - q_a) x_a \Delta_a + \sum_{p_b = q_b = 1} r_b \Delta_b.$$

Here $(x_a)$ are dual flat coordinates, and $-K_V = c_1(\mathcal{T}_V) = \sum_b r_b \Delta_b$.

This completes the proof.

**1.4. Dubrovin's conjecture and related insights.** In [Du] (p. 321) the problem of characterization of varieties $V$ with semisimple quantum cohomology was formulated explicitly. It was also stated there that a necessary condition for such $V$ is to be Fano. This was disproved by A. Bayer [Ba], who established that blowing up points on such a variety does not destroy semisimplicity. In particular, not only del Pezzo surfaces have semisimple quantum cohomology, but arbitrary blowups of $\mathbf{P}^2$ as well.

A. Bayer has later conjectured that the maximal length of a semi–orthogonal decomposition of $D^b(V)$ must coincide with the generic number of idempotents in $H^*_{quant}(V)$.

Combining the results of [Ba], of this note, and the further part of Dubrovin's conjecture stated on p. 322 of [Du] (cf. also [Z]), one can now guess that a necessary and sufficient condition for semisimplicity is that $V$ is of Hodge–Tate type, whose bounded derived coherent category admits a full exceptional collection $(E_i)$. Moreover, after adjusting some arbitrary choices, in this case one should be able to identify the Stokes matrix of its second structure connection with the matrix $(\chi(E_i, E_j))$.

This last statement is now checked, in particular, for three–dimensional Fano varieties with minimal cohomology in [Go2]. The reader can find there more details and explanations about the involvement of the vanishing cycles in the mirror Landau–Ginzburg model.

All these constructions reflect some facets of Kontsevich's homological mirror symmetry program. However, one should keep in mind that in this note we are concerned almost exclusively with a multiplication on the tangent bundle, i.e. with the structure of an $F$–manifold (see below). In order to invoke mirror symmetry, we need also to take in consideration a compatible flat metric. In quantum cohomology, it comes "for free" at the start; it is multiplication that requires a special construction. In various contexts relevant for mirror symmetry, the metric can be described implicitly by at least five different kinds of data which we list here for reader's convenience.

(a) Values of the diagonal coefficients of the flat metric $\sum_i \eta_i (du_i)^2$ in canonical coordinates and values of their first derivatives $\eta_{ij}$ at a tame semi–simple point. This is initial data for the second structure connection (cf. [Ma1], II.3).

(b) Monodromy data for the first structure connection and oscillating integrals for the deformed flat coordinates (cf. [Gi], [Du], [Sa] and the references therein).

(c) Choice of one of K. Saito's primitive forms.

(d) Choice of a filtration on the cohomology space of the Milnor fiber (M. Saito, cf. [He2] and the references therein).

(e) Use of the semi–infinite Hodge structure. This is a refinement of (c), described by S. Barannikov ([Bar1], [Bar2]).

## §2. $F$–geometry and symplectic geometry

**2.1. $F$–structure and Poisson structure.** Manifolds $M$ considered in this section can be $C^\infty$, analytic, or formal, eventually with even and odd coordintes (supermanifolds). The ground field $K$ of characteristic zero is most often $\mathbf{C}$ or $\mathbf{R}$. Each such manifold, by definition, is endowed with the structure sheaf $\mathcal{O}_M$ which is a sheaf of (super)commutative $K$–algebras, and the tangent sheaf $\mathcal{T}_M$ which is a locally free $\mathcal{O}_M$–module of (super)rank equal to the (super)dimension of $M$. $\mathcal{T}_M$

acts on $\mathcal{O}_M$ by derivations, and is a sheaf of Lie (super)algebras with an intrinsically defined Lie bracket $[\,,]$.

There is a classical notion of *Poisson structure* on $M$ which endows $\mathcal{O}_{\mathcal{M}}$ as well with a Lie bracket $\{\,,\}$ constrained by a well known identity.

