

# FORMAL COMPLETIONS AND IDEMPOTENT COMPLETIONS OF TRIANGULATED CATEGORIES OF SINGULARITIES

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*Dedicated to the blessed memory of my adviser Vasily Alexeevich Iskovskikh*

ABSTRACT. The main goal of this paper is to prove that the idempotent completions of the triangulated categories of singularities of two schemes are equivalent if the formal completions of these schemes along singularities are isomorphic. We also discuss Thomason theorem on dense subcategories and a relation to the negative K-theory.

## 1. INTRODUCTION

Let  $X$  be a noetherian scheme over a field  $k$ . Denote by  $\mathbf{D}^b(\text{coh } X)$  the bounded derived categories of coherent sheaves on  $X$ . Since  $X$  is noetherian the natural functor from  $\mathbf{D}^b(\text{coh } X)$  to the unbounded derived category of quasi-coherent sheaves  $\mathbf{D}(\text{Qcoh } X)$  is fully faithful and realizes an equivalence of  $\mathbf{D}^b(\text{coh } X)$  with the full subcategory  $\mathbf{D}_{\text{coh}}^{\emptyset, b}(\text{Qcoh } X)$  consisting of all cohomologically bounded complexes with coherent cohomologies ([5], Ex.II, 2.2.2).

Denote by  $\mathfrak{P}erf(X) \subseteq \mathbf{D}(\text{Qcoh } X)$  the full triangulated subcategory of perfect complexes. Recall that a complex on a scheme is said to be perfect if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank.

The derived category  $\mathbf{D}(\text{Qcoh } X)$  admits all coproducts and it is well-known that subcategory of perfect complexes  $\mathfrak{P}erf(X)$  coincides with subcategory of compact objects in  $\mathbf{D}(\text{Qcoh } X)$ , i.e. all objects  $C \in \mathbf{D}(\text{Qcoh } X)$  for which the functor  $\text{Hom}(C, -)$  commutes with arbitrary coproducts. The category  $\mathfrak{P}erf(X)$  can be considered as a full triangulated subcategory of  $\mathbf{D}^b(\text{coh } X)$ .

**Definition 1.1.** *We define a triangulated category of singularities of  $X$ , denoted by  $\mathbf{D}_{\text{Sg}}(X)$ , as the quotient of the triangulated category  $\mathbf{D}^b(\text{coh } X)$  by the full triangulated subcategory  $\mathfrak{P}erf(X)$ .*

We say that  $X$  satisfies a condition (ELF) if  $X$  is separated noetherian of finite Krull dimension and has enough locally free sheaves, i.e. for any coherent sheaf  $\mathcal{F}$  there is an epimorphism  $\mathcal{E} \rightarrow \mathcal{F}$  with a locally free sheaf  $\mathcal{E}$ .

The last condition implies that any perfect complex is also globally (not only locally) quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. For example, any quasi-projective scheme satisfies these conditions. Note that any closed and any open subscheme of  $X$  is also noetherian, finite dimensional and has enough locally free sheaves. It is clear for a closed

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This work is done under partial financial support by RFFI grant 08-01-00297, INTAS grant 05-100008-8118, and NSh grant 9969.2006.1.

subscheme while for an open subscheme  $U$  it follows from the fact that any coherent sheaf on  $U$  can be obtained as the restriction of a coherent sheaf on  $X$  ([7], ex.5.15).

Further in the paper we usually assume that a scheme  $X$  satisfies condition (ELF).

It is known that if a scheme  $X$  is regular then the category  $\mathfrak{P}erf(X)$  coincides with  $\mathbf{D}^b(\text{coh } X)$ . In this case, the triangulated category of singularities is trivial.

Let  $f : X \rightarrow Y$  be a morphism of finite Tor-dimension (for example, a flat morphism or a regular closed embedding). In this case we have an inverse image functor  $\mathbf{L}f^* : \mathbf{D}^b(\text{coh } Y) \rightarrow \mathbf{D}^b(\text{coh } X)$ . It is clear that the functor  $\mathbf{L}f^*$  sends perfect complexes on  $Y$  to perfect complexes on  $X$ . Therefore, the functor  $\mathbf{L}f^*$  induces an exact functor  $\mathbf{L}\bar{f}^* : \mathbf{D}_{\text{Sg}}(Y) \rightarrow \mathbf{D}_{\text{Sg}}(X)$ .

A fundamental property of triangulated categories of singularities is a property of locality in Zarisky topology. It says that for any open embedding  $j : U \hookrightarrow X$ , for which  $\text{Sing}(X) \subset U$ , the functor  $\bar{j}^* : \mathbf{D}_{\text{Sg}}(X) \rightarrow \mathbf{D}_{\text{Sg}}(U)$  is an equivalence of triangulated categories [12].

