MULTIPLE FIBERS OF DEL PEZZO FIBRATIONS

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ABSTRACT. We prove that a terminal three-dimensional del Pezzo fibration has no fibers of multiplicity ≥ 6. We also obtain a rough classification possible configurations of singular points on multiple fibers and give some examples.

1. INTRODUCTION

Throughout this paper a *weak del Pezzo fibration* is a projective morphism $f\colon X \to Z$ with connected fibers from a threefold X with terminal singularities to a smooth curve Z such that $-K_X$ is f-nef and f-big near a general fiber. If additionally $-K_X$ is f-ample, we say that $f\colon X \to Z$ is a *del Pezzo bundle*. (We do not assume that X is $\mathbb{Q}$-factorial nor $\rho(X/Z) = 1$). The main reason to study del Pezzo fibrations comes from the three-dimensional birational geometry, namely the class of del Pezzo bundles with $\mathbb{Q}$-factorial singularities and relative Picard number one is one of three possible outcomes of the minimal model program for threefolds of negative Kodaira dimension.

Our main result is the following.

Theorem 1.1. *Let $f\colon X \to Z$ be a weak del Pezzo fibration and let $f^*(o) = m_oF_o$ be a special fiber of multiplicity m_o. Then $m_o \leq 6$. Moreover, all the cases $1 \leq m_o \leq 6$ occur. Furthermore, let $\mathbf{B}(F_o) = (r_1, \dots, r_n)$ be the basket of singular points of X at which F_o is not Cartier. Then, in the case $m_o \geq 2$, there are only the following possibilities:*

(i) $m_o = 2$, $\mathbf{B}(F_o) = (8)$, $(2, 6)$, $(4, 4)$, $(2, 2, 4)$, *or* $(2, 2, 2, 2)$,
(ii) $m_o = 3$, $\mathbf{B}(F_o) = (9)$, $(3, 3, 3)$, *or* $(3, 6)$,
(iii) $m_o = 4$, $\mathbf{B}(F_o) = (2, 4, 4)$,
(iv) $m_o = 5$, $\mathbf{B}(F_o) = (5, 5)$,
(v) $m_o = 6$, $\mathbf{B}(F_o) = (2, 3, 6)$.

The possible types of singularities in $\mathbf{B}(F_o)$, the local behavior of F_o near singular points, and the possible types of a general fiber are collected in Table 1.

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Warning. In the statement of Theorem 1.1 and Table 1 we do not assert that the basket $\mathbf{B}(F_o)$ contains all the singularities along F_o. It is possible that F_o is Cartier at some non-Gorenstein points (see Example 5.6).

TABLE 1

type	m_o	$\mathbf{B}(F_o) = (r_1, \dots, r_n)$	$(b_1, \dots, b_n)$	q_i	$K^2_{F_g}$
$I_{2,3,6}$	6	$(2,3,6)$	$(1, \pm 1, \pm 1)$	$q_i \equiv -1$	6
$I_{5,5}$	5	$(5,5)$	$b_1^2 + b_2^2 \equiv 0$	$q_i \equiv -1$	5
$I_{2,4,4}$	4	$(2,4,4)$	$(1, \pm 1, \pm 1)$	$q_i \equiv -1$	4, 8
$I_{3,3,3}$	3	$(3,3,3)$	$(\pm 1, \pm 1, \pm 1)$	$q_i \equiv -1$	3, 6, 9
$I_{2,2,2,2}$	2	$(2,2,2,2)$	$(1,1,1,1)$	$q_i \equiv 1$	even
$I_{3,6}$	3	$(3,6)$	$(\pm 1, \pm 1)$	$q_i \equiv 4$	3, 6, 9
I_9	3	(9)	$b_1 = \pm 2q_1/3$	$q_1 = 3$ or 6	$\equiv q_1/3 \mod 3$
$I_{2,2,4}$	2	$(2,2,4)$	$(1,1,\pm 1)$	$q_i \equiv r_i/2$	odd
$I_{4,4}$	2	$(4,4)$	$(\pm 1, \pm 1)$	$q_i \equiv r_i/2$	even
$I_{2,6}$	2	$(2,6)$	$(1, \pm 1)$	$q_i \equiv r_i/2$	even
I_8	2	(8)	(± 1) or (± 3)	$q_i \equiv r_i/2$	odd

The idea of the proof is easy. In fact, it is sufficient to compute dimensions of linear systems $\dim |lF_o|$ by using the orbifold Riemann-Roch formula [Rei87]. The main theorem is proved in §§3 – 4. In §5 we give some examples. In fact, it will be shown that all cases in Table 1 except possibly for cases $I_{2,6}$ and I_8 occur. Finally, in §6 we discuss fibers of multiplicity 5 and 6.

Notation in Table 1. The number b_k in the fourth column is the weight which appears in a singularity $\frac{1}{r_k}(1, -1, b_k) \in \mathbf{B}(F_o)$, q_k in the fifth column is an integer such that $F_o \sim q_k K_X$ near $P_k \in \mathbf{B}(F_o)$. F_g in the final column denotes a general fiber of f. We say that the fiber $f^*(o) = m_o F_o$ is of type $I_{r_1,\dots,r_n}$ if $\mathbf{B}(F_o) = (r_1, \dots, r_n)$.

We also say that the fiber F_o is *regular* if it is of type $I_{2,3,6}$, $I_{5,5}$, $I_{2,4,4}$, $I_{3,3,3}$, or $I_{2,2,2,2}$. Otherwise, F_o is said to be *irregular.*

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2. Preliminaries

2.1. Terminal singularities [Mor85]**,** [Rei87]**.** Let (X, P) be a three-dimensional terminal singularity of index r and let D be a Weil $\mathbb{Q}$-Cartier divisor on X.

Lemma 2.2 ([Kaw88, Corollary 5.2])**.** *In the above notation, there is an integer i such that $D \sim iK_X$ near P. In particular, rD is Cartier.*

2.3. Notation as above. There is a deformation X_λ of X such that X_λ has only cyclic quotient singularities $(X_\lambda, P_{\lambda,k}) \simeq \frac{1}{r_k}(1, -1, b_k)$, $0 < b_k < r_k$, $\gcd(b_k, r_k) = 1$. Thus, to every theefold X with terminal singularities, one can associate a collection $\mathbf{B} = ((r_{P,k}, b_{P,k}))$, where $P_{\lambda,k} \in X_{\lambda,k}$ is a singularity of type $\frac{1}{r_{P,k}}(1, -1, b_{P,k})$. This collection is called the *basket* of singularities of X. By abuse of notation, we also will write $\mathbf{B} = (r_{P,k})$ instead of $\mathbf{B} = ((r_{P,k}, b_{P,k}))$. The index of (X, P) is the least common multiple of indices of points $P_{\lambda,k}$. For any Weil divisor D, $\mathbf{B}(D) \subset \mathbf{B}$ denotes the collection of points where D is not Cartier.

Deforming D with (X, P) we obtain Weil divisors D_λ on X_λ. Thus we have a collection of numbers q_k such that $0 \le q_k < r_k$ and $D_\lambda \sim q_k K_{X_\lambda}$ near $P_{\lambda,k}$.

2.4. Orbifold Riemann-Roch formula [Rei87]**.** Let X be a threefold with terminal singularities and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. Then

$$\chi(D) = \frac{1}{12} D \cdot (D - K_X) \cdot (2D - K_X) + \frac{1}{12} D \cdot c_2(X) + \chi(\mathscr{O}_X) + \sum_{P \in \mathbf{B}} c_P(D), \tag{2.5}$$

where

$$c_P(D) = -q_P \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{q_P - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}, \tag{2.6}$$

q_P is such as in 2.3, and $\overline{}$ denotes the smallest residue $\mod r_P$.