Similarly, *an $F$–structure* on $M$ endows $\mathcal{T}_M$ with an extra operation: (super)commutative and associative $\mathcal{O}_M$–bilinear multiplication. We denote it always $\circ$ and assume that it is endowed with identity: an even vector field $e$. Then $\mathcal{O}_M$ is embedded in $\mathcal{T}_M$ as a subalgebra: $f \mapsto fe$.

Given such a multiplication on the tangent sheaf, we can define its *spectral cover* $\widetilde{M}$ which is a closed ringed (super)subspace (generally not a submanifold) in the cotangent (super)manifold $T^*M$. In the Grothendieck language, it is simply the relative affine spectrum of the sheaf of algebras $(\mathcal{T}_M, \circ)$ on $M$.

More precisely, consider $Symm_{\mathcal{O}_M}(\mathcal{T}_M)$ as the sheaf of algebras of those functions on the cotangent (super)space $T^*_M$ that are polynomial along the fibres of the projection $T^*_M \to M$. The multiplication in this sheaf will be denoted $\cdot$. For example, for two local vector fields $X, Y \in \mathcal{T}_M(U)$, $X \cdot Y$ denotes their product as an element of $Symm^2_{\mathcal{O}_M}(\mathcal{T}_M)$.

Consider the canonical surjective morphism of sheaves of $\mathcal{O}_M$–algebras

$$Symm_{\mathcal{O}_M}(\mathcal{T}_M) \to (\mathcal{T}_M, \circ)$$

sending, say, $X \cdot Y$ to $X \circ Y$. Denote its kernel by $J(M, \circ)$, and let $\widetilde{M}$ be defined by the sheaf of ideals $J(M, \circ)$.

The spectral cover $\widetilde{M} \to M$ is flat, because $\mathcal{T}_M$ is locally free.

Now we will describe the structure identities imposed onto $\{\,,\}$ on $\mathcal{O}_M$, resp. $\circ$ on $\mathcal{T}_M$. To this end, recall the notion of the Poisson tensor. Let generally $A$ be a $K$–linear superspace (or a sheaf of superspaces) endowed with a $K$–bilinear multiplication and a $K$–bilinear Lie bracket $[\,,]$. Then for any $a, b, c \in A$ put

$$P_a(b, c) := [a, bc] - [a, b]c - (-1)^{ab} b[a, c]. \qquad (2.1)$$

(From here on, $(-1)^{ab}$ and similar notation refers to the sign occuring in superalgebra when the two neighboring elements get permuted.)

This tensor will be written for $A = (\mathcal{O}_M, \cdot, \{\,,\})$ in case of the Poisson structure, and for $A = (\mathcal{T}_M, \circ, [\,,])$ in case of an $F$–structure.

We will now present parallel lists of basic properties of Poisson, resp. $F$–manifolds.

**2.2. Poisson (super)manifolds.** (i)$_P$. *Structure identity:* for all local functions $f, g, h$ on $M$

$$P_f(g,h) \equiv 0. \tag{2.2}$$

(ii)$_P$. *Each local function $f$ on $M$ becomes a local vector field $X_f$ (of the same parity as $f$) on $M$ via $X_f(g) := \{f, g\}$.*

This is a reformulation of (2.2).

(iii)$_P$. *Maximally nondegenerate case: symplectic structure.* There exist local canonical coordinates $(q_i, p_i)$ such that for any $f, g$

$$\{f,g\} = \sum_{i=1}^{n} (\partial_{q_i} f \partial_{p_i} g - \partial_{q_i} g \partial_{p_i} f).$$

Thus, locally all symplectic manifolds of the same dimension are isomorphic. The local group of symplectomorphisms is, however, infinite dimensional.