On the other hand, two analytically isomorphic singularities can have non-equivalent triangulated categories of singularities. Even two different double points given by equations  $f = y^2 - x^2$  and  $g = y^2 - x^2 - x^3$  have non-equivalent categories of singularities. The main reason here is that a triangulated category of singularities is not necessary idempotent complete. This means that not for each projector  $p : C \rightarrow C$ ,  $p^2 = p$  there is a decomposition of the form  $C = \text{Ker } p \oplus \text{Im } p$ .

For any triangulated category  $\mathcal{T}$  we can consider its so called idempotent completion (or Karoubian envelope)  $\overline{\mathcal{T}}$ . This is a category that consists of all kernels of all projectors. It has a natural structure of a triangulated category and the canonical functor  $\mathcal{T} \rightarrow \overline{\mathcal{T}}$  is an exact full embedding [4]. Moreover, the category  $\overline{\mathcal{T}}$  is idempotent complete now, i.e. each idempotent  $p : C \rightarrow C$  in  $\overline{\mathcal{T}}$  arises from a splitting  $\text{Ker } p \oplus \text{Im } p$ . We denote by  $\overline{\mathbf{D}_{\text{Sg}}(X)}$  the idempotent completion of the triangulated categories of singularities.

For any closed subscheme  $Z \subset X$  we can define the formal completion of  $X$  along  $Z$  as a ringed space  $(Z, \varinjlim \mathcal{O}_X/\mathcal{J}^n)$ , where  $\mathcal{J}$  is the ideal sheaf corresponding to  $Z$ . The formal completion actually depends only on the closed subset  $\text{Supp } Z$  and does not depend on a scheme structure on  $Z$ . We denote by  $\mathfrak{X}$  the formal completion of  $X$  along its singularities  $\text{Sing}(X)$ .

The main goal of this paper is to prove that for any two schemes  $X$  and  $X'$  satisfying (ELF), if the formal completions  $\mathfrak{X}$  and  $\mathfrak{X}'$  along singularities are isomorphic, then the idempotent completions of the triangulated categories of singularities  $\overline{\mathbf{D}_{\text{Sg}}(X)}$  and  $\overline{\mathbf{D}_{\text{Sg}}(X')}$  are equivalent (Theorem 2.10). Actually, we show a little bit more. We prove that any object of  $\mathbf{D}_{\text{Sg}}(X)$  is a direct summand of an object from its full subcategory  $\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X)/\mathfrak{P}erf_{\text{Sing}(X)}(X)$ , where  $\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X)$  and  $\mathfrak{P}erf_{\text{Sing}(X)}(X)$  are subcategories of  $\mathbf{D}^b(\text{coh } X)$  and  $\mathfrak{P}erf(X)$  respectively, consisting of complexes with cohomologies supported on  $\text{Sing } X$  (Proposition 2.7).

Thus, to any scheme  $X$  we can attach the category  $\overline{\mathbf{D}_{\text{Sg}}(X)}$  and two subgroups in  $K_0(\overline{\mathbf{D}_{\text{Sg}}(X)})$  that by Thomason theorem [16] (see Theorem 4.1) one-to-one corresponds to the dense subcategories  $\mathbf{D}_{\text{Sg}}(X)$  and  $\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X)/\mathfrak{P}erf_{\text{Sing}(X)}(X)$  respectively. We discuss this correspondence and a relation to the negative K-theory of the category of perfect complexes in the last section.

## 2. COMPLETIONS

Let  $X$  be a noetherian scheme and let  $i : Z \hookrightarrow X$  be a closed subspace. Let  $\text{coh}_Z X \subset \text{coh } X$  be the abelian subcategory of coherent sheaves on  $X$  with support on  $Z$ .

Consider the natural functor from  $\mathbf{D}^b(\text{coh}_Z X)$  to  $\mathbf{D}^b(\text{coh } X)$ . It can be easily shown that this functor is fully faithful and gives an equivalence with the full subcategory  $\mathbf{D}_Z^b(\text{coh } X) \subset \mathbf{D}^b(\text{coh } X)$  consisting of all complexes cohomologies of which are supported on  $Z$ . (In other words, the subcategory  $\mathbf{D}_Z^b(\text{coh } X)$  consists of all complexes restriction of which on the open subset  $U = X \setminus Z$  is acyclic.)

