

REMARKS ON GENERATORS AND DIMENSIONS OF TRIANGULATED CATEGORIES

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ABSTRACT. In this paper we prove that the dimension of the bounded derived category of coherent sheaves on a smooth quasi-projective curve is equal to one. We also discuss dimension spectrums of these categories.

Let \mathcal{T} be a triangulated category. We say that an object $E \in \mathcal{T}$ is a **classical generator** for \mathcal{T} if the category \mathcal{T} coincides with the smallest triangulated subcategory of \mathcal{T} which contains E and is closed under direct summands.

If a classical generator generates the whole category for a finite number of steps then it is called a **strong generator**. More precisely, let \mathcal{I}_1 and \mathcal{I}_2 be two full subcategories of \mathcal{T} . We denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of \mathcal{T} consisting of all objects such that there is a distinguished triangle $M_1 \rightarrow M \rightarrow M_2$ with $M_i \in \mathcal{I}_i$. For any subcategory $\mathcal{I} \subset \mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of \mathcal{T} containing \mathcal{I} and closed under finite direct sums, direct summands and shifts. We put $\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$ and we define by induction $\langle \mathcal{I} \rangle_k = \langle \mathcal{I} \rangle_{k-1} \diamond \langle \mathcal{I} \rangle$. If \mathcal{I} consists of an object E we denote $\langle \mathcal{I} \rangle$ as $\langle E \rangle_1$ and put by induction $\langle E \rangle_k = \langle E \rangle_{k-1} \diamond \langle E \rangle_1$.

Definition 1. *Now we say that E is a strong generator if $\langle E \rangle_n = \mathcal{T}$ for some $n \in \mathbb{N}$.*

Note that E is classical generator if and only if $\bigcup_{k \in \mathbb{Z}} \langle E \rangle_k = \mathcal{T}$. It is also easy to see that if a triangulated category \mathcal{T} has a strong generator then any classical generator of \mathcal{T} is strong as well.

Following to [2] we define the dimension of a triangulated category.

Definition 2. *The dimension of a triangulated category \mathcal{T} , denoted by $\dim \mathcal{T}$, is the minimal integer $d \geq 0$ such that there is $E \in \mathcal{T}$ with $\langle E \rangle_{d+1} = \mathcal{T}$.*

We also can define the dimension spectrum of a triangulated category as follows.

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Definition 3. *The dimension spectrum of a triangulated category \mathcal{T} , denoted by $\sigma(\mathcal{T})$, is a subset of \mathbb{Z} , which consists of all integer $d \geq 0$ such that there is $E \in \mathcal{T}$ with $\langle E \rangle_{d+1} = \mathcal{T}$ and $\langle E \rangle_d \neq \mathcal{T}$.*

A. Bondal and M. Van den Bergh showed in [1] that the triangulated category of perfect complexes $\mathfrak{P}erf(X)$ on a quasi-compact quasi-separated scheme X has a classical generator. (Recall that a complex of \mathcal{O}_X -modules is called perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles.)

For the triangulated category of perfect complexes on a quasi-projective scheme we can present a classical generator directly.

Theorem 4. *Let X be a quasi-projective scheme of dimension d and let \mathcal{L} be a very ample line bundle on X . Then the object $\mathcal{E} = \bigoplus_{i=k-d}^k \mathcal{L}^i$ is a classical generator for the triangulated category of perfect complexes $\mathfrak{P}erf(X)$.*

