Stable Higgs Bundles with trivial Chern Classes — Several Examples —

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1 Introduction

The Bogomolov inequality for semistable vector bundes on smooth complex projective n-folds X reads

$$c_2(\mathcal{E})A^{n-2} \geq \frac{r-1}{2r} c_1(\mathcal{E})^2 A^{n-2},$$

where A is an ample divisor and $\mathcal{E}$ is an A-semistable vector bundle of rank r on X. In case $\mathcal{E}$ is A-stable with vanishing $c_1(\mathcal{E})$, the lower bound of this inequality $c_2(\mathcal{E}) \geq 0$ is attained if and only if $\mathcal{E}$ admits the structure of a flat hermitian bundle associated with an irreducible unitary representation of the fundamental group $\pi_1(X)$, thereby establishing the one-to-one *Kobayashi-Hitchin correspondence* between the stable bundles with vanishing Chern classes and the irreducible unitary representation of $\pi_1(X)$ [2]. The Bogomolov inequality is natural enough to have several proofs by completely different approaches (geometric invariant theory [1]; characteristic p method [3]; the theory of effective cones on ruled surfaces [8]; Yang-Mills theory of connections [2]). Because of this naturality, the Bogomolov inequality generalizes to certain classes of generalized vector bundles, including parabolic bundles and orbibundles.

Another important class of generalized vector bundles is that of Higgs bundles (see [9]), and the Bogomolov inequality was as well extended to this class by Simpson [9] through a modified version of Yang-Mills theory. [1]

[1] In contrast to the aforementioned classes, an algebro-geometric proof of the Bogomolov inequality is so far not available for Higgs bundles except for very special cases: when several standard examples listed in Section 1 as Examples 0-1, 1 and 2, and the bundles of small ranks 2, 3 [7]. For instance, Simpson's theorem implies that, if X is a ball quotient surface and n is a positive integer, then $\mathrm{Sym}^{3n}\Omega^1_X(-nK_X)$ has no non-zero global section (see Proposition 2.1 below); this vanishing does not seem to follow from the algebro-geometric method we have at hand.

One of the implications of Simpson's result is that, if a stable Higgs bundle has trivial Chern classes, then it comes from an irreducible representation of $\pi_1(X)$ to the special linear group $SL(r, \mathbf{C})$. However, we do not have many such examples except for some standard ones. In this note, we give a sequence of examples of stable Higgs bundles with trivial Chern classes or, equivalently, of stable flat Higgs bundles.

2 Higgs bundles: definition and basic examples

Let $\mathcal{E}$ be a vector bundle on a comlex manifold X and $\theta : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ an $\mathcal{O}_X$-linear mapping. The pair $(\mathcal{E}, \theta)$ is said to be a *Higgs bundle* if the natural composite map $\theta \wedge \theta : \mathcal{E} \to \Omega^2_X \otimes \mathcal{E}$ identically vanishes. Alternatively, $\mathcal{E}$ is a Higgs bundle if an $\mathcal{O}_X$-linear action of the sheaf of the local vector fields Θ_X on $\mathcal{E}$ is given in such a way such that $\xi_1(\xi_2(e)) = \xi_2(\xi_1(e))$ for arbitrary $\xi_i \in \Theta_X$ and $e \in \mathcal{E}$. In other words, a Higgs bundle is a vector bundle with a Sym Θ_X-module structure, where

$$\mathrm{Sym}\ \Theta_X = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i\,\Theta_X$$

is the symmetric tensor algebra generated by Θ_X. Higgs subsheaves are, by definition, Sym Θ_X-submodules.

Given an ample divisor A on X, the notion of A-(semi)stable Higgs bundles is naturally defined. Namely, a Hiiggs bundle $\mathcal{E}$ is A-stable if

$$\frac{c_1(\mathcal{S})A^{n-1}}{\mathrm{rank}\ \mathcal{S}} < \frac{c_1(\mathcal{E})A^{n-1}}{\mathrm{rank}\ \mathcal{E}}$$

for any nontrivial saturated Higgs subsheaf $\mathcal{S} \subset \mathcal{E}$, $\mathcal{S} \neq 0, \mathcal{E}$, where $n = \dim X$.