At first, let us consider abelian category of quasi-coherent sheaves  $\text{Qcoh } X$  and its abelian subcategory  $\text{Qcoh}_Z X$  of quasi-coherent sheaves with support on  $Z$  (or  $Z$ -torsion sheaves), i.e. all quasi-coherent sheaves  $\mathcal{F}$  such that  $j^* \mathcal{F} = 0$ , where  $j : U \rightarrow X$  is the open embedding of the complement  $U = X \setminus Z$ . The inclusion functor  $i : \text{Qcoh}_Z X \rightarrow \text{Qcoh } X$  has a right adjoint  $\Gamma_Z$  which associates to each quasi-coherent sheaf  $\mathcal{F}$  its subsheaf of sections with support in  $Z$ . It can be shown that for a quasi-coherent sheaf  $\mathcal{F}$  we have an isomorphism

$$\Gamma_Z(\mathcal{F}) \cong \varinjlim_n \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{F}),$$

where  $\mathcal{J}$  is a some ideal sheaf such that  $Z = \text{Supp}(\mathcal{O}_X/\mathcal{J})$ . The functor  $\Gamma_Z$  has a right-derived functor  $\mathbf{R}\Gamma_Z : \mathbf{D}(\text{Qcoh } X) \rightarrow \mathbf{D}(\text{Qcoh}_Z X)$  via h-injective resolutions [15].

It is known that the canonical functor  $i : \mathbf{D}(\text{Qcoh}_Z X) \rightarrow \mathbf{D}(\text{Qcoh } X)$  is fully faithful and realizes equivalences of  $\mathbf{D}(\text{Qcoh}_Z X)$  with the full subcategory  $\mathbf{D}_Z(\text{Qcoh } X)$  consisting of all complexes cohomologies of which are supported on  $Z$ . It is proved for noetherian schemes for example in [3] (Prop. 5.2.1 and Prop. 5.3.1). (It is also true for quasi-compact and separated  $X$  and proregular embedded  $Z \subset X$  as shown in [2].) To prove this fact it is sufficient to show that for any  $C \in \mathbf{D}_Z(\text{Qcoh } X)$  the natural map  $i\mathbf{R}\Gamma_Z(C) \rightarrow C$  is an isomorphism (see, for example, [3] Lemma 5.2.2). Since the functor  $\mathbf{R}\Gamma_Z$  is bounded for noetherian schemes by usual "way out" argument ([8], §7) it is sufficient to check the isomorphism  $i\mathbf{R}\Gamma_Z(\mathcal{F}) \rightarrow \mathcal{F}$  only for sheaves  $\mathcal{F} \in \text{Qcoh}_Z X$ . That is evident, because  $\mathbf{R}\Gamma_Z \mathcal{F} \cong \Gamma_Z \mathcal{F}$ , when  $\mathcal{F}$  is  $Z$ -torsion.

Thus, for any object  $C \in \mathbf{D}(\text{Qcoh } X)$  there is a distinguished triangle of the form

$$i\mathbf{R}\Gamma C \rightarrow C \rightarrow \mathbf{R}j_* j^* C,$$

which shows that the categories  $\mathbf{D}(\text{Qcoh } U)$  and  $\mathbf{D}(\text{Qcoh}_Z X)$  are equivalent to the quotient categories  $\mathbf{D}(\text{Qcoh } X)/\mathbf{D}_Z(\text{Qcoh } X)$  and  $\mathbf{D}(\text{Qcoh } X)/\mathbf{D}(\text{Qcoh } U)$  respectively.

**Lemma 2.1.** *Let  $X$  be a noetherian scheme and let  $Z$  be a closed subspace. Then the natural functor  $\mathbf{D}^b(\text{coh}_Z X) \rightarrow \mathbf{D}^b(\text{coh } X)$  is fully faithful and gives an equivalence with the full subcategory  $\mathbf{D}_Z^b(\text{coh } X) \subset \mathbf{D}^b(\text{coh } X)$  consisting of all complexes cohomologies of which are supported on  $Z$ .*

**Proof.** We know that the natural functors  $\mathbf{D}^b(\text{coh } X) \hookrightarrow \mathbf{D}(\text{Qcoh } X)$  and  $\mathbf{D}(\text{Qcoh}_Z X) \hookrightarrow \mathbf{D}(\text{Qcoh } X)$  are fully faithful. This implies that the functor  $\mathbf{D}^b(\text{coh}_Z X) \rightarrow \mathbf{D}^b(\text{coh } X)$  is fully