Proof. The scheme X is an open subscheme of a projective scheme $X' \subset \mathbb{P}^N$ and \mathcal{L} is the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ on X . Let us take $N+1$ linear independent hyperplanes $H_i \subset \mathbb{P}^N, i = 0, \dots, N$. In this case the intersection $H_0 \cap \dots \cap H_N$ is empty. The hyperplanes H_i give a section s of the vector bundle $U = \mathcal{O}(1)^{\oplus(N+1)}$ which does not have zeros. This implies that the Koszul complex induced by s

$$0 \longrightarrow \Lambda^{N+1}(U^*) \longrightarrow \Lambda^N(U^*) \longrightarrow \dots \longrightarrow \Lambda^2(U^*) \longrightarrow U^* \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0$$

is exact on \mathbb{P}^N . Consider the restriction of the truncated complex on X

$$\Lambda^{d+1}(U_X^*) \longrightarrow \dots \longrightarrow \Lambda^2(U_X^*) \longrightarrow U_X^*.$$

It has two nontrivial cohomologies, one of which is \mathcal{O}_X . And, moreover, since the dimension of X is equal to d the sheaf \mathcal{O}_X is a direct summand of this complex. Tensoring this complex with \mathcal{L}^{k+1} we obtain that the triangulated subcategory which contains \mathcal{L}^i for $i = k-d, \dots, k$ also contains \mathcal{L}^{k+1} . Thus, it contains \mathcal{L}^i for all $i \geq k-d$. By duality this category contains also all \mathcal{L}^i for all $i \leq k$. Thus we have all powers \mathcal{L}^i , where $i \in \mathbb{Z}$.

Finally, it is easy to see that $\{\mathcal{L}^i\}_{i \in \mathbb{Z}}$ classically generate the triangulated category of perfect complexes $\mathfrak{P}erf(X)$. Indeed, for any perfect complex E we can construct a bounded above complex P^\cdot , where all P^k are direct sums of line bundles \mathcal{L}^i , together with a quasi-isomorphism $P^\cdot \xrightarrow{\sim} E$. Consider the brutal truncation $\sigma^{\geq -m} P^\cdot$ for sufficiently large m and the map $\sigma^{\geq -m} P^\cdot \longrightarrow E$. The cone of this map is isomorphic to $\mathcal{F}[m+1]$, where \mathcal{F} is a vector bundle. And since the $\text{Hom}(E, \mathcal{F}[m+1]) = 0$ for sufficiently large m we get that E is a direct summand of $\sigma^{\geq -m} P^\cdot$. \square

A. Bondal and M. Van den Bergh also proved that for any smooth separated scheme X the triangulated category of perfect complexes $\mathfrak{P}erf(X)$ has a strong generator ([1], Th.3.1.4). Furthermore, R. Rouquier showed that for quasi-projective scheme X the property to be regular is equivalent to the property that the triangulated category of perfect complexes $\mathfrak{P}erf(X)$ has a strong generator (see [2], Prop 7.35). On the other hand, there is a remarkable result of R. Rouquier which says that under some general conditions the bounded derived category of coherent sheaves $\mathbf{D}^b(\text{coh}(X))$ has a strong generator. More precisely it says

Theorem 5. (R. Rouquier, [2] Th.7.39) Let X be a separated scheme of finite type. Then there are an object $E \in \mathbf{D}^b(\text{coh}(X))$ and an integer $d \in \mathbb{Z}$ such that $\mathbf{D}^b(\text{coh}(X)) \cong \langle E \rangle_{d+1}$. In particular, $\dim \mathbf{D}^b(\text{coh}(X)) < \infty$.

Keeping in mind this theorem we can ask about the dimension of the derived category of coherent sheaves on a separated scheme of finite type. It is proved in [2] that

- for a reduced separated scheme X of finite type $\dim \mathbf{D}^b(\text{coh}(X)) \geq \dim X$;
- for a smooth affine scheme $\dim \mathbf{D}^b(\text{coh}(X)) = \dim X$;
- for a smooth quasi-projective scheme $\dim \mathbf{D}^b(\text{coh}(X)) \leq 2 \dim X$.

In this paper we show that the dimension of the derived category of coherent sheaves on a smooth quasi-projective curve C is equal to 1. For affine curve it is known and for \mathbb{P}^1 it is evident. Thus, it is sufficient to consider a smooth projective curve of genus $g \geq 1$.