Historically Higgs structure was introduced in the study of moduli of integrable connections [5]. Let $\mathcal{E}$ be a vector bundle with an integrabale connection $\nabla_0 : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$. Given another integrable connection ∇, the difference $\theta = \nabla - \nabla_0$ is a Higgs bundle structure and this correspondence translates the moduli of the integrable connections on a flat vector bundle $\mathcal{E}$ into the moduli of the Higgs bundle structures.

With the above definition in mind, we give below several standard examples of Higgs bundles.

Example 0-1. An ordinary vector bundle is viewed as a Higgs bundle with zero action of Θ_X (trivial Higgs structure). In this case, the Higgs stability is nothing but the usual stability. Usually a line bundle is thought of as a Higgs bundle with trivial Higgs structure.

Example 0-2. Starting from given Higgs bundles, we can construct new Hiiggs bundles by taking tensor products, duals and pull-backs.

Given two Higgs bundles $\mathcal{E}_1$, $\mathcal{E}_2$, the tensor bundle $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a Higgs bundle by defining $\xi(e_1 \otimes e_2) = \xi(e_1) \otimes e_2 + e_1 \otimes \xi(e_2)$ for $\xi \in \Theta_X$.

The dual bundle $\mathcal{E}^\vee$ of a Higgs bundle is again a Higgs bundle by $\langle e|\xi(e^\vee)\rangle = -\langle \xi(e)|e^\vee\rangle$, where $e \in \mathcal{E}$, $e^\vee \in \mathcal{E}^\vee$, $\xi \in \Theta_X$. The canonical coupling $\mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{O}_X$ is a Higgs bundle homomorphism.

If $g : X \to Y$ is a morphism between complex manifolds and $\mathcal{E}$ is a Higgs bundle on Y, then the pull-back $g^*\mathcal{E}$ is naturally a Sym $g^*\Theta_Y$-module. Then the natural $\mathcal{O}_X$-algebra homomorphism Sym $\Theta_X \to$ Sym $g^*\Theta_Y$ defines a canonical Higgs bundle structure of $g^*\mathcal{E}$ in an obvious manner.

Thanks to Simpson's theorem, the tensor product of two stable Higgs bundle is semi-stable and a direct sum of stable Higgs bundles.

Example 1. Let X be a complex manifold. The symmetric tensor algebra $\mathbf{E}_0^\infty(X) = \mathrm{Sym}\ \Theta_X$ is naturally a Higgs bundle of infinite rank and so are its ideals. In particular, the graded ideal $\mathbf{E}_{l+1}^\infty(X) = \bigoplus_{i=l+1}^{\infty} \mathrm{Sym}^i \Theta_X$ is a Higgs subbundle of infinite rank. Given $l \geq k \geq 0$, the subquotient $\mathbf{E}_k^l(X) = \mathbf{E}_k^\infty(X)/\mathbf{E}_{l+1}^\infty(X)$ is a coherent Higgs bundle isomorphic to $\bigoplus_{i=k}^{l} \mathrm{Sym}^i \Theta_X$. The action of Θ_X on $\mathbf{E}_k^l(X)$ is given by zero on $\mathrm{Sym}^l \Theta_X$ and by the standard multiplication $\Theta \otimes \mathrm{Sym}^i \Theta_X \to \mathrm{Sym}^{i+1} \Theta_X$ on the other components.

If $K_X A^{n-1} > 0$ and Θ_X is A-semistable as an ordinary vector bundle [resp. If $K_X A^{n-1} \geq 0$ and Θ_X is A-semistable], then $\mathbf{E}_0^1(X) = \mathcal{O}_X \oplus \Theta_X$ is A-stable [resp. A-semistable]. After some easy arguments, this implies that $\mathbf{E}_0^l(X) = \mathrm{Sym}^l(\mathbf{E}_0^1)$ is also an A-stable [resp. A-semistable] Higgs bundle. If K_X is ample and $A = K_X$, then the Yau inequality [10]

$$c_2(X)K_X^{n-2} \geq \frac{\dim X - 1}{2 \dim X} K_X^n$$

yields the Bogomolov inequality for $\mathbf{E}_k^l(X)$.