faithful iff the functor  $\mathbf{D}^b(\mathrm{coh}_Z X) \rightarrow \mathbf{D}(\mathrm{Qcoh}_Z X)$  is fully faithful. Denote by  $\phi$  the natural embedding of  $\mathrm{coh}_Z X$  to  $\mathrm{Qcoh}_Z X$ . Since coherent sheaves generate the category  $\mathbf{D}^b(\mathrm{coh}_Z X)$  it is enough to show that for any two coherent sheaves  $\mathcal{F}, \mathcal{G} \in \mathrm{coh}_Z X$  the natural maps  $\mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) \rightarrow \mathrm{Ext}^n(\phi(\mathcal{F}), \phi(\mathcal{G}))$  are isomorphisms. We know that it is evidently true for  $n = 0$ . Now to apply induction, it is sufficient to check that for any  $e \in \mathrm{Ext}^n(\phi(\mathcal{F}), \phi(\mathcal{G}))$ ,  $n \geq 1$ , there is an epimorphism  $\mathcal{F}' \rightarrow \mathcal{F}$  which erases  $e$  ([5], Ex. II, Lemma 2.1.3). Any such element  $e$  can be represented by an exact sequence in  $\mathrm{Qcoh}_Z X$

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{E}_{n-1} \rightarrow \cdots \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0,$$

where  $\mathcal{E}_i$  are quasi-coherent sheaves with support on  $Z$ . The epimorphism  $\mathcal{E}_0 \rightarrow \mathcal{F}$  erases  $e$ . Since any quasi-coherent sheaf on a noetherian scheme is a direct colimit of its coherent subsheaves there is a coherent subsheaf  $\mathcal{F}' \subset \mathcal{E}_0$  that covers  $\mathcal{F}$ . As a subsheaf of  $\mathcal{E}_0$  it erases  $e$  and belongs to  $\mathrm{coh}_Z X$ . Thus, the natural functor  $\mathbf{D}^b(\mathrm{coh}_Z X) \rightarrow \mathbf{D}^b(\mathrm{coh} X)$  is fully faithful and gives an equivalence with the full subcategory  $\mathbf{D}_Z^b(\mathrm{coh} X) \subset \mathbf{D}^b(\mathrm{coh} X)$  consisting of all complexes cohomologies of which are supported on  $Z$ .  $\square$

The restriction functor  $j^*$  sends coherent sheaves to coherent and we get a functor from the quotient category  $\mathbf{D}^b(\mathrm{coh} X)/\mathbf{D}_Z^b(\mathrm{coh} X)$  to the derived category  $\mathbf{D}^b(\mathrm{coh} U)$ . This functor establishes an equivalence between these categories.

**Lemma 2.2.** *Let  $X$  be a noetherian scheme. Then the natural functor  $\mathbf{D}^b(\mathrm{coh} X)/\mathbf{D}_Z^b(\mathrm{coh} X) \rightarrow \mathbf{D}^b(\mathrm{coh} U)$  is an equivalence.*

**Proof.** This fact is known and we omit a proof. There are few different ways to get it.

First, we know this fact for quasi-coherent sheaves and by Lemma 2.5 it is enough to show that any map from a bounded complex of coherent sheaves to an object of  $\mathbf{D}_Z(\mathrm{Qcoh} X)$  admits a factorization through an object from  $\mathbf{D}_Z^b(\mathrm{coh} X)$ .

Second, since  $\mathrm{coh} U$  is the quotient of the abelian category  $\mathrm{coh} X$  by the Serre subcategory  $\mathrm{coh}_Z X$  it is possible to show that the functor  $\mathbf{D}^b(\mathrm{coh} X)/\mathbf{D}_Z^b(\mathrm{coh} X) \rightarrow \mathbf{D}^b(\mathrm{coh} U)$  is surjective on objects and on morphisms. This implies an equivalence as well.  $\square$

**Remark 2.3.** Note that the second way allows us to prove a more general result. For any Serre subcategory  $\mathcal{B}$  of an abelian category  $\mathcal{A}$  the functor  $F : \mathbf{D}^b(\mathcal{A})/\mathbf{D}_{\mathcal{B}}^b(\mathcal{A}) \rightarrow \mathbf{D}^b(\mathcal{A}/\mathcal{B})$  is an equivalence of triangulated categories, where  $\mathbf{D}_{\mathcal{B}}^b(\mathcal{A})$  is the full subcategory in  $\mathbf{D}^b(\mathcal{A})$  consisting of all complexes with cohomologies in  $\mathcal{B}$  (it is known but unpublished [6]).

Denote by  $\mathfrak{P}erf_Z(X)$  the intersection  $\mathfrak{P}erf(X) \cap \mathbf{D}_Z^b(\mathrm{coh} X)$ .