Theorem 6. *Let C be a smooth projective curve of genus $g \geq 1$. Then $\dim \mathbf{D}^b(\text{coh}(C)) = 1$.*

At first, we should bring an object which generates $\mathbf{D}^b(\text{coh}(C))$ for one step. Let \mathcal{L} be a line bundle on C such that $\deg \mathcal{L} \geq 8g$. Let us consider $\mathcal{E} = \mathcal{L}^{-1} \oplus \mathcal{O}_C \oplus \mathcal{L} \oplus \mathcal{L}^2$. We are going to show that \mathcal{E} generates the bounded derived category of coherent sheaves on C for one step, i.e. $\langle \mathcal{E} \rangle_2 = \mathbf{D}^b(\text{coh } X)$.

Since any object of $\mathbf{D}^b(\text{coh}(X))$ is a direct sum of its cohomologies it is sufficient to prove that any coherent sheaf \mathcal{G} belongs to $\langle \mathcal{E} \rangle_2$. Further, each coherent sheaf \mathcal{G} on a curve is a direct sum of a torsion sheaf T and a vector bundle \mathcal{F} .

Lemma 7. *Let C be a smooth projective curve of genus $g \geq 1$ and let \mathcal{L} be a line bundle on C as above. Then there is an exact sequence of the form*

$$(\mathcal{L}^{-1})^{\oplus r_1} \longrightarrow \mathcal{O}_C^{\oplus r_0} \longrightarrow T \longrightarrow 0$$

for any torsion coherent sheaf T on C .

Let \mathcal{F} be a vector bundle on the curve C . Consider the Harder-Narasimhan filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$. It is such filtration that every quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semi-stable and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$ for all $0 < i < n$, where $\mu(\mathcal{G})$ is the slope of a vector bundle \mathcal{G} and is equal to $c_1(\mathcal{G})/r(\mathcal{G})$.

Main Lemma 8. *Let \mathcal{L} be a line bundle with $\deg \mathcal{L} \geq 8g$. Let \mathcal{F} be a vector bundle on C and let $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ be its Harder-Narasimhan filtration. Choose $0 \leq i \leq n$ such that $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \geq 4g > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$. Then there are exact sequences of the form*

$$a) \quad (\mathcal{L}^{-1})^{\oplus r_1} \xrightarrow{\alpha} \mathcal{O}_C^{\oplus r_0} \longrightarrow \mathcal{F}_i \longrightarrow 0, \quad b) \quad 0 \longrightarrow \mathcal{F}/\mathcal{F}_i \longrightarrow \mathcal{L}^{\oplus s_0} \xrightarrow{\beta} (\mathcal{L}^2)^{\oplus s_1}.$$

To prove Lemma 7 and the Main Lemma 8 we need the following lemma which is well-known.

Lemma 9. *Let \mathcal{G} be a vector bundle on a smooth projective curve C over a field k . Denote by $\overline{\mathcal{G}}$ its pullback on $\overline{C} = C \otimes_k \overline{k}$. Assume that for any line bundle \mathcal{M} on \overline{C} with $\deg \mathcal{M} = d$ we have $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{M}) = 0$. Then*

- i) $H^1(C, \mathcal{G} \otimes \mathcal{N}) = 0$ for any \mathcal{N} on C with $\deg \mathcal{N} \geq d$;
- ii) any sheaf $\mathcal{G} \otimes \mathcal{N}$ is generated by the global sections for all \mathcal{N} with $\deg \mathcal{N} > d$.