When X is a two-dimensional compact ball quotient, then the Hirzebruch proportionality theorem yields $c_1^2(X) = 3c_2(X)$, so that $\mathbf{E}_0^1(X)$ has $c_1 = -K_X$, $c_2 = \frac{1}{3}K_X^2$; namely $\mathbf{E}_0^1(X)$ attains the equality of the Bogomolov inequality. In particular, $\mathbf{E}_0^{3l}(lK_X)$ has trivial Chern classes ($\mathcal{O}_X(lK_X)$ is viewed as a trivial Higgs bundle) and, by Simpson's theorem, it carries an integrable connection ∇. More precisely, there is a $\mathbf{C}$-vector space V of dimension $\begin{pmatrix} 3l+2 \\ 2 \end{pmatrix}$ such that $\mathcal{V} = V \otimes_{\mathbf{C}} \mathcal{O}_X$ has a filtration $0 = \mathcal{V}^{3l+1} \subset \mathcal{V}^{3l} \subset \cdots \subset \mathcal{V}^0 = \mathcal{V}$ with the p-th graded quotient $\mathbf{Gr}^p$ is isomorphic to $\mathrm{Sym}^{3l-p}\Theta_X(lK_X)$. Furthermore, the Higgs action of $\xi \in \Theta_X$ on $\mathrm{Sym}^{3l-p}\Theta_X(lK_X)$ is identified with the $\mathcal{O}_X$-linear map (Kodaira-Spencer map) $\mathbf{Gr}^p \to \mathbf{Gr}^{p-1}$ induced by the natural derivation ∇_ξ of $\mathcal{V} \otimes \mathcal{O}_X$ by ξ. Thus $\mathbf{E}_0^{3l}$ looks very like a variation of Hodge structure (VHS) of weight $3l$. However, it does not have the Hodge symmetry (the ranks of the successive quotients are $3l+1, 3l, 3l-1, \ldots, 1$), and the underlying vector space V may not have a $\mathbf{Q}$-vector space structure.

Example 2. Given integers $l \geq k \geq 0$, we define the Higgs bundle $\mathbf{F}_l^k(X)$ as the vector bundle $\bigoplus_{i=k}^{l} \mathrm{Sym}^i \,\Omega_X^1$ with the Θ_X-action defined by 0 on $\mathrm{Sym}^k \,\Omega_X^1$ and by (-1) times the standard contraction map $\Theta_X \otimes \mathrm{Sym}^i \,\Omega_X^1 \to \mathrm{Sym}^{i-1} \,\Omega_X$. $\mathbf{F}_l^k(X)$ is the dual $\mathbf{E}_0^\infty(X)$-module $\mathcal{H}om_{\mathbf{E}_0^\infty(X)}(\mathbf{E}_k^l(X), \mathcal{O}_X)$ of $\mathbf{E}_k^l(X)$, where $\mathcal{O}_X$ is viewed as a Higgs bundle with trivial Θ_X-action. For $l \geq m \geq k$, $\mathbf{F}_m^k(X)$ is naturally a Higgs subbundle of $\mathbf{F}_l^k(X)$ with the quotient $\mathbf{F}_l^k(X)/\mathbf{F}_m^k(X)$ isomorphic to $\mathbf{F}_l^{m+1}(X)$. The stability condition and the Bogomolov inequality for $\mathbf{F}_l^k$ are similar as for $\mathbf{E}_k^l$.

Let $p : X \to Y$ be a surjective morphism between smooth projective surfaces. Assume that K_Y is ample and that p^*K_Y is divisible by 3 in $\mathrm{Pic}(X)$. Then the *normalized Higgs bundle*

$$\tilde{\mathbf{F}}_1^0(Y) = \left(\mathcal{O}_X \oplus p^*\Omega_Y^1\right)\left(-\frac{\pi^*K_Y}{3}\right)$$

is a Higgs bundle $\subset \mathbf{F}_1^0(X)(-p^*K_Y/3)$ on X with trivial first Chern class. Its l-th symmetric power $\tilde{\mathbf{F}}_l^0(Y)$ is also a Higgs bundle $\subset \mathbf{F}_l^0(X)(-lp^*K_Y/3)$. By easy computation, we get

$$c_2(\tilde{\mathbf{F}}_1^0(Y)) = p^*\left(c_2(Y) - \frac{K_Y^2}{3}\right).$$

For simplicity, we consider only two dimensional cases from now on; a complex manifold X will be a smooth projective surface unless otherwise mentioned.