**Lemma 2.4.** *Let  $X$  satisfies (ELF). Then an object  $A \in \mathbf{D}_Z^b(\mathrm{coh} X)$  belongs to  $\mathfrak{P}erf_Z(X)$  iff for any object  $B \in \mathbf{D}_Z^b(\mathrm{coh} X)$  all  $\mathrm{Hom}(A, B[i])$  are trivial except for finite number of  $i \in \mathbb{Z}$ .*

**Proof.** Denote by  $\mathcal{D}_{hf} \subset \mathbf{D}_Z^b(\mathrm{coh} X)$  the full subcategory consisting of all objects  $A$  such that for any object  $B \in \mathbf{D}_Z^b(\mathrm{coh} X)$  the spaces  $\mathrm{Hom}(A, B[i])$  are trivial except for finite number of

$i \in \mathbb{Z}$ . If an object  $A \in \mathfrak{P}erf_Z(X)$  then it is quasi-isomorphic to a bounded complex of vector bundles. Since the cohomologies of any coherent sheaf is bounded by the Krull dimension of the scheme we have that for any vector bundle  $\mathcal{P}$  and any coherent sheaf  $\mathcal{F}$  there is an equality  $\text{Ext}^i(\mathcal{P}, \mathcal{F}) = 0$  when  $i$  is greater than Krull dimension of  $X$ . Therefore,  $A$  belongs to the subcategory  $\mathcal{D}_{hf}$ .

Suppose now that  $A \in \mathcal{D}_{hf}$ . The object  $A$  is a bounded complex of coherent sheaves. Let us take locally free bounded above resolution  $P^\cdot \xrightarrow{\sim} A$  and consider a good truncation  $\tau^{\geq -k} P^\cdot$  for sufficient large  $k \gg 0$  which is clearly isomorphic to  $A$  in  $\mathcal{D}$ .

Since  $A \in \mathcal{D}_{hf}$ , for any closed point  $t : x \hookrightarrow X$  the groups  $\text{Hom}(A, t_* \mathcal{O}_x[i])$  are zero for  $|i| \gg 0$ . This means that for sufficiently large  $k \gg 0$  the truncation  $\tau^{\geq -k} P^\cdot$  is a complex of locally free sheaves at the point  $x$ , and, hence, in some neighborhood of  $x$ . The scheme  $X$  is quasi-compact. This implies that there is a common sufficiently large  $k$  such that the truncation  $\tau^{\geq -k} P^\cdot$  is a complex of locally free sheaves everywhere on  $X$ , i.e.  $A$  is perfect.  $\square$

The natural embedding  $\mathbf{D}_Z^b(\text{coh } X) \hookrightarrow \mathbf{D}^b(\text{coh } X)$  induces a functor between quotient categories

$$\mathbf{D}_Z^b(\text{coh } X) / \mathfrak{P}erf_Z(X) \longrightarrow \mathbf{D}_{\text{Sg}}(X) := \mathbf{D}^b(\text{coh } X) / \mathfrak{P}erf(X).$$

It can be proved that this functor between quotient categories is fully faithful too. To prove it we need the following well-known lemma.

**Lemma 2.5.** ([18, 9]) *Let  $\mathcal{D}$  be a triangulated category and  $\mathcal{D}', \mathcal{N}$  be full triangulated subcategories. Let  $\mathcal{N}' = \mathcal{D}' \cap \mathcal{N}$ . Assume that any morphism  $N \rightarrow X'$  (resp. any morphism  $X' \rightarrow N$ ) with  $N \in \mathcal{N}'$  and  $X' \in \mathcal{D}'$  admits a factorization  $N \rightarrow N' \rightarrow X'$  (resp.  $X' \rightarrow N' \rightarrow N$ ) with  $N' \in \mathcal{N}'$ . Then the natural functor  $\mathcal{D}' / \mathcal{N}' \rightarrow \mathcal{D} / \mathcal{N}$  is fully faithful.*

**Lemma 2.6.** *The functor  $\mathbf{D}_Z^b(\text{coh } X) / \mathfrak{P}erf_Z(X) \rightarrow \mathbf{D}_{\text{Sg}}(X)$  is fully faithful.*