Proof. i) Since any field extension is strictly flat it is sufficient to check that $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \overline{\mathcal{N}}) = 0$. From an exact sequence

$$(1) \quad 0 \rightarrow \overline{\mathcal{G}} \otimes \overline{\mathcal{N}}(-x) \rightarrow \overline{\mathcal{G}} \otimes \overline{\mathcal{N}} \rightarrow (\overline{\mathcal{G}} \otimes \overline{\mathcal{N}})_x \rightarrow 0$$

on \overline{C} we deduce that if $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \overline{\mathcal{N}}(-x)) = 0$ then $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \overline{\mathcal{N}}) = 0$. This implies i).

ii) By the same reason as above it is enough to show that the sheaf $\overline{\mathcal{G}} \otimes \overline{\mathcal{N}}$ is generated by the global sections. Since by $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \overline{\mathcal{N}}(-x)) = 0$ the map

$$H^0(\overline{C}, \overline{\mathcal{G}} \otimes \overline{\mathcal{N}}) \rightarrow H^0(\overline{C}, (\overline{\mathcal{G}} \otimes \overline{\mathcal{N}})_x)$$

is surjective for any $x \in \overline{C}$. Hence, $\overline{\mathcal{G}} \otimes \overline{\mathcal{N}}$ and $\mathcal{G} \otimes \mathcal{N}$ are generated by the global sections for all \mathcal{N} of degree greater than d . \square

Proof of Lemma 7. Any torsion sheaf T is generated by the global sections. Consider the surjective map $\mathcal{O}_C^{\oplus r_0} \rightarrow T$, where $r_0 = \dim H^0(T)$. Denote by U the kernel of this map. Now it is evident that $H^1(\overline{U} \otimes \mathcal{M}) = 0$ for any line bundle \mathcal{M} on \overline{C} with $\deg \mathcal{M} \geq 2g - 1$, because $H^1(\mathcal{M}) = 0$. Applying Lemma 9 we get that $U \otimes \mathcal{L}$ is generated by the global sections. Hence, there is an exact sequence of the form

$$(\mathcal{L}^{-1})^{\oplus r_1} \longrightarrow \mathcal{O}_C^{\oplus r_0} \longrightarrow T \longrightarrow 0.$$

for any torsion sheaf T . \square

Proof of the Main Lemma. If \mathcal{G} is a semi-stable vector bundle on C with $\mu(\mathcal{G}) \geq 2g$ then by Serre duality we have $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{M}) = 0$ for all \mathcal{M} with $\deg \mathcal{M} \geq -1$. Therefore, by Lemma 9 the bundle \mathcal{G} is generated by the global sections.

Now $\mathcal{F}_i \subseteq \mathcal{F}$ as an extension of semi-stable sheaves with $\mu \geq 4g$ is generated by the global sections as well. Consider the short exact sequence

$$0 \longrightarrow U \longrightarrow \mathcal{O}_C^{\oplus r_0} \longrightarrow \mathcal{F}_i \longrightarrow 0,$$

where r_0 is the dimension of $H^0(\mathcal{F}_i)$. Take a line bundle \mathcal{M} on \overline{C} of degree $2g$ and consider the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{U} \otimes \mathcal{M}^{-1} & \longrightarrow & (\mathcal{M}^{-1})^{\oplus r_0} & \longrightarrow & \overline{\mathcal{F}}_i \otimes \mathcal{M}^{-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{U}^{\oplus 2} & \longrightarrow & \mathcal{O}_{\overline{C}}^{\oplus 2r_0} & \longrightarrow & \overline{\mathcal{F}}_i^{\oplus 2} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{U} \otimes \mathcal{M} & \longrightarrow & \mathcal{M}^{\oplus r_0} & \longrightarrow & \overline{\mathcal{F}}_i \otimes \mathcal{M} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Since the sheaf $\overline{\mathcal{F}}_i \otimes \mathcal{M}^{-1}$ is the extension of semi-stable sheaves with $\mu \geq 2g$ we have $H^1(\overline{\mathcal{F}}_i \otimes \mathcal{M}^{-1}) = 0$. Hence, the map $H^0(\overline{\mathcal{F}}_i^{\oplus 2}) \rightarrow H^0(\overline{\mathcal{F}}_i \otimes \mathcal{M})$ is surjective. Further, we know that the map $H^0(\mathcal{O}_{\overline{C}}^{\oplus 2r_0}) \rightarrow H^0(\overline{\mathcal{F}}_i^{\oplus 2})$ is surjective and the map $H^0(\mathcal{O}_{\overline{C}}^{\oplus 2r_0}) \rightarrow H^0(\mathcal{M}^{\oplus r_0})$ is injective. This implies that the map $H^0(\mathcal{M}^{\oplus r_0}) \rightarrow H^0(\overline{\mathcal{F}}_i \otimes \mathcal{M})$ is surjective as well. Hence $H^1(\overline{U} \otimes \mathcal{M}) = 0$. Therefore, by Lemma 9 the bundle $\overline{U} \otimes \mathcal{M}'$ is generated by the global sections for all \mathcal{M}' with $\deg \mathcal{M}' \geq 2g + 1$. In particular, $U \otimes \mathcal{L}$ is generated by the global sections. Thus, we get an exact sequence