**2.3.** ***F*–manifolds.** (i)$_F$. *Structure identity:* for all local vector fields $X, Y, Z, U$

$$P_{X \circ Y}(Z,U) - X \circ P_Y(Z,U) - (-1)^{XY} Y \circ P_X(Z,U) = 0. \tag{2.3}$$

(ii)$_F$. *Each local vector field on $M$ becomes a local function on the spectral cover $\widetilde{M}$ of $M$.*

As we already mentioned, generally $\widetilde{M}$ is not a (super)manifold. In the pure even case this often happens because of nilpotents in $\mathcal{O}_{\widetilde{M}}$ and/or singularities. In the presence of odd coordinates on $M$ nilpotents by themselves are always present, but typically they cannot form an exterior algebra over functions of even coordinates because ranks do not match.

A theorem due to Hertling describes certain important cases when $\widetilde{M}$ is a manifold.

(iii)$_F$. *Maximally nondegenerate case: semisimple $F$–manifolds.* $\widetilde{M}$ will be a manifold and even an unramified covering of $M$ in the appropriate "maximally nondegenerate case", namely, when $M$ is pure even, and locally $(\mathcal{T}_M, \circ)$ is isomorphic to $(\mathcal{O}_M^d)$ as algebra, $d = \dim M$.

In this case there exist local canonical coordinates $(u_a)$ (Dubrovin's coordinates) such that the respective vector fields $\partial_a := \partial/\partial_a$ are orthogonal idempotents:

$$\partial_a \circ \partial_a = \delta_{ab} \partial_a.$$

Thus, locally all semisimple $F$–manifolds of the same dimension are isomorphic. Local automorphisms of an $F$–semisimple structure are generated by renumberings and shifts of canonical coordinates:

$$u_a \mapsto u_{\sigma(a)} + c_a$$

so that this structure is more rigid than the symplectic one.

**2.4. Spectral cover as a subspace in symplectic supermanifold.** There is a structure of sheaf of Lie algebras on $Symm_{\mathcal{O}_M}(\mathcal{T}_M)$. It is given by the Poisson brackets $\{\,,\}$ with respect to the canonical (super)symplectic structure on $T^*_M$.

It is easy to check that the ideal $J = J(M,\circ) \subset Symm_{\mathcal{O}_M}(\mathcal{T}_M)$ defining $\widetilde{M}$ in this sheaf of supercommutative algebras is generated by all expressions:

$$e - 1, \quad X \circ Y - X \cdot Y, \quad X, Y \in \mathcal{T}_M. \tag{2.4}$$

**2.5. Theorem.** *The multiplication $\circ$ satisfies the structure identity of $F$–manifolds (2.3), iff the ideal $J(M,\circ)$ is stable with respect to the Poisson brackets.*

**Proof.** From (2.2), one easily infers that stability of an ideal in a Poisson algebra with respect to the brackets can be checked on any system of generators of this ideal. In our case we choose (2.4).

Clearly, $\{e-1, e-1\} = 0$.

If $X, Y$ are local vector fields, then $\{X, Y\} = [X, Y]$

We will establish by a direct computation that for all $X, Y, Z, W$ as above,

$$\{X \circ Y - X \cdot Y,\, Z \circ W - Z \cdot W\} \equiv$$

$$P_{X\circ Y}(Z,W) - X \circ P_Y(Z,W) - (-1)^{XY} Y \circ P_X(Z,W) \bmod J(M,\circ) \tag{2.5}$$

and

$$\{e-1, X \circ Y - X \cdot Y\} = [e, X \circ Y] - X \cdot [e, Y] - [e, X] \cdot Y. \tag{2.6}$$

Assume that this is done. From (2.5) and (2.6) it follows that if (2.3) holds, then $J(M,\circ)$ is stable with respect to the Poisson brackets. For (2.6), one uses the identity $[e, X \circ Y] = X \circ [e, Y] + [e, X] \circ Y$ which follows from (2.3) by choosing $X = Y = e$ and renaming $Z, U$.

Conversely, if $J(M,\circ)$ is stable with respect to the brackets, then the right–hand side of (2.5) must belong to $J(M,\circ)$. But it lies in the degree 1 part of the symmetric algebra of $\mathcal{T}_M$, which projects onto $\mathcal{T}_M$. Hence it must vanish, and as a result, the right hand side of (2.6) must belong to $J(M,\circ)$ as well.