**Proof.** By Lemma 2.5 we should show that any morphism  $P^\cdot \rightarrow C^\cdot$ , where  $P^\cdot$  is a perfect complex and  $C^\cdot$  is an object of  $\mathbf{D}_Z^b(\text{coh } X)$ , can be factorized through an object  $P'^\cdot \in \mathfrak{P}erf_Z(X)$ . Since there is an equivalence  $\mathbf{D}^b(\text{coh}_Z X) \cong \mathbf{D}_Z^b(\text{coh } X)$  we can assume that the object  $C^\cdot$  is a bounded complex of coherent sheaves with support on  $Z$ . This implies that there is a subscheme structure  $i_S : S \hookrightarrow X$  with support on  $Z$  such that  $C^\cdot \cong \mathbf{R}i_{S*} C'^\cdot$ . Consider a vector bundle  $\mathcal{E}$  on  $X$  that covers the ideal sheaf  $\mathcal{I}_S$ . It exists by (ELF) condition. Denote by  $\mathcal{K}^\cdot$  the Koszul complex  $0 \rightarrow \det(\mathcal{E}) \rightarrow \cdots \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X \rightarrow 0$ . We have canonical maps  $P^\cdot \rightarrow P^\cdot \otimes \mathcal{K}^\cdot \rightarrow P^\cdot \otimes \mathcal{O}_S$ . Now any map from  $P^\cdot$  to  $C^\cdot \cong \mathbf{R}i_{S*} C'^\cdot$  is factorized through  $P^\cdot \otimes \mathcal{O}_S$  and, hence, through  $P^\cdot \otimes \mathcal{K}^\cdot$ . But the object  $P^\cdot \otimes \mathcal{K}^\cdot$  is perfect as a tensor product of two perfect complexes and has cohomologies with supports on  $Z$ . Thus, it belongs to  $\mathfrak{P}erf_Z(X)$ .  $\square$

Now consider the case when  $Z$  is exactly the subset of singularities of  $X$ , i.e.  $Z = \text{Sing}(X)$ .

**Proposition 2.7.** *Any object of  $\mathbf{D}_{\text{Sg}}(X)$  is a direct summand of an object from its full subcategory  $\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X) / \mathfrak{P}erf_{\text{Sing}(X)}(X)$ . In particular, idempotent completions of these categories are equivalent.*

**Proof.** Since any object of the category  $\mathbf{D}_{\text{Sg}}(X)$  is represented by a coherent sheaf up to shift ([12], Lemma 1.11) it is sufficient to consider the coherent sheaf  $\mathcal{F}$ . Let us take the locally free resolution  $P \rightarrow \mathcal{F}$  and consider the brutal truncation  $\sigma^{\geq -n}P$  for sufficiently large  $n > \dim X$ . Denote by  $\mathcal{G}$  the  $(-n)$ -th cohomology of  $\sigma^{\geq -n}P$ . Let  $\alpha : \mathcal{F} \rightarrow \mathcal{G}[n+1]$  be the corresponding map in  $\mathbf{D}^b(\text{coh } X)$ . Its image in the category  $\mathbf{D}_{\text{Sg}}(X)$  is an isomorphism. On the other hand, consider the functor  $j^* : \mathbf{D}^b(\text{coh } X) \rightarrow \mathbf{D}^b(\text{coh } U)$ , where  $U = X \setminus \text{Sing}(X)$ . Since  $U$  is smooth and  $n > \dim U$  the image  $j^*(\alpha)$  is zero. But the category  $\mathbf{D}^b(\text{coh } U)$  is the quotient of the category of  $\mathbf{D}^b(\text{coh } X)$  by the subcategory  $\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X)$ . Hence the morphism  $\alpha$  is factorized through an object  $A$  of  $\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X)$ . Therefore, in the quotient category  $\mathbf{D}_{\text{Sg}}(X)$  the object  $\mathcal{F}$  is a direct summand of the image of the object  $A$  in  $\mathbf{D}_{\text{Sg}}(X)$ .  $\square$

For any scheme  $X$  we denote by  $\mathfrak{X}_Z$  the formal completion of  $X$  along a closed subspace  $Z$  and denote by  $\kappa : \mathfrak{X}_Z \rightarrow X$  the canonical morphism. Let  $\mathcal{J}$  be an ideal sheaf such that  $\text{Supp}(\mathcal{O}_X/\mathcal{J}) = Z$  and let  $\mathfrak{J}$  be a corresponding ideal of definition of the formal noetherian scheme  $\mathfrak{X}_Z$ . We set

$$\Gamma_{\mathfrak{X}}(\mathfrak{F}) := \varinjlim_n \underline{\text{Hom}}_{\mathcal{O}_{\mathfrak{X}_Z}}(\mathcal{O}_{\mathfrak{X}_Z}/\mathfrak{J}^n, \mathfrak{F}),$$

for any quasi-coherent sheaf  $\mathfrak{F}$  on  $\mathfrak{X}_Z$ . This functor depends only on  $\mathcal{O}_{\mathfrak{X}_Z}$  and does not depend on the ideal  $\mathfrak{J}$ . We say that  $\mathfrak{F} \in \text{Qcoh } \mathfrak{X}_Z$  is a torsion sheaf if  $\Gamma_{\mathfrak{X}}(\mathfrak{F}) = \mathfrak{F}$ . We denote by  $\text{coh}_t \mathfrak{X}_Z$  (resp.  $\text{Qcoh}_t \mathfrak{X}_Z$ ) the full subcategory of  $\text{Qcoh } \mathfrak{X}_Z$  whose objects are the (quasi)-coherent torsion sheaves. It is easy to see that under the inverse image functor  $\kappa^*$  a  $Z$ -torsion (quasi)-coherent sheaf on  $X$  goes to a torsion (quasi)-coherent sheaf on  $\mathfrak{X}_Z$ . Indeed, applying  $\varinjlim$  to the isomorphisms