$$(\mathcal{L}^{-1})^{\oplus r_1} \xrightarrow{\alpha} \mathcal{O}_C^{\oplus r_0} \longrightarrow \mathcal{F}_i \longrightarrow 0.$$

Sequence b) can be obtained by dualizing of sequence a) applied for the sheaf $\mathcal{F}^* \otimes \mathcal{L}$. \square

Proof of Theorem 6. At first, since the category of coherent sheaves on C has homological dimension one we see that any torsion sheaf T is a direct summand of the complex of the form $(\mathcal{L}^{-1})^{\oplus r_1} \rightarrow \mathcal{O}_C^{\oplus r_0}$. Hence, it belongs to $\langle \mathcal{E} \rangle_2$.

Now consider a vector bundle \mathcal{F} on C with the Harder-Narasimhan filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_n = \mathcal{F}$. As above let us fix $0 \leq i \leq n$ such that $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \geq 4g > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$.

Applying the Main Lemma we obtain the following long exact sequence

$$0 \longrightarrow \text{Ker } \alpha \longrightarrow (\mathcal{L}^{-1})^{\oplus r_1} \xrightarrow{\alpha} \mathcal{O}_C^{\oplus r_0} \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^{\oplus s_0} \xrightarrow{\beta} (\mathcal{L}^2)^{\oplus s_1} \longrightarrow \text{Coker } \beta \longrightarrow 0.$$

Furthermore, it is easy to see that the canonical map $\text{Ext}^1(\mathcal{L}^{\oplus s_0}, \mathcal{O}_C^{\oplus r_0}) \longrightarrow \text{Ext}^1(\mathcal{F}/\mathcal{F}_i, \mathcal{F}_i)$ is surjective. Let us fix $e \in \text{Ext}^1(\mathcal{F}/\mathcal{F}_i, \mathcal{F}_i)$ which defines \mathcal{F} as the extension and choose some its pull back $e' \in \text{Ext}^1(\mathcal{L}^{\oplus s_0}, \mathcal{O}_C^{\oplus r_0})$.

Now let us consider the map

$$(2) \quad \phi : (\mathcal{L}^{-1})^{\oplus r_0} \oplus \mathcal{L}^{\oplus s_0}[-1] \longrightarrow \mathcal{O}_C^{\oplus r_0} \oplus (\mathcal{L}^2)^{\oplus s_1}[-1], \quad \text{where } \phi = \begin{pmatrix} \alpha & e' \\ 0 & \beta \end{pmatrix}$$

and take a cone $C(\phi)$ of ϕ . The cone $C(\phi)$ is isomorphic to a complex that has three nontrivial cohomologies $H^{-1}(C(\phi)) \cong \text{Ker } \alpha$, $H^1(C(\phi)) \cong \text{Coker } \beta$ and, finally, $H^0(C(\phi)) \cong \mathcal{F}$. Thus, \mathcal{F} is a direct summand of $C(\phi)$ and, consequently, it belongs to $\langle \mathcal{E} \rangle_2$. This implies that the whole bounded derived category of coherent sheaves on C coincides with $\langle \mathcal{E} \rangle_2$ and the dimension of $\mathbf{D}^b(\text{coh}(C))$ is equal to 1. \square

Having in view of the given theorem we may assume, that the following conjecture can be true.