Assume that $D^2 \equiv 0$. Then

$$\chi(D) = \frac{1}{12} D \cdot K_X^2 + \frac{1}{12} D \cdot c_2(X) + \chi(\mathscr{O}_X) + \sum c_P(D). \tag{2.7}$$

We have (see, e. g., [Ale94, proof of 2.13])

$$c_P(-K) = \frac{r_P^2 - 1}{12 r_P} - \frac{b_P(r_P - b_P)}{2 r_P}, \qquad c_P(K) = -\frac{r_P^2 - 1}{12 r_P}. \tag{2.8}$$

Construction 2.9 (Base change)**.** Let $f\colon X \to Z$ be a weak del Pezzo fibration and let $f^*(o) = m_o F_o$ be a special fiber of multiplicity m_o. Regard $f\colon X \to (Z, o)$ as a germ. Let $(\mathbb{C}, 0) \simeq (Z', o') \to (Z, o) \simeq (\mathbb{C}, 0)$ is given by $t \mapsto t^{m_o}$ and let X' be the normalization of $X \times_Z Z'$. We obtain the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\ \pi\ } & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Z' & \longrightarrow & Z \end{array} \tag{2.10}$$

Here f' is a weak del Pezzo fibration with special fiber $F'_o = f'^* o' = \pi^* F_o$ of multiplicity 1 and π is a $\boldsymbol{\mu}_{m_o}$-cover which is étale outside of the set M of points where F_o is not Cartier. Hence there is a $\boldsymbol{\mu}_{m_o}$-action on X' such that $X = X'/\boldsymbol{\mu}_{m_o}$ and the action is free outside of M.

Conversely, let $f'\colon X' \to Z' \ni o'$ be a weak del Pezzo fibration with central fiber of multiplicity 1. Assume that f equipped with an equivariant $\boldsymbol{\mu}_{m_o}$-action such that the action on X' is étale in codimension two. If the quotient $X'/\boldsymbol{\mu}_{m_o}$ has only terminal singularities, then $X'/\boldsymbol{\mu}_{m_o} \to Z'/\boldsymbol{\mu}_{m_o}$ is a weak del Pezzo fibration with special fiber of multiplicity m_o.

Proposition 2.11. *Let $f\colon X \to Z$ be a weak del Pezzo fibration. Let $f^*(o) = m_o F_o$ be a special fiber of multiplicity m_o. There is a point $P \in F_o$ such that the index of F_o at P is divisible by m_o.*

Proof. Regard $f\colon X \to (Z, o)$ as a germ and apply Construction 2.9. It is sufficient to show that $\boldsymbol{\mu}_{m_o}$ has a fixed point on F'_o (see Lemma 2.2).

First we consider the case of del Pezzo bundle, i.e., the case where $-K_X$ is ample. Let γ be the log canonical threshold of (X', F'_o) and let $W' \subset X'$ be a minimal center of log canonical singularities of $(X', \gamma F'_o)$ (see [Kaw97a, §1]).

Assume that $\dim W' \le 1$. Let H be a general hyperplane section of X passing through $\pi(W')$ and let $H' := \pi^* H$. For $0 < \varepsilon \ll 1$, the pair $(X',\ \gamma F'_o + \varepsilon H')$ is not LC along $\pi^{-1}\pi(W')$ and LC outside. Therefore for some $0 < \delta \ll \epsilon$ the pair $(X',\ (\gamma-\delta)F'_o+\varepsilon H')$ is not KLT along $\pi^{-1}\pi(W')$ and KLT outside. Moreover, W' is a minimal LC center for $(X',\ (\gamma-\delta)F'_o+\varepsilon H')$. Recall that any irreducible component of the intersection of two LC centers is also an LC center [Kaw97a, Proposition 1.5]. Hence W' is the only LC center in its neighborhood. Since the boundary $(\gamma - \delta)F'_o + \varepsilon H'$ is $\boldsymbol{\mu}_{m_o}$-invariant, all the gW' for $g \in \boldsymbol{\mu}_{m_o}$ are also centers of log canonical singularities for the pair $(X',\ (\gamma - \delta)F'_o + \varepsilon H')$. On the other hand, the locus $\boldsymbol{\mu}_{m_o} W'$ of log canonical singularities for the pair $(X',\, (\gamma-\delta)F'_o+\varepsilon H')$ is connected, see [Sho93, §5], [Kol92, 17.4]. Hence $\boldsymbol{\mu}_{m_o}W'$ is irreducible and so W' is $\boldsymbol{\mu}_{m_o}$-invariant. If W' is a point, we are done. Otherwise W' is a smooth rational curve [Kaw97a, Th. 1.6], [Kaw97b]. But any cyclic group acting on $\mathbb{P}^1$ has a fixed points.

Assume that $\dim W'=2$, that is, $W'=\lfloor \gamma F'_o\rfloor$ and the pair $(X',\gamma F'_o)$ is PLT. By the inversion of adjunction [Sho93, 3.3], [Kol92, 17.6] and Connectedness Lemma [Sho93, §5], [Kol92, 17.4] the surface W' is irreducible, normal and has only KLT singularities. Hence W' is a KLT log del Pezzo surface. In particular, W' is rational. Then the assertion follows by Lemma 2.12 below.

Now we consider the general case. We apply $\boldsymbol{\mu}_{m_o}$-equivariant MMP in the category $\boldsymbol{\mu}_{m_o}$-threefolds (i.e., threefolds with terminal singularities and such that every $\boldsymbol{\mu}_{m_o}$-invariant Weil divisor is $\mathbb{Q}$-Cartier, see e.g. [Mor88, 0.3.14]). Let $X_1\to X'$ be a $\boldsymbol{\mu}_{m_o}$-equivariant $\mathbb{Q}$-factorialization. Run $\boldsymbol{\mu}_{m_o}$-equivariant MMP over Z':

$$X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_N.$$

These maps induce a sequence maps

$$X_1/\boldsymbol{\mu}_{m_o} \dashrightarrow X_2/\boldsymbol{\mu}_{m_o} \dashrightarrow \cdots \dashrightarrow X_N/\boldsymbol{\mu}_{m_o}.$$

where each step is either K-negative divisorial contraction or a flip (both are not neseccarily extremal). Hence, on each step the quotient $X_i/\boldsymbol{\mu}_{m_o}$ has only terminal singularities and the action of $\boldsymbol{\mu}_{m_o}$ on X_i is free in codimension two. On the last step X_N is either a del Pezzo bundle over Z' with $\rho^{\boldsymbol{\mu}_{m_o}}(X_N/Z')=1$ or a $\mathbb{Q}$-conic bundle over a surface S and S/Z' is a rational curve fibration. In both cases $\boldsymbol{\mu}_{m_o}$ has a fixed point on X_N. We prove the existence of fixed point on X_i by a descending induction on i. So we assume that X_{i+1} has a fixed point, say P. If $\psi_i\colon X_i \dashrightarrow X_{i+1}$ is a flip, we may assume that P is contained in the flipped curve $C_{i+1}\subset X_{i+1}$. In this case $\boldsymbol{\mu}_{m_o}$ acts on a connected closed subset of the flipping curve $C_i\subset X_i$. Since C_i is a tree of rational curves, $\boldsymbol{\mu}_{m_o}$ has a fixed point on C_i. Similar argument works in the case where $\psi_i\colon X_i\to X_{i+1}$ is a contraction of a $\boldsymbol{\mu}_{m_o}$-invariant divisor to a curve. Thus we may assume that $\psi_i\colon X_i\to X_{i+1}$ is a divisorial contraction that contracts a $\boldsymbol{\mu}_{m_o}$-invariant divisor $E\subset X_i$ to P. Let γ be the log canonical threshold of (X_i,E) and let $W_i\subset X_i$ be a minimal center of log canonical singularities of $(X_i,\gamma E)$. As in the del Pezzo bundle case above, considering the action of $\boldsymbol{\mu}_{m_o}$ on W_i we find a fixed point. This proves our proposition. □

Lemma 2.12. *Let S be a rational surface. Then any action of a finite cyclic group on S has a fixed point.*

Proof. Let $\boldsymbol{\mu}_m$ be the cyclic group acting on S. Replacing S with its normalization and the minimal resolution, we may assume that S is smooth.