Example 3. Let Y be a smooth projective surface and $p : X \to Y$ a surjective morphism such that p^*K_Y is divisible by 3. If L be a line bundle contained in $\left(\mathrm{Sym}^{l+1}\pi^*\Omega^1_Y\right)\left(-\dfrac{l\pi^*K_Y}{3}\right)$, then $L\oplus\tilde{F}^0_l(Y)$ is a Higgs bundle contained in

$$\left(\mathrm{Sym}^{l+1}\pi^*\Omega^1_Y\right)\left(-\frac{l\pi^*K_Y}{3}\right)\oplus\tilde{\mathbf{F}}^0_l(Y)=\tilde{\mathbf{F}}^0_{l+1}(Y)\otimes\mathcal{O}_X\left(\frac{\pi*K_Y}{3}\right).$$

Its Chern classes are: $c_1 = c_1(L)$, $c_2 = c_2(\tilde{\mathbf{F}}^0_l(Y))$. This bundle is A-semistable if and only if $3c_1(L)A\geq 0$, and under this semistability condition we get $c_1(L)^2\leq c_2(\tilde{\mathbf{F}}^0_l(A))$.

When Y is a ball quotient and $A=\pi^*K_Y$, we have $c_1(L)^2\leq 0$ whenever $c_1(L)\pi^*K_Y\geq 0$. In particular, a line bundle which is numerically equivalent to $\mathcal{O}_X((l+1)\pi^*K_Y/3)$ cannot be contained in $\pi^*\mathrm{Sym}^{l+1}\Omega^1_Y$. Hence

Proposition 2.1. *Let Y be a compact ball quotient surface and $\mathbf{1}$ be the tautological divisor on the projective bundle $\pi:\mathbf{P}(\Omega^1_Y)\to Y$. Then the numerical equivalence class of $\mathbf{1}-(1/3)\pi^*K_Y$ is not effective in the rational Néron-Severi group* $\mathrm{NS}(\mathbf{P}(\Omega_Y)\otimes\mathbf{Q}$. (On the other hand, $\mathbf{1}-(1/3)\pi^*K_Y$ is known to be *pseudo-effective*, i.e. a limit of effective classes.)

An algebro-geometric proof of the proposition above is not known.

Example 4. Let $m\geq l\geq 0$ be integers. We define a Higgs bundle structure on

$$\left(\bigoplus_{i=0}^{l}\mathrm{Sym}^i\,\Omega^1_X\right)\oplus\left(\bigoplus_{i=m-l}^{0}\mathrm{Sym}^i\,\Theta_X\otimes\mathrm{Sym}^{m+1}\,\Omega^1_X\right)$$

by defining the action of $\xi\in\Theta_X$ as follows:

- For $\alpha\in\mathrm{Sym}^i\,\Omega^1_X$, $l\geq i\geq 0$, $\xi(\alpha)$ is the $(-1)\times$ the natural contraction $\in\mathrm{Sym}^{i-1}\,\Omega^1_X$.
- For $\alpha\in\mathrm{Sym}^{l-m}\,\Theta_X\otimes\mathrm{Sym}^{m+1}\,\Omega^1_X$, $\xi(\alpha)$ is defined by the composition of the natural product $\xi\alpha\in\mathrm{Sym}^{m-l+1}\,\Theta_X\otimes\mathrm{Sym}^{m+1}\,\Omega_X$ and $(-1)^{m-l+1}$ times the contraction map $\mathrm{Sym}^{m-l+1}\,\Theta_X\otimes\mathrm{Sym}^{m+1}\,\Omega_X\to\mathrm{Sym}^l\,\Omega^1_X$.
- For $\alpha\in\mathrm{Sym}^i\,\Theta_X\otimes\mathrm{Sym}^{m+1}\,\Omega^1_X$, $i<m-l$, $\xi(\alpha)$ is the natural product $\xi\alpha\in\mathrm{Sym}^{i+1}\,\Theta_X\otimes\mathrm{Sym}^{m+1}$.

This Higgs bundle is an extension of $\mathbf{E}^{m-l}_0(X)\otimes\mathrm{Sym}^{m+1}\,\Omega^1_X$ by $\mathbf{F}^0_l(X)$.