Conjecture 10. *Let X be a smooth quasi-projective scheme of dimension n . Then $\dim \mathbf{D}^b(\text{coh}(X)) = n$.*

Remark 11. For a non regular scheme it is evidently not true. For example, the dimension of the bounded derived category of coherent sheaves on the zero-dimension scheme $\text{Spec}(k[x]/x^2)$ equals to 1.

It is also very interesting to understand what the spectrum $\sigma(\mathbf{D}^b(\text{coh}(X)))$ forms. In particular we can ask the following questions

Question 12. *Is the spectrum of the bounded derived category of coherent sheaves on a smooth quasi-projective scheme bounded? Is it bounded for a non smooth scheme?*

Question 13. *Does the spectrum of the bounded derived category of coherent sheaves on a (smooth) quasi-projective scheme form an integer interval?*

Let us try to calculate the dimension spectra of the derived categories of coherent sheaves on some smooth curves.

Proposition 14. *Let C be a smooth affine curve. Then the dimension spectrum $\sigma(\mathbf{D}^b(\text{coh } C))$ coincides with $\{1\}$.*

Proof. If \mathcal{E} is a strong generator then it has a some locally free sheaf \mathcal{F} as a direct summand. Now since C is affine then there is an exact sequence of the form

$$\mathcal{F}^{\oplus r_1} \longrightarrow \mathcal{F}^{\oplus r_0} \longrightarrow \mathcal{G} \longrightarrow 0$$

for any coherent sheaf \mathcal{G} on C . Hence, any coherent sheaf \mathcal{G} belongs to $\langle \mathcal{E} \rangle_2$. Since the global dimension of $\text{coh } C$ is equal to 1 we obtain that $\langle \mathcal{E} \rangle_2 = \mathbf{D}^b(\text{coh } C)$. \square

We can also find the dimension spectrum of the projective line.

Proposition 15. *The dimension spectrum $\sigma(\mathbf{D}^b(\text{coh } \mathbb{P}^1))$ coincides with the set $\{1, 2\}$.*

Proof. Indeed, 1 is the dimension. And, for example, the object $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}$ generate the whole category $\mathbf{D}^b(\text{coh } \mathbb{P}^1)$ for one step. Now, the object $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_p$, where p is a point, is a generator, because $\mathcal{O}(-1)$ belongs to $\langle \mathcal{E} \rangle_2$. This also implies that $\langle \mathcal{E} \rangle_3 \cong \mathbf{D}^b(\text{coh } \mathbb{P}^1)$. On the other hand, $\langle \mathcal{E} \rangle_2 \not\cong \mathbf{D}^b(\text{coh } \mathbb{P}^1)$. To see it we can check that an object \mathcal{O}_q , where $q \neq p$, doesn't belong to $\langle \mathcal{E} \rangle_2$. Indeed, \mathcal{O}_q is completely orthogonal to \mathcal{O}_p and doesn't belong to subcategory generated by \mathcal{O} . Finally, it easy to see that any object \mathcal{E} , which generates the whole category, generates it at least for two steps, i.e. $\langle \mathcal{E} \rangle_3 \cong \mathbf{D}^b(\text{coh } \mathbb{P}^1)$. If \mathcal{E} contains as direct summands two different line bundles than it generates the whole category for one step. If \mathcal{E} has only one line bundle as a direct summand then it also has a torsion sheaf as a direct summand. This implies that $\langle \mathcal{E} \rangle_2$ has another line bundle. Therefore, $\langle \mathcal{E} \rangle_3$ is the whole category. \square