Since S is rational, $H^i(S,\mathbb{C})=0$ if i is odd. Then the assertion follows by the Lefschetz fixed point formula. $\square$

3. Preparations

Notation 3.1. Let $f\colon X\to Z$ be a weak del Pezzo fibration. Compactify X and Z and resolve X only above the added points of Z. Thus we may assume that both X and Z are projective. Let F_g be a general fiber and let $f^*(o)=m_oF_o$ be a special fiber of multiplicity m_o. Write $m_o=m\alpha$, where m and α are positive integers and put $D:=\alpha F_o$. Then $m_oF_o=mD=f^*(o)$.

3.2. By a variant of J. Kollár's Higher Direct Images Theorem (see [KMM87, 1-2-7], [Nak86]), one has that $R^if_*\mathscr{O}_X(K_X-jD)$ is torsion free for all i. But its restriction to the general fiber F_g is zero for $i\neq 2$ because $-K_{F_g}$ is nef and big. Hence $R^if_*\mathscr{O}(K_X-jD)=0$ for $i\neq 2$. Further, the Leray spectral sequence yields

$$H^q(X,K_X-jD)=H^{q-2}(Z,R^2f_*\mathscr{O}(K_X-jD))=0$$

for $q-2\neq 1$ and $j\gg 0$ because $R^2f_*\mathscr{O}(K_X-jD)$ is very negative. By Serre duality

$$H^{3-q}(X,jD)\simeq H^q(X,K_X-jD)^\vee=0$$

for $q\neq 3$ and $j\gg 0$.

Finally, $H^i(X,jD)=0$ for all $i>0$, $j>j_0\gg 0$. We also have

$$H^0(X,jf^*(o)+lD)\simeq H^0(X,jf^*(o))$$

for $l=0,\dots,m-1$. Put $j_1:=\lfloor j_0/m\rfloor$ and

$$\Theta_l:=\frac{1}{mj_1}h^0(X,j_1f^*(o))-\frac{1}{mj_1+l}h^0(X,j_1f^*(o)+lD).$$

Thus for $l=0,\dots,m-1$ we have

$$\Theta_l=\frac{l}{mj_1(mj_1+l)}h^0(X,j_1f^*(o))=\frac{l(j_1-p_a+1)}{mj_1(mj_1+l)}, \tag{3.3}$$

where p_a is the genus of Z. On the other hand, by (2.7)

$$\Theta_l=-\frac{1}{mj_1+l}\sum_{P\in\mathbf{B}}c_P(lD)+\frac{l}{mj_1(mj_1+l)}\chi(\mathscr{O}_X). \tag{3.4}$$

Comparing (3.3) and (3.4) we get

$$m\sum_{P\in\mathbf{B}}c_P(lD)=-l,\qquad l=0,\dots,m-1. \tag{3.5}$$

3.6. Denote

$$\begin{aligned}\Delta_a &:= \chi(\mathscr{O}_X(-K-aF_o))-\chi(\mathscr{O}_X(-K-(a+1)F_o)),\\ \delta_a &:= \sum_{P\in\mathbf{B}}c_P(-K-aF_o)-\sum_{P\in\mathbf{B}}c_P(-K-(a+1)F_o).\end{aligned}$$

As above, for $a = 0, \dots, m_o - 2$, the following equality holds

$$\Delta_a = \frac{13}{12}K^2 \cdot F_o + \frac{1}{12}F_o \cdot c_2(X) + \sum_{P \in \mathbf{B}} c_P(-K - aF_o) - \\ - \sum_{P \in \mathbf{B}} c_P(-K - (a+1)F_o) = \frac{13}{12m_o}K^2 \cdot F_g + \frac{1}{12m_o}F_g \cdot c_2(X) + \delta_a.$$

Since $K^2 \cdot F_g = K_{F_g}^2$ and $F_g \cdot c_2(X) = c_2(F_g) = 12 - K_{F_g}^2$, we have

$$\Delta_a = \frac{K_{F_g}^2 + 1}{m_o} + \delta_a. \tag{3.7}$$

3.8. Some computations. Let (X, P) be a cyclic quotient singularity of type $\frac{1}{r}(a, -a, 1)$, let D be a Weil divisor on X, and let m be a natural number. We have $D \sim qK_X$ for some $0 \le q < r$. Denote

$$\Xi_{P,m} := \sum_{l=1}^{m-1} c_P(lD). \tag{3.9}$$

We also will write Ξ_P or Ξ instead of $\Xi_{P,m}$ if no confusion is likely. By definition

$$\Xi_{P,m} = \sum_{l=1}^{m-1} \left(-\overline{ql}\frac{r^2-1}{12r} + \sum_{j=1}^{\overline{ql}-1} \frac{\overline{bj}(r - \overline{bj})}{2r} \right). \tag{3.10}$$

We compute Ξ in some special situation:

Lemma 3.11. *Let $s := \gcd(r, q)$. Write $r = sm$ and $q = sk$ for some s, $k \in \mathbb{Z}_{>0}$ (so that $\gcd(m, k) = 1$). Then*

$$\Xi_{P,m} = -\frac{m^2-1}{24m}r. \tag{3.12}$$

Proof. By our assumption $\gcd(m, k) = 1$ the parameter $\overline{ql} = \overline{skl}$ runs through all the values sl, $l = 1, \dots, m-1$. Hence,

$$\Xi = -\sum_{l=1}^{m-1} sl\frac{r^2-1}{12r} + \sum_{l=1}^{m-1}\sum_{j=1}^{sl-1} \frac{\overline{bj}(r - \overline{bj})}{2r}.$$

Since $\overline{bj}(r - \overline{bj}) = \overline{bj'}(r - \overline{bj'})$ for $j + j' = r$, we have

$$\sum_{j=1}^{sl-1} \frac{\overline{bj}(r - \overline{bj})}{2r} = \sum_{j=r-sl+1}^{r-1} \frac{\overline{bj}(r - \overline{bj})}{2r}.$$

Therefore,

$$\Xi = -\frac{m(m-1)s}{2}\frac{r^2-1}{12r} + \frac{m-1}{2}\sum_{j=1}^{r-1}\frac{\overline{bj}(r-\overline{bj})}{2r} - \frac{1}{2}\sum_{l=1}^{m-1}\frac{\overline{bsl}(r-\overline{bsl})}{2r} =$$

$$-\frac{(m-1)r}{2}\frac{r^2-1}{12r} + \frac{m-1}{2}\sum_{j=1}^{r-1}\frac{\overline{bj}(r-\overline{bj})}{2r} - \frac{1}{2}\sum_{l=1}^{m-1}\frac{sl(r-sl)}{2r} = \frac{m-1}{2}c_P(rK)$$

$$-\frac{s}{4}\sum_{l=1}^{m-1} l + \frac{s^2}{4r}\sum_{l=1}^{m-1} l^2 = -\frac{sm(m-1)}{8} + \frac{s^2}{24r}(m-1)m(2m-1) =$$

$$= -\frac{m-1}{8}\left(r - \frac{s}{3}(2m-1)\right) = -\frac{m-1}{24}\left(r + \frac{r}{m}\right).$$

(We used $sm = r$ and $c_P(rK) = 0$.) This proves our lemma. □

Lemma 3.13. *If $m = m_1 m_2$, where $m_1 D$ is Cartier, then*

$$\Xi_{P,m} = m_2 \Xi_{P,m_1}.$$

Proof. Follows by (3.9) because $c_P(tD)$ is r-periodic. □

4. Proof of Theorem 1.1

Notation as in 3.1. Near each singular point $P \in X$ of index r_P we write

$$D \sim q_P K_X.$$

Then $mq_P K_X \sim mD$ is Cartier near P. Hence,

$$mq_P \equiv 0 \mod r_P. \tag{4.1}$$

From (3.5) we have

$$\sum_{P\in \mathbf{B}} \Xi_{P,m} = -\sum_{l=1}^{m-1}\frac{l}{m} = -\frac{m-1}{2}. \tag{4.2}$$

Proposition 4.3. *Notation as above. If m is prime, then we have one of the following possibilities:*

(4.3.1) $m = 2$, $\mathbf{B}(D) = (8)$,
(4.3.2) $m = 2$, $\mathbf{B}(D) = (2, 6)$,
(4.3.3) $m = 2$, $\mathbf{B}(D) = (4, 4)$,
(4.3.4) $m = 2$, $\mathbf{B}(D) = (2, 2, 4)$,
(4.3.5) $m = 2$, $\mathbf{B}(D) = (2, 2, 2, 2)$,
(4.3.6) $m = 3$, $\mathbf{B}(D) = (9)$,
(4.3.7) $m = 3$, $\mathbf{B}(D) = (3, 3, 3)$,
(4.3.8) $m = 3$, $\mathbf{B}(D) = (3, 6)$,
(4.3.9) $m = 5$, $\mathbf{B}(D) = (5, 5)$,
(4.3.10) $m = 5$, $\mathbf{B}(D) = (10)$,
(4.3.11) $m = 11$, $\mathbf{B}(D) = (11)$.