Example 5. Let L be an invertible subsheaf of Ω^1_X. Then $\mathcal{O}_X\oplus L$ is a Higgs subsheaf of $\mathbf{F}^0_1$. In this case, the A-stability condition is $c_1(L)A>0$ and the Bogomolov inequality is equivalent to the classical de Franchis lemma

$c_1(L)^2 \leq 0$. Hence we get a stable flat Higgs bundle $L^{-1/2} \oplus L^{1/2}$ once we have $L \subset \Omega^1_X$ with $c_1(L)A > 0$, $L^2 = 0$. If X has a fibration over a curve C of genus ≥ 2, the pull back of ω_C gives an example of such L. In this case, our flat Higgs bundle is the pull back of the *theta Higgs bundle*

$$\omega_C^{-1/2} \oplus \omega_C^{1/2}$$

on C.

If X has several fibrations over curves of genus ≥ 2, then the tensor product of the pull back of the theta Higgs bundles is again a flat Higgs bundle.

In the five examples above, flat Higgs bundles on surfaces were produced via tensor products, duals and pull backs from the following three basic examples:

(1) Stable vector bundle (of arbitrary rank) with trivial Higgs structure.

(2) Pull back of the theta Higgs bundle of rank two on curves of genus ≥ 2 (which is uniformaized by the upper half plane and has projectively flat connection).

(3) The 3-bundle $\tilde{\mathbf{F}}^0_1(Y) = \mathbf{F}^0_1(Y) \otimes \mathcal{O}\left(-\frac{K_Y}{3}\right)$, where Y is a two-dimensional ball quotient (again a projectively flat manifold);

In Section 3 and 4, we construct new examples which do not directly derive from the above basic ones.

3 Hirzebruch's Kummer covers $X^{(n)}$ attached to the complete qradrilateral on $\mathbb{P}^2$ and construction of stable Higgs bundles with vanishing Chern classes

We briefly review Hirzebruch's construction of Kummer covers of projective plane branching along a complete quadrilateral [4].

Take general four points $P_1, \ldots, P_4$ on $\mathbb{P}^2$, and let $L_{ij} = L_{ji}$ denote the line connecting P_i and P_j $(i \neq j)$. The reduced divisor $D = \bigcup L_{ij}$ is the so called complete quadrilateral consisting of six lines, the P_i being the triple points of D. D has extra three double points of the form $L_{i_1,i_2} \cap L_{j_1,j_2}$, where $\{i_1, i_2, j_1, j_2\} = \{1, 2, 3, 4\}$. Exactly three singlular points of D lies on each L_{ij}, two of which are the triple points P_i, P_j and one a double point. Thus the Euler number of the non-singular locus of D is $6 \times (2-3) = -6$, while that of D is $-6+4+3 = 1$. Therefore the Euler number of of the complement of D is given by $e(\mathbb{P}^2 \setminus D) = 3 - 1 = 2$.

Let $\mu : X \to \mathbb{P}^2$ be the blowing up at the four triple points $P_1, \dots, P_4$ and let $E_i \subset X$ denote the exceptional divisor over P_i. X is a Del Pezzo surface of degree five with very ample anticanonical divisor $-K_X \sim 3H - \sum E_i$, where H stands for the pullback of the hyperplane of $\mathbb{P}^2$. The effective divisor $\mu^* D$ is supported by a reduced effective divisor

$$\tilde{D} \sim \mu^* \sum L_{ij} - 2 \sum E_i \sim 6\mu^* H - 2 \sum E_i \sim -2K_X.$$

$\tilde{D}$ has only simple normal crossings as singularities and consists of ten irreducible components: four exceptional curve E_i and six strict transforms $\tilde{L}_{ij}$. The $\tilde{L}_{ij}$ meet each other at the three points lying over the double points of D, while each E_i contains three singular points of $\tilde{D}$. Hence $\tilde{D}$ has exactly $3+4\times 3 = 15$ double points, so that $e(\tilde{D}) = 4\times(2-3)+6\times(2-3)+15 = 5$.

Given a positive integer n, there exists a finite Kummer covering $\pi^{(n)} : X^{(n)} \to X$ of degree n^5 branching along $\tilde{D}$ [4]. The function field of $X^{(n)}$ is simply obtained by adjoining the n-th roots $\sqrt[n]{l_{ij}/l_{12}}$ ($\{i,j\} \neq \{1,2\} \in \{1,2,3,4\}$) to $\mathbb{C}(\mathbb{P}^2)$, where l_{ij} is a linear differential equation of the line L_{ij}.