It remains to check (2.5) and (2.6). We will briefly indicate how to do it, restricting ourselves to the clumsier case (2.5).

First of all, the right hand side of (2.5) can be rewritten as follows:

$$P_{X\circ Y}(Z,W) - X \circ P_Y(Z,W) - (-1)^{XY} Y \circ P_X(Z,W) =$$

$$[X\circ Y, Z\circ W]-[X\circ Y,Z]\circ W-(-1)^{(X+Y)Z}Z\circ[X\circ Y,W]$$
$$-X\circ[Y,Z\circ W]-(-1)^{XY}Y\circ[X,Z\circ W]+X\circ[Y,Z]\circ W+(-1)^{YZ}X\circ Z\circ[Y,W]$$
$$+(-1)^{XY}Y\circ[X,Z]\circ W+(-1)^{X(Y+Z)}Y\circ Z\circ[X,W]. \tag{2.7}$$

It turns out that (2.7) is in fact a tensor, that is $\mathcal{O}_M$–polylinear in $X,Y,Z,W$. See [Me1], [Me2] for a discussion and operadic generalizations of the condition of its vanishing.

In our context, this formula is convenient, because a straightforward decomposition of the left hand side of (2.5) into Poisson monomials (constructed using two operations) gives exactly the same list of monomials as in (2.7) modulo $J(M,\circ)$, with the same signs.

Here are samples of calculations.

The first term $\{X\circ Y, Z\circ W\}$ at the left hand side of (2.5) coincides with the first term in (2.7).

Using the Poisson identity (2.2), we find further:

$$-\{X\circ Y, Z\cdot W\}=-\{X\circ Y,Z\}\cdot W-(-1)^{(X+Y)Z}Z\cdot\{X\circ Y,W\}.$$

Modulo $J(M,\circ)$, this can be replaced by

$$-[X\circ Y,Z]\circ W-(-1)^{(X+Y)Z}Z\circ[X\circ Y,W]$$

which corresponds to the second and third terms of (2.7).

We leave the rest as an exercise to the reader.

**2.5.1. Reduced spectral cover.** Contrary to what might be expected, the condition

$$\{J(M,\circ),J(M,\circ)\}\subset J(M,\circ)$$

*does not* imply the respective condition for the radical of $J(M,\circ)$ even in the pure even case. This means that $\widetilde{M}_{red}$ need not be a Lagrange subvariety, even if it comes from an $F$–manifold.

This can be shown on the following explicit examples.

We will construct two families of everywhere indecomposable (see 2.6 below) $F$-manifolds in terms of the ideals $J$, defining (nonreduced) subspaces $\widetilde{M}\subset T^*M$. In order to give rise to $F$-manifolds with $\pi:\widetilde{M}\to M$ as their spectral cover, they have to satisfy the following conditions:

(a) The projection $\widetilde{M}\to M$ is flat of degree $n=\dim M$ and the canonical map $\mathcal{T}_M\to\pi_*(\mathcal{O}_{\widetilde{M}})$ is an isomorphism.

To check this by direct calculations, we will choose (pure even) local coordinates $(t_1, \ldots, t_n)$ on $M$ in such a way that $e = \partial/\partial t_1$. By $(y_1, \ldots, y_n)$ we will denote the conjugate coordinates along the fibres of $T^*(M)$.

(b) $\{J, J\} \subset J$.