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Proof. By (4.1) we have $mq_P \equiv 0 \mod r_P$ and $r_P \equiv 0 \mod m$ for all $P \in \mathbf{B}(D)$ (otherwise $q_P \equiv 0 \mod r_P$ and $P \notin \mathbf{B}(D)$). Put $s_P := r_P/m$. Then $q_P = s_P k_P$ for some $k_P \in \mathbb{Z}_{>0}$. Since $\gcd(k_P, q_P) = 1$, the assumption of Lemma 3.11 holds for each point $P \in \mathbf{B}(D)$. Combining (3.12) with (4.2) we obtain

$$(m+1)\sum_{P\in\mathbf{B}} r_P = 12m.$$

Hence, $m \in \{2, 3, 5, 11\}$. Using the fact $r_P \equiv 0 \mod m$ we get the statement. □

Proposition 4.4. *Cases* (4.3.10) *and* (4.3.11) *do not occur. In particular, the assertion of Theorem* 1.1 *holds if* m_o *is prime.*

Proof. Consider the case (4.3.11). Since $\gcd(q, m) = 1$, there is $0 < l < r = m$ such that $ql \equiv 1 \mod m$. Then by (3.5) and (2.8) we have

$$-\frac{l}{11} = c_P(lD) = c_P(K) = -\frac{r^2-1}{12r} = -\frac{10}{11},$$

so $l = q = 10$. Then again by (3.5) and (2.8)

$$-\frac{1}{11} = c_P(D) = c_P(-K) = \frac{r^2-1}{12r} - \frac{b(r-b)}{2r} = \frac{10}{11} - \frac{b(11-b)}{22}.$$

Hence, $b(11-b) = 22$ and b cannot be coprime to 11, a contradiction.

Consider the case (4.3.10). Since $mq = 5q \equiv 0 \mod r = 10$, q is even. There is $0 < l < 5$ such that $ql \equiv 2 \mod r$. Then by (3.5) we have

$$-\frac{l}{5} = c_P(lD) = c_P(2K) = -\frac{2(r^2-1)}{12r} + \frac{b(r-b)}{2r} = \frac{b(10-b)-33}{20}.$$

Thus $b(10-b) + 4l = 33$, $b \in \{3, 7\}$, $l = 3$, and $q = 4$. Again by (3.5)

$$-\frac{1}{5} = c_P(D) = -\frac{4(r^2-1)}{12r} + \sum_{j=1}^{3} \frac{\overline{bj}(r-\overline{bj})}{2r} = -\frac{33}{10} + \sum_{j=1}^{3} \frac{\overline{3j}(10-\overline{3j})}{20} = -\frac{3}{5},$$

a contradiction. This proves our lemma. □

Corollary 4.5. *For every prime divisor* d *of* m_o *we have* $d \in \{2, 3, 5\}$.

Proof. Apply Propositions 4.3 and 4.4 with $D = \frac{m_o}{d} F_o$. □

Let P_i be points of $\mathbf{B}(F_o)$. Let $P = P_1$ be a point in $\mathbf{B}(F_o)$ whose index r_{P_1} is divisible by m_o (see Proposition 2.11). For short, below we will write r_i, b_i, q_i, etc instead of r_{P_i}, b_{P_i}, q_{P_i}, respectively.

Corollary 4.6. m_o *is not divisible by* $m \in \{16, 27, 25, 10, 15, 12, 18\}$.

Proof. Let $d = 2$, 3 or 5 be a prime divisor of m_o and let $D = \frac{m_o}{d} F_o$. Then $dD = f^*(o)$ and D is not Cartier at P_1. In this case, by Propositions 4.3 and 4.4 the index of (X, P_1) is at most 9, a contradiction. □

Corollary 4.7. *If m_o is not prime, then $m_o \in \{4, 6, 8, 9\}$.*

Lemma 4.8. *If $m_o = 6$, then $\mathbf{B}(F_o) = (2,3,6)$. Moreover,* $\gcd(r_P, q_P) = 1$ *for all $P \in \mathbf{B}(F_o)$.*

Proof. Take $D = 3F_o$. Then $2D \sim f^*(o)$ but D is not Cartier at P_1. Hence (X, P_1) is of index 6 and for D we are in the case (4.3.2), that is, $\mathbf{B}(3F_0) = (2,\, 6)$. At all points $P_i \notin \mathbf{B}(3F_0)$ the divisor $3F_o$ is Cartier. Similarly, take $D' = 2F_o$. Then for D we get the case (4.3.8), that is, $\mathbf{B}(2F_o) = (3,\, 6)$. Hence $\mathbf{B}(F_o)$ contains three points P_1, P_2, P_3 of indices 6, 2, 3, respectively, and in all other points both $D' = 2F_o$ and $D = 3F_o$ are Cartier. Hence $F_o = D - D'$ is Cartier outside of P_1, P_2, P_3 and $\mathbf{B}(F_o) = (2,3,6)$. □

Lemma 4.9. *If $m_o = 4$, then $\mathbf{B}(F_o) = (2,4,4)$. Moreover,* $\gcd(r_P, q_P) = 1$ *for all $P \in \mathbf{B}(F_o)$.*

Proof. Clearly, $2F_o$ is Cartier at all points of index 2. Hence $\mathbf{B}(2F_o)$ contains no such points and for $\mathbf{B}(2F_o)$ we are in the case (4.3.1) or (4.3.3). For all points $P_i \notin \mathbf{B}(2F_o)$ the divisor $2F_o$ is Cartier at P_i. Hence, $q_i = r_i/2$.

Assume that $\mathbf{B}(2F_o) = (8)$. Let $P \in \mathbf{B}(2F_o)$. Since $4F_o$ is Cartier, $4q_P \equiv 0 \mod 8$ (but $2q_P \not\equiv 0 \mod 8$). By Lemma 3.11 and 3.13 we have

$$\Xi_{P_1,4} = -\frac{5}{4}, \qquad \Xi_{P_j,8} = 4\Xi_{P_j,2} = -\frac{r_j}{4}, \quad j \neq 1.$$

Therefore, by (4.2) the following holds $\sum_{i\neq 1} r_i = 1$, a contradiction.

Hence $\mathbf{B}(2F_o) = (4,4)$. At both points $P_i \in \mathbf{B}(2F_o)$ we have $F_o \sim \pm K_X$ near P_i. Again by Lemma 3.11 and 3.13

$$\Xi_{P_i,4} = -\frac{5}{8}, \quad i = 1,\, 2 \qquad \Xi_{P_j,4} = 2\Xi_{P_j,2} = -\frac{r_j}{8}, \quad j \neq 1,\, 2.$$

Therefore, by (4.2) we have $\sum\limits_{i\neq 1,\,2} r_i = 2$ and there is only one solution $\mathbf{B}(F_o) = (4,4,2)$. □

Corollary 4.10. $m_o \neq 8$

Proof. Indeed, if $m_o = 8$, then for $\mathbf{B}(2F_o)$ there is only one possibility from Lemma 4.9. This contradicts Proposition 2.11. □

Lemma 4.11. $m_o \neq 9$.