$X^{(n)}$ is a smooth projective surface and the local description of $X^{(n)}$ is quite simple: if $\tilde{D}$ is locally defined by the equation $x = 0$ or $xy = 0$, then $\pi^{(n)*} : \mathcal{O}_X \to \mathcal{O}_{X^{(n)}}$ is given by $(x,y) \mapsto (t^n, u)$ or $(x,y) \mapsto (t^n, u^n)$, where (x,y) and (t,u) are local coordinates of X and $X^{(n)}$. In particular, the inverse image $\left(\pi^{(n)}\right)^{-1}(p) \subset X^{(n)}$ of a closed point $p \in X$ consists of n^5 [resp. n^4, n^3] points when $p \in X \setminus \tilde{D}$ [resp. $p \in \tilde{D} \setminus \mathrm{Sing}(\tilde{D})$, $p \in \mathrm{Sing}(\tilde{D})$]. The topological Euler number of $\tilde{X}^{(n)}$ of $X^{(n)}$ is thus given by

$$\begin{aligned} \frac{c_2(X^{(n)})}{n^5} = \frac{e(X^{(n)})}{n^5} &= e(X \setminus \tilde{D}) + \frac{e(\tilde{D} \setminus \mathrm{Sing}(\tilde{D})}{n} + \frac{e(\mathrm{Sing}(\tilde{D})}{n^2} \\ &= 2 - \frac{10}{n} + \frac{15}{n^2}. \end{aligned}$$

On the other hand we calculate $K_{X^{(n)}}$ by

$$K_{X^{(n)}} \sim \pi^{(n)*}\left(K_X + \left(1 - \frac{1}{n}\right)\tilde{D}\right) \sim \left(1 - \frac{2}{n}\right)\pi^{(n)*}(-K_X),$$

and hence

$$\frac{c_1(X^{(n)})^2}{n^5} = 5\left(1 - \frac{2}{n}\right)^2.$$

$X^{(n)}$ has ample canonical divisor if $n \geq 3$ ($X^{(2)}$ is a K3 surface). When $n = 5$, we have $c_1(X^{(5)})^2 = 5^4 \times 9$, $c_2(X^{(5)}) = 5^4 \times 3$, meaning that $X^{(5)}$ is a surface of general type which attains the upper bound of the Miyaoka-Yau inequality $K \leq 3c_2$.

The Del Pezzo surface X carries five linear pencils $|2H - \sum E_i|$, $|H - E_1|$, $\dots$, $|H - E_4|$, defining five surjective morphisms $f_0, f_1, \dots f_4$ from X onto $\mathbb{P}^1$. Each f_i of these morphisms has exactly three fibres contained in $\tilde{D}$, which are the singular fibres of f_i. For f_1 associated with $|H - E_1$, $\tilde{L}_{1j} + E_j$, $j = 2, 3, 4$ are such fibres, and so are the three curves $\tilde{L}_{12} + \tilde{L}_{34}$, $\tilde{L}_{13} + \tilde{L}_{24}$, $\tilde{L}_{14} + \tilde{L}_{23}$ for f_0 associated with $|2H - \sum E_i|$.

Upstairs on $X^{(n)}$, there are thus five morphisms $f_0^{(n)}, f_1^{(n)}, \dots, f_4^{(n)}$ onto the curve $C^{(n)}$, an n^2-sheeted Kummer cover of $\mathbb{P}^1$ branching at three points, $0, 1, \infty$, say. The pullback line bundle $\mathcal{L}_i^{(n)} = f_i^{(n)*}\omega_{C^{(n)}}$ is an invertible subshef of $\Omega^1_{X^{(n)}}$. We easily check that $\mathcal{L}_i^{(n)}$ is saturated in $\Omega^1_{X^{(n)}}$ and that

$$\begin{aligned}
\mathcal{L}_0^{(n)} &\sim \left(1 - \frac{3}{n}\right) \pi^{(n)*} \left(2H - \sum_{i=1}^{4} E_i\right) \\
\mathcal{L}_i^{(n)} &\sim \left(1 - \frac{3}{n}\right) \pi^{(n)*}(H - E_i), \;\; i = 1, 2, 3, 4.
\end{aligned}$$

Ishida [6] showed that the natural map

$$\bigoplus_{j=0}^{4} f_j^{(n)*} H^0(C^{(n)}, \Omega^1_{C^{(n)}}) \to H^0(X^{(n)}, \Omega^1_{X^{(n)}})$$

is an isomorphism. In particular, the irregularity of $X^{(n)}$ is given by

$$q(X^{(n)}) = \frac{5(n-2)(n-1)}{2}.$$

Now let us construct a stable Higgs bundle of rank 12 on $X^{(n)}$ ($n \geq 6$).