We will see that in these examples

$$\{\sqrt{J}, \sqrt{J}\} \not\subset \sqrt{J}.$$

**2.5.2. The first family.** Here we put

$$J = (y_1 - 1,\ (y_i - \rho_i)(y_j - \rho_j)),$$

with $\rho_1 = 1$ and $\rho_i \in \mathcal{O}_M$ for $i \geq 2$ such that $\partial_1 \rho_i = 0$. Clearly, (a) and (b) are satisfied. The radical of $J$ is

$$\sqrt{J} = (y_1 - 1; y_2 - \rho_2, \ldots, y_n - \rho_n).$$

We have $\{\sqrt{J}, \sqrt{J}\} \not\subset \sqrt{J}$, if

$$\partial_i \rho_j \neq \partial_j \rho_i \quad \text{for some } i, j \geq 2 \text{ with } i \neq j.$$

The algebra $T_t M$ at any point $t \in M$ is isomorphic to $\mathbf{C}[x_1, ..., x_{n-1}]/(x_i x_j)$.

**2.5.3. The second family.** Here we put for any $n \geq 3$

$$J = (y_1 - 1,\ (y_2 - \rho_2)^2,\ (y_2 - \rho_2) \cdot y_3,\ y_3^{n-1},\ y_4 - y_3^2,\ y_5 - y_3^3, ...,\ y_n - y_3^{n-2}),$$

with

$$\rho_2(y, t) = t_3 y_1 + \sum_{k=3}^{n-1} (k-1) t_{k+1} \cdot y_k.$$

Now, (a) is rather obvious, but checking (b) requires a calculation which we omit. The radical of $J$ is

$$\sqrt{J} = (y_1 - 1,\ y_2 - t_3 \cdot y_1,\ y_3, y_4, y_5, ..., y_n).$$

The algebra $T_t M$ at any point $t \in M$ is isomorphic to $\mathbf{C}[x_2, x_3]/(x_2^2, x_2 x_3, x_3^{n-1})$.

We will now explain in which context the considerations of this section can be related to the problems, arising in the study of semisimple quantum cohomology

**2.6. Hertling's local decomposition theorem.** For any point $x$ of a pure even $F$–manifold $M$, the tangent space $T_xM$ is endowed with the structure of a $K$–algebra. This $K$–algebra can be represented as a direct sum of local $K$–algebras. The decomposition is unique in the following sense: the set of pairwise orthogonal idempotent tangent vectors determining it is well defined.

C. Hertling has shown that this decomposition extends to a neighborhood of $x$.

More precisely, define the sum of two $F$–manifolds:

$$(M_1, \circ_1, e_1) \oplus (M_2, \circ_2, e_2) := (M_1 \times M_2, \circ_1 \boxplus \circ_2, e_1 \boxplus e_2)$$

A manifold is called *indecomposable* if it cannot be represented as a sum in a nontrivial way.

**2.6.1. Theorem.** *Every germ $(M, x)$ of a complex analytic $F$–manifold decomposes into a direct sum of indecomposable germs such that for each summand, the tangent algebra at $x$ is a local algebra.*

*This decomposition is unique in the following sense: the set of pairwise orthogonal idempotent vector fields determining it is well defined.*

For a proof, see [He], Theorem 2.11.

Furthermore, we have ([He], Theorems 5.3 and 5.6):

**2.7. Theorem.** *(i) The spectral cover space $\widetilde{M}$ of the $F$–structure on the germ of the unfolding space of an isolated hypersurface singularity is smooth.*

*(ii) Conversely, let $M$ be an irreducible germ of a generically semisimple $F$–manifold with the smooth spectral cover $\widetilde{M}$. Then it is (isomorphic to) the germ of the unfolding space of an isolated hypersurface singularity. Moreover, any isomorphism of germs of such unfolding spaces compatible with their $F$–structure comes from a stable right equivalence of the germs of the respective singularities.*

Recall that the stable right equivalence is generated by adding sums of squares of coordinates and making invertible analytic coordinate changes.

In view of this result, it would be important to understand the following

**2.8. Problem.** *Characterize those varieties $V$ for which the quantum cohomology Frobenius spaces $H^*_{quant}(V)$ have smooth spectral covers.*

Theorem 2.7 produces for such manifolds a weak version of Landau–Ginzburg model, and thus gives a partial solution of the mirror problem for them.

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