Proof. Assume that $m_o = 9$. Take $D := 3F_o$. Then $3D \sim f^*(o)$ but D is not Cartier at P_1. Hence, $\gcd(q_1, r_1) = 1$, (X, P_1) is of index 9 and for D we are in the case (4.3.6), that is, $\mathbf{B}(D) = (9) \subset \mathbf{B}(F_o)$. In all points $P_i \in \mathbf{B}(F_o)$, $P_i \neq P_1$ the divisor $D = 3F_o$ is Cartier. Hence by Lemma 3.11 and 3.13 we have

$$\Xi_{P_1,9} = -\frac{10}{3}, \qquad \Xi_{P_i,9} = 3\Xi_{P_i,3} = -\frac{r_i}{3}, \quad i \neq 1.$$

Therefore, by (4.2)

$$-4 = \sum \Xi_{P_i,m} = -\frac{10}{3} - \frac{1}{3}\sum_{i\neq 1} r_i, \qquad r_i = 2.$$

This contradicts $r_i \equiv 0 \mod 3$. □

4.12. The last lemma finishes the proof of Theorem 1.1. It remains to compute values b_k, q_k, and $K_{F_g}^2$ in Table 1.

First we compute the possible values of q_i. We may assume that $1 \le q_i < r_i$. In regular cases ($I_{2,3,6}$, $I_{5,5}$, $I_{3,3,3}$, $I_{2,4,4}$, $I_{2,2,2,2}$) we have $\gcd(q_i, r_i) = 1$ (see Lemmas 4.8 and 4.9) and $m_o \ge r_i$ for all i. Take $1 \le l \le m_o - 1$ so that $q_i l \equiv 1 \mod r_i$. Then by (2.8) and (3.5) the following equality holds

$$\sum_i c_{P_i}(lF_o) = \sum_i c_{P_i}(K) = -\sum_i \frac{r_i^2 - 1}{12 r_i} = -\frac{l}{m_o}.$$

From this we immediately obtain $l \equiv q_i \equiv -1 \mod r_i$ for all i.

If $m_o = 2$ (cases $I_{4\times 2}$, $I_{2,2,4}$, $I_{4,4}$, $I_{2,6}$, I_8), then $2F_o$ is Cartier. Hence $q_i = r_i/2$. It remains to consider only cases I_9 and $I_{3,6}$. In case I_9, since $3F_o$ is Cartier, we have $q := q_1 = 3$ or 6. If $q = 3$, then by (3.5) we have

$$-1 = 3c_P(F_o) = 3c_P(3K) = -\frac{40}{6} + \frac{b(9-b)}{6} + \frac{\overline{2b}(9-\overline{2b})}{6}.$$

Hence, $34 = b(9-b) + \overline{2b}(9-\overline{2b})$ and $5b^2 \equiv 2 \mod 9$. This immediately implies $b \equiv \pm 2$. Similarly, if $q = 6$, then $b^2 \equiv -2 \mod 9$ and $b \equiv \pm 4$.

Finally consider the case $I_{3,6}$. Then by (2.8) and (2.6)

$$c_{P_1}(F_o) = \begin{cases} -2/9 & \text{if } q_1 = 1 \\ -1/9 & \text{if } q_1 = 2 \end{cases} \qquad c_{P_2}(F_o) = \begin{cases} -5/9 & \text{if } q_1 = 2 \\ -1/9 & \text{if } q_1 = 4 \end{cases}$$

The equality $c_{P_1}(F_o) + c_{P_2}(F_o) = -1/3$ (see (3.5)) holds only if $q_1 = 1$, $q_2 = 4$.

Corollary 4.13. *The fiber F_o is regular if and only if $q_i \equiv -1 \mod r_i$ for all i. In particular, for regular F_o near each point $P \in F_o$ where F_o is not Cartier we have $K_X + F_o \sim 0$.*

4.14. Now we find the possible values of b_i. In all cases except for $I_{5,5}$ and I_9 the relations $\gcd(r_i, q_i) = 1$ is sufficient to get the conclusion. The case I_9 was treated above. Consider the case $I_{5,5}$. Then by (2.8) and (3.5) we have $10 = b_1(5 - b_1) + b_2(5 - b_2)$. Hence $b_1^2 + b_2^2 \equiv 0 \mod 5$.

4.15. To obtain the possible values for $K_{F_g}^2$ we use (3.7) with $a = 0$. Since Δ_a is an integer, it is sufficient to compute $\delta_0 = c_P(-K) - c_P(-K - F_o)$. Table 2 gives all values of δ_0. For example, if F_o is regular, then $q_P \equiv -1$

mod r_P for all P and $\delta_0 = \sum c_P(-K) = \sum c_P(F_o)$. So by (2.8) and (3.5) we have $\delta_0 = -1/m_o$. Assume that $q_P = r_P/2$ (and all the r_p are even). Then

$$\delta_0 = \sum_{P \in \mathbf{B}} \left(c_P(-K) - c_P\left(\frac{r_P - 2}{2} K\right)\right).$$

Hence by (2.8) and (2.6)

$r_P = 2 \Longrightarrow \delta_0 = c_P(-K) = -1/8$,
$r_P = 4 \Longrightarrow \delta_0 = c_P(-K) - c_P(K) = 1/4$,
$r_P = 6 \Longrightarrow \delta_0 = c_P(-K) - c_P(2K) = 5/8$,
$r_P = 8 \Longrightarrow \delta_0 = c_P(-K) - c_P(3K) = 1$ or 0 if $b_P = 1$ or 3, respectively.

This immediately gives the values of δ_0 in cases $I_{2,2,4}$, $I_{4,4}$, $I_{2,6}$, and I_8. Cases $I_{3,6}$ and I_9 are similar.

Table 2

	regular	$I_{3,6}$	I_9	$I_{2,2,4}$	$I_{4,4}$	$I_{2,6}$	I_8
δ_0	$-\frac{1}{m_o}$	$\frac{2}{3}$	$\frac{6-q_1}{9}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3-\lvert b_1\rvert}{2}$

5. Examples

In this section we construct some examples of del Pezzo bundles with multiple fibers. We use notation of Construction 2.9. We start with regular case.

Proposition 5.1. *Let $f' \colon X' \to Z' \ni o'$ be a Gorenstein del Pezzo bundle. Assume that the central fiber $F'_o := f'^{-1}(o')$ has only Du Val singularities. Assume also that the cyclic group $\boldsymbol{\mu}_{m_o}$ acts on X' and Z' so that*

(i) *the action on Z' is free outside of o',*
(ii) *f' is $\boldsymbol{\mu}_{m_o}$-equivariant,*
(iii) *the action on F'_o is free in codimension one,*
(iv) *the quotient $F_o := F'_o/\boldsymbol{\mu}_{m_o}$ has only Du Val singularities.*

Then $f \colon X = X'/\boldsymbol{\mu}_{m_o} \to Z = Z'/\boldsymbol{\mu}_{m_o}$ is a del Pezzo bundle with regular central fiber of multiplicity m_o and, moreover, $F_o \sim -K_X$ near each point $P \in X$.

Proof. In notation of Construction 2.9 it is sufficient to show that X has only terminal singularities. Since X' has only terminal singularities and the action of $\boldsymbol{\mu}_{m_o}$ is free outside of a finite number of points P'_k lying on F'_o, the quotient X is smooth outside of $\pi(P'_k) \in F_o$. By the inversion of adjunction [Kol92, 17.6] the pair (X, F_o) is PLT near F_o. Since F_o is Gorenstein, the divisor $K_X + F_o$ is Cartier. Hence the pair (X, F_o) is canonical near F_o and so X has only terminal singularities. $\square$

Now we apply Proposition 5.1 to construct concrete examples.