The tensor product $\mathcal{L}^{(n)} = \bigotimes_{i=0}^{4} f_i^* \Omega^1_{C^{(n)}}$ is an invertible subsheaf of $\mathrm{Sym}^5 \Omega^1_{X^{(n)}}$. Easy calculation shows that

$$c_1(\mathcal{L}^{(n)}) = \frac{n-3}{n} \pi^{(n)*} \tilde{D},$$

while

$$K_{X^{(n)}} = \frac{n-2}{2n} \pi^{(n)*} \tilde{D}.$$

Hence

$$\mathcal{L}^{(n)} \equiv \frac{2(n-3)}{n-2} K_{X^{(n)}}.$$

This subsheaf $\subset \mathrm{Sym}^5\Omega$ induces a rank 12 Higgs subsheaf

$$\begin{aligned}\mathcal{E}^{(n)} &= \left(\bigoplus_{i=0}^{2} \mathrm{Sym}^i\,\Omega_{X^{(n)}}\right) \oplus \left(\bigoplus_{i=2}^{0} \mathrm{Sym}^i\,\Theta_X \otimes \mathcal{L}^{(n)}\right) \\ &\subset \left(\bigoplus_{i=0}^{2} \mathrm{Sym}^i\,\Omega_{X^{(n)}}\right) \oplus \left(\bigoplus_{i=2}^{0} \mathrm{Sym}^i\,\Theta_X \otimes \mathrm{Sym}^5\,\Omega^1_{X^{(n)}}\right)\end{aligned}$$

(see Section 2, Examples 4 and 5). We claim:

Proposition 3.1. (1) $c_2(\mathcal{E}^{(n)}) = \dfrac{11}{24} c_1(\mathcal{E}^{(n)})^2$.

(2) *If $n \geq 5$, then $\mathcal{E}^{(n)}$ is $K_{X^{(n)}}$-semistable. If $n \geq 6$, then $\mathcal{E}^{(n)}$ is $K_{X^{(n)}}$-stable.*

Proof. The first statement follows from direct computation, which is a little messy. An alternative proof is the following: $c_2(\mathcal{E}^{(n)})/n^5$ and $c_1(\mathcal{E}^{(n)})^2/n^5$ are both quadratic polynomials in $1/n$. Hence the equality holds for every n if we check it for three special values of $1/n$ (1, 1/3, 1/5, for example).

The Higgs subbundle $\bigoplus_{i=0}^{2} \mathrm{Sym}^i\,\Omega_{X^{(n)}}$ and the quotient Higgs bundle $\bigoplus_{i=2}^{0} \mathrm{Sym}^i\,\Theta_X \otimes \mathcal{L}^{(n)}$ are both stable Higgs bundles of rank 6 with first Chern class $4K_X$ and $6c_1(L^{(n)}) - 4K_X$, respectively. Hence the semistability of the bundle is equivalent to the inequality

$$\frac{n-3}{n-2} \geq \frac{2}{3}$$

or, equivalently, $n \geq 5$.

Corollary 3.2. *If $n \geq 6$, then $\mathcal{E}^{(n)}$ is projectively flat. The flat bundle $\mathcal{E}^{(n)}\left(-\dfrac{n-3}{2n}\pi^{(n)*}\tilde{D}\right)$ of rank 12 looks like a variation of Hodge structure of weight five with Hodge numbers $(1, 2, 3, 3, 2, 1)$.*

4 Example of a flat Higgs bundle over a family

In the previous section, we constructed countably many examples of stable Higgs bundles of rank 12 defined over a dicrete series of surfaces. In this section, we construct a stable Higgs bundle of rank 6 defined over a one-parameter family of surfaces.