Another simple result says

Proposition 16. *Let C be a smooth projective curve of genus $g > 0$ over a field k . Assume that C has at least two different points over k . Then the dimension spectrum $\sigma(\mathbf{D}^b(\text{coh } C))$ contains $\{1, 2\}$ as a proper subset, i.e. $\{1, 2\}$ is strictly contained in the dimension spectrum.*

Proof. The spectrum contains 1 as the dimension of the category. Let us now take a line bundle \mathcal{L} on C which satisfies the condition as in Theorem 8, i.e. $\deg \mathcal{L} \geq 8g$ and consider the object $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}^2$. It is easy to see that the line bundles \mathcal{L}^{-1} and \mathcal{L} belong to $\langle \mathcal{E} \rangle_2$, because there are exact sequences

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{L}^{\oplus 2} \longrightarrow \mathcal{L}^2 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow (\mathcal{L}^{-1})^{\oplus 2} \longrightarrow \mathcal{O}_C^{\oplus 3} \longrightarrow \mathcal{L}^2 \longrightarrow 0.$$

The proof of Theorem 8 (see the map 2) implies that $\langle \mathcal{E} \rangle_3 \cong \mathbf{D}^b(\text{coh } C)$. On the other hand, the subcategory $\langle \mathcal{E} \rangle_2$ doesn't coincides with the whole $\mathbf{D}^b(\text{coh } C)$. For example, a nontrivial line bundle \mathcal{M} from $\text{Pic}^0 C$ doesn't belong to $\langle \mathcal{E} \rangle_2$, because it is completely orthogonal to the structure sheaf \mathcal{O}_C and, evidently, could not be obtained from the line bundle \mathcal{L}^2 .

Let us take a point $p \in C$ and consider the object $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_p$, where \mathcal{O}_p is the skyscraper in p . This object is a strong generator and we can show that $\langle \mathcal{E} \rangle_3 \neq \mathbf{D}^b(\text{coh } C)$. Take another point $q \neq p$ and consider the skyscraper sheaf \mathcal{O}_q . It is completely orthogonal to \mathcal{O}_p and have only one-dimensional 1-st Ext to \mathcal{O}_C . Hence, if \mathcal{O}_q belongs to $\langle \mathcal{E} \rangle_3$ then it should be a direct summand of an object M which is included in an exact triangle of the form

$$\mathcal{O}_C^{\oplus k} \longrightarrow N \longrightarrow M \longrightarrow \mathcal{O}_C^{\oplus k}[1],$$

where $N \in \langle \mathcal{E} \rangle_2$. Since the 1-st Ext from \mathcal{O}_q to \mathcal{O}_C is one-dimensional we can take $k = 1$. The composition of the map $\mathcal{O}_q \rightarrow M$ with $M \rightarrow \mathcal{O}_C$ should be the nontrivial 1-st Ext from \mathcal{O}_q to \mathcal{O}_C . Now object N is a direct sum of indecomposable objects from $\langle \mathcal{E} \rangle_2$. It is easy to see that we can consider only objects for which there are nontrivial homomorphisms from \mathcal{O}_C and nontrivial homomorphisms to \mathcal{O}_q . All other can be removed from N . Thus N is a direct sum of $\mathcal{O}(p)$ and objects U that are extensions

$$(3) \quad 0 \longrightarrow \mathcal{O}_C^{\oplus r_1} \longrightarrow U \longrightarrow \mathcal{O}_C^{\oplus r_2} \longrightarrow 0.$$

Finally, split embedding $\mathcal{O}_q \rightarrow M$ gives us a nontrivial map from $\mathcal{O}(q)$ to N . But there are no nontrivial maps from $\mathcal{O}(q)$ to $\mathcal{O}(p)$ and to U of the form (3). Therefore, \mathcal{O}_q can not belong to $\langle \mathcal{E} \rangle_3$. \square

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