Example 5.2. Let F'_o be a del Pezzo surface of degree $d := K^2_{F'_o}$ with at worst Du Val singularities. Assume that the group $\boldsymbol{\mu}_{m_o}$, $m_o \geq 2$ acts on F'_o freely in codimension one and so that the quotient $F_o := F'_o/\boldsymbol{\mu}_{m_o}$ has again only Du Val singularities. Clearly, F_o is del Pezzo surface and $m_o K^2_{F_o} = d$. Hence, $d \geq m_o \geq 2$. For $d = 2$, 3, 4, and 8, according to [HW81] there is an embedding

$$\begin{array}{lcll} F'_o & \subset & \mathbb{P} := \mathbb{P}(1,1,1,2) & \text{if } d = 2 \\ F'_o & \subset & \mathbb{P} := \mathbb{P}^3 & \text{if } d = 3 \\ F'_o & \subset & \mathbb{P} := \mathbb{P}^4 & \text{if } d = 4 \\ F'_o & \subset & \mathbb{P} := \mathbb{P}^3 & \text{if } d = 8 \end{array}$$

Moreover, if $d = 2$, 3, 8, then F_o is a (weighted) hypersurface of degree 4, 3, 2, respectively and if $d = 4$, then F'_o is an intersection of two quadrics. The action of $\boldsymbol{\mu}_{m_o}$ on F'_o induces the action on $\mathbb{P}$. We fix a linearization of this action and take semi-invariant coordinates x_i in $\mathbb{P}$. Now we define $\boldsymbol{\mu}_{m_o}$-equivariant del Pezzo bundle $f' \colon X' \to Z'$. If F'_o is smooth, we can take $X' = F'_o \times \mathbb{C}_t$. In general case, X' is embedded into $\mathbb{P} \times \mathbb{C}_t$, $Z' = \mathbb{C}_t$ and f' is the projection, where t is a coordinate in $\mathbb{C}$ with $\operatorname{wt} t = 1$. Consider for example the case $d \leq 3$ (case $d = 4$ is similar). Let $\phi = \phi(x_1, x_2, x_3, x_4)$ be the defining equation of F'_o and let γ_k be all monomials of weighted degree d. For each γ_k, let n_k be the smallest positive integer such that $n_k \equiv -\operatorname{wt} \gamma_k \mod m_o$. Then the polynomial $\psi(x_1, \dots, x_4; t) := \phi + \sum c_k t^{n_k} \gamma_k$, $c_k \in \mathbb{C}$ is $\boldsymbol{\mu}_{m_o}$-semi-invariant. Let $X' = \{\psi = 0\} \subset \mathbb{P} \times \mathbb{C}_t$. By Betrtini's theorem, for sufficiently general constants c_k, fibers F'_t of f' over $t \neq 0$ are smooth del Pezzo surfaces. Hence we can apply Proposition 5.1 and get a del Pezzo bundle with a regular fiber of multiplicity m_o.

Note that the map $F'_o \to F_o$ is étale outside of $\operatorname{Sing} F_o$. Hence there is a surjection $\pi_1(F_o \setminus \operatorname{Sing} F_o) \twoheadrightarrow \boldsymbol{\mu}_{m_o}$. Conversely, assume that F_o is a del Pezzo surface with Du Val singularities such that $\pi_1(F_o \setminus \operatorname{Sing} F_o) \twoheadrightarrow \boldsymbol{\mu}_{m_o}$. Then there is an étale outside of $\operatorname{Sing} F_o$ cyclic $\boldsymbol{\mu}_{m_o}$-cover $\upsilon \colon F'_o \to F_o$. Since $K_{F'_o} = \upsilon^* K_{F_o}$, F'_o is also a del Pezzo surface with Du Val singularities. The fundamental groups of smooth loci of Du Val del Pezzo surfaces are described in [MZ88], [MZ93]. For example, from [MZ88] we have the following examples with $\rho(F_o) = 1$ (we do not list all the possibilities):

$K^2_{F_o}$	$\operatorname{Sing} F_o$	m_o	$K^2_{F'_o} = K^2_{F_g}$	$\rho(F'_o)$	F'_o, $\operatorname{Sing} F'_o$	type
1	$A_1A_2A_5$	6	6	4	smooth	$I_{2,3,6}$

$K^2_{F_o}$	$\operatorname{Sing} F_o$	m_o	$K^2_{F'_o} = K^2_{F_g}$	$\rho(F'_o)$	F'_o, $\operatorname{Sing} F'_o$	type
1	$2A_4$	5	5	5	smooth	$I_{5,5}$
2	$A_1 2A_3$	4	8	2	$\mathbb{P}^1 \times \mathbb{P}^1$	$I_{2,4,4}$
1	$A_3 D_5$	4	4	4	A_2	$I_{2,4,4}$
3	$3A_2$	3	9	1	$\mathbb{P}^2$	$I_{3,3,3}$
2	$A_2 A_5$	3	6	3	A_1	$I_{3,3,3}$
1	A_8	3	3	5	A_2	$I_{3,3,3}$
4	$2A_1 A_3$	2	8	1	$\mathbb{P}(1,1,2)$	$I_{2,2,2,2}$
3	$A_1 A_5$	2	6	2	A_2	$I_{2,2,2,2}$
2	A_7	2	4	3	A_3	$I_{2,2,2,2}$
1	D_8	2	2	3	D_5	$I_{2,2,2,2}$

Example 5.3. In some cases we can give more explicit construction. As was mentioned above, if F'_o is smooth, we can take $X' = Z' \times F'_o$. Consider the following cases:

- $F'_o = \mathbb{P}^2$, $\boldsymbol{\mu}_3$ acts on $\mathbb{P}^2_{x,y}$ by $x \mapsto \epsilon x$, $y \mapsto \epsilon^{-1} y$ (here x, y are non-homogeneous coordinates on $\mathbb{P}^2$ and $\epsilon^3 = 1$). Then $\mathbb{P}^2/\boldsymbol{\mu}_3$ is a toric del Pezzo surface of degree 3 having three singular points of type A_2. The quotient $f\colon X \to Z$ is a del Pezzo bundle with special fiber of type $I_{3,3,3}$.
- $F'_o = \mathbb{P}^1 \times \mathbb{P}^1$, $\boldsymbol{\mu}_2$ acts on $\mathbb{P}^1_x \times \mathbb{P}^1_y$ by $x \mapsto -x$, $y \mapsto -y$. Then $\mathbb{P}^1 \times \mathbb{P}^1/\boldsymbol{\mu}_2$ is a del Pezzo surface of degree 4 having four singular points of type A_1. The quotient $f\colon X \to Z$ is a del Pezzo bundle with special fiber of type $I_{2,2,2,2}$.
- $F'_o = \mathbb{P}^1 \times \mathbb{P}^1$, $\boldsymbol{\mu}_4$ acts by $x \mapsto y$, $y \mapsto -x$. Then $\mathbb{P}^1 \times \mathbb{P}^1/\boldsymbol{\mu}_4$ is a del Pezzo surface of degree 2 having two points of type A_3 and one point of type A_1. The quotient $f\colon X \to Z$ is a del Pezzo bundle with special fiber of type $I_{2,4,4}$.

Now we give some examples of irregular multiple fibers.

Example 5.4. Recall that any smooth del Pezzo surface of degree 1 can be realized as a weighted hypersurface of degree 6 in $\mathbb{P} = \mathbb{P}(1,1,2,3)$. Let

$$\phi(x_1, x_2, y, z) = a_1 x_1^6 + a_2 x_2^6 + y^2(b_1 x_1^2 + b_2 x_2^2) + cz^2, \qquad a_i, b_j, c \in \mathbb{C}^*$$

be a polynomial of weighted degree 6, where x_1, x_2, y, z are coordinates in $\mathbb{P}$ with $\operatorname{wt} x_i = 1$, $\operatorname{wt} y = 2$, $\operatorname{wt} z = 3$. Consider the hypersurface $F'_o \subset \mathbb{P}$ given by $\phi = 0$. By Bertini's theorem, for sufficiently general a_i, b_j, c, the surface F'_o is smooth outside of $P' := (0:0:1:0)$. Consider the subvariety X' in $\mathbb{P} \times \mathbb{C}_t$ given by $\phi + ty^3 = 0$ and let $f'\colon X' \to Z' = \mathbb{C}$ be the natural projection. Since F'_o is the scheme fiber of the projection $f'\colon X' \to Z'$, the variety X' is smooth outside of P'. We identify F'_o with the fiber over $t = 0$. Then f' is a del Pezzo bundle of degree 1 having a unique singular point of type $\frac{1}{2}(1, 1, 1)$ at P'.