$$\kappa^* \underline{\text{Hom}}_X(\mathcal{O}_X/\mathcal{J}^n, \mathcal{F}) \xrightarrow{\sim} \underline{\text{Hom}}_{\mathfrak{X}_Z}(\mathcal{O}_{\mathfrak{X}_Z}/\mathfrak{J}^n, \kappa^* \mathcal{F})$$

we get a natural isomorphism  $\kappa^* \Gamma_Z \cong \Gamma_{\mathfrak{X}} \kappa^*$ . Hence, we obtain that  $\kappa^*(\text{Qcoh}_Z X) \subset \text{Qcoh}_t \mathfrak{X}_Z$  and  $\kappa^*(\text{coh}_Z X) \subset \text{coh}_t \mathfrak{X}_Z$ .

Now, if  $\mathcal{F}$  is a  $Z$ -torsion coherent sheaf on  $X$  then there is an integer  $n$  such that  $\mathcal{F}$  comes from  $Z_n = \mathbf{Spec} \mathcal{O}_X/\mathcal{J}^n = (\mathfrak{X}_Z, \mathcal{O}_{\mathfrak{X}_Z}/\mathfrak{J}^n)$  under the closed inclusion  $i_n : Z_n \rightarrow X$ , i.e.  $\mathcal{F} = i_{n*} \mathcal{F}'$  for some coherent sheaf  $\mathcal{F}' \in \text{coh } Z_n$ . Consider the cartesian diagram

$$\begin{array}{ccc} Z_n & \xrightarrow{i_n} & \mathfrak{X}_Z \\ \parallel & & \downarrow \kappa \\ Z_n & \xrightarrow{i_n} & X \end{array}$$

We have a sequence of isomorphisms  $\kappa_* \kappa^* \mathcal{F} \cong \kappa_* \kappa^* i_{n*} \mathcal{F}' \cong \kappa_* i_{n*} \mathcal{F}' \cong i_{n*} \mathcal{F}' \cong \mathcal{F}$ . If now  $\mathfrak{F}$  is a torsion sheaf on  $\mathfrak{X}_Z$  then again there is an integer  $n$  such that  $\mathfrak{F} \cong i_{n*} \mathcal{F}'$  and  $\kappa^* \kappa_* \mathfrak{F} \cong \kappa^* \kappa_* i_{n*} \mathcal{F}' \cong \kappa^* i_{n*} \mathcal{F}' \cong i_{n*} \mathcal{F}' \cong \mathfrak{F}$ . Thus, we obtain that the functors  $\kappa^*$  and  $\kappa_*$  induce inverse equivalences between the abelian categories  $\text{coh}_Z X$  and  $\text{coh}_t \mathfrak{X}_Z$ . It is also can be shown that the functor  $\kappa_*$  sends quasi-coherent torsion sheaf to quasi-coherent  $Z$ -torsion sheaves, because  $\kappa_*$  commutes with colimits (see [3], Prop. 5.1.1, 5.1.2). Thus, we get the following proposition.

**Proposition 2.8.** ([3]) *Let  $X$  be a noetherian scheme and  $\kappa : \mathfrak{X}_Z \rightarrow X$  be a formal completion of  $X$  along a closed subspace  $Z$ . Then the functors  $\kappa^*$  and  $\kappa_*$  induce inverse equivalences between the categories  $\text{coh}_Z X$  and  $\text{coh}_t \mathfrak{X}_Z$ , and between the categories  $\text{Qcoh}_Z X$  and  $\text{Qcoh}_t \mathfrak{X}_Z$ .*

**Corollary 2.9.** *Let  $X$  and  $X'$  be two schemes satisfying (ELF). Assume that the formal schemes  $\mathfrak{X}_Z$  and  $\mathfrak{X}'_{Z'}$  are isomorphic. Then the derived categories  $\mathbf{D}_Z^b(\text{coh } X)$  and  $\mathbf{D}_{Z'}^b(\text{coh } X')$  (resp.  $\mathbf{D}_Z^b(\text{Qcoh } X)$  and  $\mathbf{D}_{Z'}^b(\text{Qcoh } X')$ ) are equivalent.*