Let C be an elliptic curve and $\Delta \subset C \times C$ the diagonal. Let $\mu : C \to C$ be the multiplication by two and define three effective divisors D, F, G on $C \times C$ by

$$\begin{aligned} D &= (\mu, \mu)^* \Delta \\ F &= (\mu, \mu)^* (\{0\} \times C) \\ G &= (\mu, \mu)^* (C \times \{0\}) \,. \end{aligned}$$

These three divisors are divisible by two (actually by four) in $\mathrm{Pic}(C \times C)$ so that there are double coverings $h_i : X_i \to C \times C$ which branch along D, F and G, respectively. By forming the fibre product over $C \times C$, we get a Galois $(\mathbf{Z}/2\mathbf{Z})^{\oplus 3}$-covering $X' \to C \times C$. The resulting surface X' has 16 ordinary double points over the 2-torsion points ${}_2C \times {}_2C \subset C \times C$.

Proposition 4.1. *There exists a double covering $g : X \to X'$ ramifying exactly at the 16 double points of X' so that X is smooth.*

Proof. Let $\tilde{X}'$ be the minimal resolution of X' and $E \subset \tilde{X}'$ the sum of the sixteen (-2)-curves over the double points. We show that $E \in \mathrm{Pic}(\tilde{X}')$ is divisible by 2.

By construction, h^*G is a non-reduced Weil divisor with multiplicity two. This means g^*h^*G is of the form $2\tilde{G}_0 + E$, $\tilde{G}_0$ denoting the reduced part of the strict transform of h^*G. On the other hand G is algebraically equivalent to $4\times$ (a fibre of the first projection $C \times C \to C$) and is divisible by 4. Hence $E \equiv 2\tilde{G}_0 + E = g^*h^*G \equiv 0 \mod 2$. □

Proposition 4.2. *The Chern numbers of X is given by $c_1^2(X) = 2^7 \times 3$, $c_2(X) = 2^4 \times 3^2$.*

Proof. The projection $f = (\mu, \mu) \circ h \circ g : X \to C \times C$ has degree 2^8 with branch locus $B = \Delta \cup (\{0\} \times C) \cup (C \times \{0\})$. Ecxept at the origin $(0, 0)$, the branch index of f along B is exactly two, while it is 2^4 at the origin. The canonical divisor K_X is thus $\dfrac{1}{2} f^*B$ so that $K_X^2 = 2^6 B^2 = 2^6 \times 6$.

The Euler numbers of B and $B \backslash (0, 0)$ are -2 and -3, so that $e\,(C \times C \setminus B) = 2$. Hence

$$c_2(X) = e(X) = 2^8 \times 2 + 2^7 \times (-3) + 2^4 \times 1 = 2^4 \times 9,$$

concluding the proof. □

Our surface X has three independent pencils. The two canonical projections $C \times C \to C$ induce two fibrations $p_i : X \to \tilde{C}$, where $\tilde{C}$ is the double cover of C branching at the four points ${}_2C$. The group homomorphism $C \times C \to (C \times C)/\Delta \simeq C$ yields a third fibration $p_3; X \to \tilde{C}$. The pull-backs of $\omega_{\tilde{C}}$ via the three projections p_1, p_2, p_3 are linearly equivalent to $\dfrac{1}{2}f^*(\{0\} \times C), \dfrac{1}{2}f^*(C \times \{0\})$ and $\frac{1}{2}f^*\Delta$, respectively, so that their product gives an invertible subsheaf $L \simeq \mathcal{O}_X(K_X) \subset \mathrm{Sym}^3\Omega^1_X$.

Proposition 4.3. *The Higgs bundle*

$$\mathcal{O}_X \oplus \Omega^1_X \oplus (\Theta_X \otimes L) \oplus L$$

is stable and, by tensoring $-\dfrac{1}{2}K_X$*, we get a stable flat Higgs bundle*

$$\mathcal{O}_X(-\frac{K_X}{2}) \oplus \Omega^1_X(-\frac{K_X}{2}) \oplus \Theta_X(\frac{K_X}{2}) \oplus \mathcal{O}_X(\frac{K_X}{2}).$$

The proof is easy.

Since an elliptic curves C has one-dimensional moduli, so does our surface X. We have thus constructed a stable flat Higgs bundle on a one-parameter family of surfaces.

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