Now let $\boldsymbol{\mu}_2$ acts on $\mathbb{P} \times \mathbb{C}$ and X' by

$$(x_1, x_2, y, z; t) \longmapsto (x_1, -x_2, -y, -z; -t).$$

The locus of fixed points Λ consists of the line $L := \{x_1 = y = t = 0\}$ and two isolated points $P' := (0 : 0 : 1 : 0; 0)$ and $P_1 := (1 : 0 : 0 : 0; 0)$. Then $F'_o \cap \Lambda = \{P', Q_1, Q_2\}$, where $Q_1 \neq Q_2$ are points given by $x_1 = y = a_2x_2^6 + z^2 = t = 0$. Let $f\colon X = X'/\boldsymbol{\mu}_2 \to Z = Z'/\boldsymbol{\mu}_2$ be the quotient of f'. Since the action of $\boldsymbol{\mu}_2$ on X' is free in codimension one, $-K_X$ is f-ample and $F_o := F'_o/\boldsymbol{\mu}_2$ is a fiber of multiplicity 2. We show that X has only terminal singularities. By the above, X is smooth outside of images of P', Q_1, Q_2. Since the (X', Q_i) are smooth points, quotients $(X', Q_i)/\boldsymbol{\mu}_2$ are terminal of type $\frac{1}{2}(1, 1, 1)$. Consider the affine chart $\{y \neq 0\} \simeq \mathbb{C}^4_{x'_1,x'_2,z',t}/\boldsymbol{\mu}_2(1, 1, 1, 0)$ containing P'. Here X' is given by the equation $\phi(x'_1, x'_2, 1, z') + t = 0$ and the action of $\boldsymbol{\mu}_2$ on $\mathbb{P}$ induces the following action of $\boldsymbol{\mu}_4$:

$$(x'_1, x'_2, z', t) \longmapsto (\mathrm{i}\, x'_1, -\mathrm{i}\, x'_2, \mathrm{i}\, z', -t), \quad \mathrm{i} = \sqrt{-1}.$$

Thus the quotients $(X', P')/\boldsymbol{\mu}_2$ is a terminal cyclic quotient of type $\frac{1}{4}(1, -1, 1)$. Therefore, $f\colon X \to Z$ is a del Pezzo bundle with special fiber of type $I_{2,2,4}$.

Example 5.5. As above let $\mathbb{P} = \mathbb{P}(1, 1, 2, 3)$ and let

$$\phi(x_1, x_2, y, z) = a_1x_1^6 + a_2x_2^6 + cy^3, \qquad a_i, c \in \mathbb{C}^*$$

be a $\boldsymbol{\mu}_2$-invariant polynomial of weighted degree 6. Consider the hypersurface $F'_o \subset \mathbb{P}$ given by $\phi = 0$. Again for sufficiently general a_i, c, the surface F'_o is smooth outside of $P'' := (0 : 0 : 0 : 1)$. Consider the subvariety X' in $\mathbb{P} \times \mathbb{C}_t$ given by $\phi + tz^2 = 0$ and let $f'\colon X' \to Z' = \mathbb{C}$ be the natural projection. Then f' is a del Pezzo bundle of degree 1 having a unique singular point of type $\frac{1}{3}(1, 1, -1)$ at P''. Now let $\boldsymbol{\mu}_3$ acts on $\mathbb{P} \times \mathbb{C}$ and X' by

$$(x_1, x_2, y, z; t) \longmapsto (x_1, \epsilon x_2, \epsilon y, \epsilon z; \epsilon t), \qquad \epsilon := \exp(2\pi\,\mathrm{i}/3).$$

The only fixed point on X' is P''. As above, one can check that $(X', P'')/\boldsymbol{\mu}_3$ is a terminal point of type $\frac{1}{9}(-1, 2, 1)$. Therefore, $X/\boldsymbol{\mu}_3 \to Z'/\boldsymbol{\mu}_3$ is a del Pezzo bundle with special fiber of type I_9.

Example 5.6. Let $\mathbb{P} := \mathbb{P}(1, 1, 1, 2, 2)$, let x_1, x_2, x_3, y_1, y_2 be coordinates, and let $X' \subset \mathbb{P} \times \mathbb{C}$ be subvariety given by

$$\begin{cases} c_1y_1^2 + c_2y_2^2 & = a_1x_1^4 + a_2x_2^4 + a_3x_3^4 \\ ty_2 & = b_1x_1^2 + b_2x_2^2 + b_3x_3^2, \end{cases}$$

where t is a coordinate on $\mathbb{C}$ and a_i, b_j, c_k are sufficiently general constants. By Bertini's theorem X' is smooth outside of $\{x_1 = x_2 = x_3 = 0\} \subset \operatorname{Sing}\mathbb{P}$. It is easy to check that $X' \cap \operatorname{Sing}\mathbb{P}$ consists of two points

$$\{P_1',\, P_2'\} = \{t = x_1 = x_2 = x_3 = 0,\ c_1y_1^2 + c_2y_2^2 = 0\}$$

and these points are terminal of type $\frac{1}{2}(1, 1, 1)$. The projection $X' \to \mathbb{C}$ is a del Pezzo bundle of degree 2. Define the action of $\boldsymbol{\mu}_2$ by

$$(x_1, x_2, x_3, y_1, y_2; t) \longmapsto (x_1, x_2, -x_3, y_1, -y_2; -t).$$

There are four fixed points

$$\{Q_1', \dots, Q_4'\} = \{t = x_3 = y_2 = 0,\ c_1y_1^2 = a_1x_1^4 + a_2x_2^4,\ b_1x_1^2 + b_2x_2^2 = 0\}.$$

The quotient $f\colon X'/\boldsymbol{\mu}_2 \to \mathbb{C}/\boldsymbol{\mu}_2$ is a del Pezzo bundle of type $I_{2,2,2,2}$. Note however that the image P of $\{P_1',\, P_2'\}$ on $X'/\boldsymbol{\mu}_2$ is a point of type $\frac{1}{2}(1, 1, 1)$ and F_o is Cartier at P (i.e., $P \notin \mathbf{B}(F_o)$).

Example 5.7. In the above notation define another action of $\boldsymbol{\mu}_2$:

$$(x_1, x_2, x_3, y_1, y_2; t) \longmapsto (x_1, x_2, -x_3, -y_1, -y_2; -t).$$

Then the quotient $f\colon X'/\boldsymbol{\mu}_2 \to \mathbb{C}/\boldsymbol{\mu}_2$ is a del Pezzo bundle of type $I_{4,4}$.

Example 5.8. Let $\mathbb{P} := \mathbb{P}(1, 1, 1, 1, 2)$, let x_1, x_2, x_3, x_4, y be coordinates, and let $X' \subset \mathbb{P} \times \mathbb{C}$ be subvariety given by

$$\begin{cases} a_1x_1^2 + a_2x_2^2 + a_3x_1x_2 + a_4x_3x_4 = ty \\ b_1x_1^3 + b_2x_2^3 + b_3x_3^3 = x_4y \end{cases}$$

where t is a coordinate on $\mathbb{C}$ and a_i, b_j are sufficiently general constants. Then the variety X' is smooth outside of the point $P' = \{x_1 = x_2 = x_3 = x_4 = 0\}$ and $P' \in X'$ is of type $\frac{1}{2}(1, 1, 1)$. The projection $X' \to \mathbb{C}$ is a del Pezzo bundle of degree 3. Define the action of $\boldsymbol{\mu}_3$ by

$$(x_1, x_2, x_3, x_4, y; t) \longmapsto (\omega^{-1}x_1, \omega^{-1}x_2, \omega x_3, x_4, y; \omega t).$$

There are two fixed points $\{t = x_1 = x_2 = x_3 = x_4y = 0\}$ and quotients of these points are of types $\frac{1}{6}(1, 1, -1)$ and $\frac{1}{3}(, 1, -1)$. Hence the quotient $f\colon X'/\boldsymbol{\mu}_3 \to \mathbb{C}/\boldsymbol{\mu}_3$ is a del Pezzo bundle of type $I_{3,6}$.

6. On del Pezzo bundles with fibers of multiplicity ≥ 5.

Notation 6.1. Let $f\colon X \to Z \ni o$ be the germ of a del Pezzo bundle and let $m_oF_o = f^*(o)$ be a fiber of multiplicity m_o. In this section we assume that $m_0 \geq 5$, i.e., F_o is of type $I_{2,3,6}$ or $I_{5,5}$.

Conjecture 6.2. *In notation of* 6.1 *f is a quotient of a Gorenstein del Pezzo bundle by a cyclic group acting free in codimension* 2 *on X.*

Proposition 6.3. *Notation as in* 6.1. *If either*

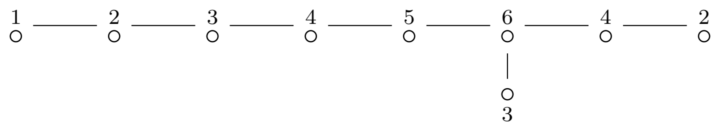
(i) $\mathbf{B}(F_o) = \mathbf{B}$, *that is, each point* $P \in F_o$ *where* F_o *is Cartier is Gorenstein on* X, *or*

(ii) *a general member* $S \in |-K_X|$ *has only Du Val singularities* (*Reid's general elephant conjecture*),

then 6.2 *holds.*

Proof. Assume that (i) holds. By Table 1 near each singular point $K_X + F_o \sim 0$. Apply Construction 2.9. Then $F'_o = \pi^* F_o$ is Cartier. Since π is étale in codimension one, $K_{X'} + F'_o \sim 0$. Hence, X' is Gorenstein.

Now assume that (ii) holds. Then $\varphi\colon S \to Z$ is an elliptic fibration with Du Val singularities. We have $K_S = (K_X + S)|_S \sim 0$. Let $\mu\colon \tilde{S} \to S$ be the minimal resolution. Since S has only Du Val singularities, $K_{\tilde{S}} \sim 0$. In particular, $\psi\colon \tilde{S} \to Z$ is a minimal elliptic fibration. By Kodaira's canonical bundle formula ψ has no multiple fibers [Kod64, Th. 12]. Since $\psi^* o$ has a component of multiplicity ≥ 5, for $\psi^* o$ we have only one possibility $\tilde{E}_8$ in the classification of singular fibers [Kod63, Th. 6.2]. More precisely, $\mathrm{Supp}(\psi^* o)$ is a tree of smooth rational curves with self-intersection number -2 and the dual graph Γ is the following:

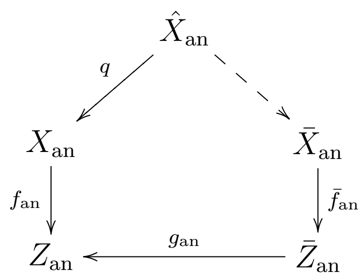


Further we consider the case $m_o = 6$ (the case $m_o = 5$ is similar). It is easy to see that the curve $S \cap F_o$ is irreducible and correspond to the central vertex v of Γ. Then $\Gamma \setminus \{v\}$ has three connected components corresponding to points of types A_1, A_2 and A_5 on S. Therefore, $\mathbf{B}(F_o) = \mathbf{B}$. □

Proposition 6.4. *In notation of* 6.1, *assume that* F_o *is irreducible. let* $f_{\mathrm{an}}\colon X_{\mathrm{an}} \to Z_{\mathrm{an}}$ *be the analytic germ near* F_o. *Then* X_{an} *is* $\mathbb{Q}$*-factorial over* Z_{an}, $\rho(X_{\mathrm{an}}/Z_{\mathrm{an}}) = 1$, *and* $\rho(F_o) = 1$.

Warning. Here the $\mathbb{Q}$-factoriality condition of X_{an} means that every global Weil divisor of the total germ X_{an} along F_o is $\mathbb{Q}$-Cartier, not that every analytic local ring of X_{an} is $\mathbb{Q}$-factorial.

Proof. Let $q\colon \hat{X}_{\mathrm{an}} \to X_{\mathrm{an}}$ be a $\mathbb{Q}$-factorialization over Z_{an}. Run the MMP over Z_{an}. So, we have the following diagram



Here $\bar{X}_{\rm an}$ is $\mathbb{Q}$-factorial over $\bar{Z}_{\rm an}$ and $\rho(\bar{X}_{\rm an}/\bar{Z}_{\rm an}) = 1$. Note that $\hat{X}_{\rm an} \dashrightarrow \bar{X}_{\rm an}$ is a composition of flips and divisorial contractions that contract divisors to curves dominating $Z_{\rm an}$. Let $\bar{F}_o$ be the proper transform of F_o on $\bar{X}_{\rm an}$. There are two possibilities:

1) $\bar{Z}_{\rm an}$ is a surface. Then $g_{\rm an}$ is a rational curve fibration with $\rho(\bar{Z}_{\rm an}/Z_{\rm an}) = 1$. Let $C := \bar{f}_{\rm an}(\bar{F}_o)$. Since $\bar{X}_{\rm an}$ has only isolated singularities, $\bar{F}_o = \bar{f}^*_{\rm an}(C)$. Further, $g^*_{\rm an}(o) = nC$ for some $n \in \mathbb{Z}_{>0}$ and $\bar{f}^*_{\rm an} g^*_{\rm an}(o) = n\bar{f}^*_{\rm an}C = n\bar{F}_o$. So, $n = m_o$. By the main result of [MP08] the surface $\bar{Z}_{\rm an}$ has only Du Val singularities. Therefore, $m_o = n \le 2$, a contradiction.

2) $\bar{Z}_{\rm an}$ is a curve. Then $g_{\rm an}$ is an isomorphism and $\bar{f}_{\rm an}\colon \bar{X}_{\rm an} \to \bar{Z}_{\rm an}$ is a del Pezzo bundle with central fiber $\bar{F}_o$ of multiplicity $\bar{m}_o = m_o \ge 5$. By Table 1 the degree of the generic fiber of $\bar{f}_{\rm an}$ (and $f_{\rm an}$) is equal to m_o. This means that degrees of generic fibers of $\bar{f}_{\rm an}$ and $f_{\rm an}$ coincide. In particular, the MMP $\hat{X}_{\rm an} \dashrightarrow \bar{X}_{\rm an}$ does not contract any divisors. Hence, $\rho(\hat{X}_{\rm an}/Z_{\rm an}) = \rho(\bar{X}_{\rm an}/Z_{\rm an}) = 1$. This implies that q is an isomorphism and $\rho(X_{\rm an}/Z_{\rm an}) = 1$. The last assertion follows from the exponential exact sequence and vanishing $R^1 f_{{\rm an}*}\mathscr{O}_{X_{\rm an}} = 0$. □

Proposition 6.5. *Notation as in* 6.1*. Conjecture* 6.2 *holds under the additional assumption that* F_o *has only log terminal singularities.*

Proof. Assume that F_o has only log terminal singularities. By Table 1 near each point $P \in \mathbf{B}(F_o)$ we have $K_X + F_o \sim 0$. By Adjunction F_o has only Du Val singularities at these points. In points $P \notin \mathbf{B}(F_o)$ the divisor F_o is Cartier. Hence F_o has only singularities of type T [KSB88]. By Noether's formula [HP, Prop. 3.5]

$$K^2_{F_o} + \rho(F_o) + \sum_{P\in F_o} \mu_P = 10.$$

Since points in $\mathbf{B}(F_o)$ correspond to distinct points on X, we have $\sum_{P\in\mathbf{B}(F_o)} \mu_P \ge 8$. Hence, $K^2_{F_o} = 1$, $\rho(F_o) = 1$, and $\mathbf{B}(F_o) = \mathbf{B}$. Now the assertion folows by Proposition 6.3. □

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