**Proof.** By Lemma 2.1 there is an equivalence  $\mathbf{D}_Z^b(\text{coh } X) \cong \mathbf{D}^b(\text{coh}_Z X)$  (resp.  $\mathbf{D}_Z^b(\text{Qcoh } X) \cong \mathbf{D}^b(\text{Qcoh}_Z X)$ ) by and by Proposition 2.8 we have  $\text{coh}_Z X \cong \text{coh}_t \mathfrak{X}_Z$  (resp.  $\text{Qcoh}_Z X \cong \text{Qcoh}_t \mathfrak{X}_Z$ ). Since  $\mathfrak{X}_Z \cong \mathfrak{X}'_{Z'}$ , we obtain that  $\text{coh}_Z X \cong \text{coh}_{Z'} X'$  (resp.  $\text{Qcoh}_Z X \cong \text{Qcoh}_{Z'} X'$ ). Therefore, the derived categories are equivalent as well.  $\square$

**Theorem 2.10.** *Let  $X$  and  $X'$  be two schemes satisfying (ELF). Assume that the formal completions  $\mathfrak{X}$  and  $\mathfrak{X}'$  along singularities are isomorphic. Then the idempotent completions of the triangulated categories of singularities  $\overline{\mathbf{D}}_{\text{Sg}}(X)$  and  $\overline{\mathbf{D}}_{\text{Sg}}(X')$  are equivalent.*

**Proof.** By Corollary 2.9 there is an equivalence between  $\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X)$  and  $\mathbf{D}_{\text{Sing}(X')}^b(\text{coh } X')$ . By Lemma 2.4 the subcategories  $\mathfrak{P}\text{erf}(X)$  and  $\mathfrak{P}\text{erf}(X')$  are also equivalent, because they can be defined in the internal terms of the bounded derived categories of coherent sheaves with support on  $Z$  and  $Z'$ . Hence, there is an equivalence between quotient categories

$$\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X)/\mathfrak{P}\text{erf}_{\text{Sing}(X)}(X) \xrightarrow{\sim} \mathbf{D}_{\text{Sing}(X')}^b(\text{coh } X')/\mathfrak{P}\text{erf}_{\text{Sing}(X')}(X'),$$

It induces an equivalence between their idempotent completions, which by Proposition 2.7 coincide with the idempotent completions of the triangulated categories of singularities.  $\square$

### 3. LOCALIZATION IN NISNEVICH TOPOLOGY AND ISOMORPHISMS INFINITELY NEAR SINGULARITIES

Let  $X$  be a noetherian scheme and  $i : Z \hookrightarrow X$  be a closed subscheme. Consider the pair  $(Z, X)$ . Let  $f : X' \rightarrow X$  be a map of schemes.

**Definition 3.1.** *We say that  $f$  is an isomorphism infinitely near  $Z$  if it is flat over  $Z$  and the fiber product  $Z' = Z \times_X X'$  is isomorphic to  $Z$ .*

It can be proved that this condition does not depend on a choice of a closed subscheme with the underline subspace  $\text{Supp } Z$  ([17], Lemma 2.6.2.2). In particular, if it holds for  $Z = \mathbf{Spec } \mathcal{O}_X/\mathcal{J}$  it also holds for infinitesimal thickenings  $Z_n := \mathbf{Spec } \mathcal{O}_X/\mathcal{J}^n$ . Thus, we may say that  $f$  is an isomorphism infinitely near the closed subspace  $\text{Supp } Z$ .

This implies that any morphism  $f : X' \rightarrow X$ , which is an isomorphism infinitely near  $Z$ , induces an isomorphism between the formal completions  $\mathfrak{f} : \mathfrak{X}'_{Z'} \xrightarrow{\sim} \mathfrak{X}_Z$ . Hence, by Corollary 2.9 we obtain that the derived categories  $\mathbf{D}_Z^b(\text{coh } X)$  and  $\mathbf{D}_{Z'}^b(\text{coh } X')$  are equivalent for any morphism  $f : X' \rightarrow X$  that is an isomorphism infinitely near  $Z$  (see [17], Th. 2.6.3).

Important examples of such morphisms are Nisnevich neighborhoods of  $Z$  in  $X$ .

**Definition 3.2.** An  $X$ -scheme  $\pi : Y \rightarrow X$  is called a Nisnevich neighborhood of  $Z$  in  $X$  if the morphism  $\pi$  is etale and the fiber product  $Z \times_X Y$  is isomorphic to  $Z$ .

**Proposition 3.3.** Let a scheme  $X$  satisfy (ELF) and let  $Z \subset X$  be a closed subscheme. Then for any morphism  $f : X' \rightarrow X$ , that is an isomorphism infinitely near of  $Z$ , the functors  $f^* : \mathbf{D}_Z(\mathrm{Qcoh} X) \rightarrow \mathbf{D}_Z(\mathrm{Qcoh} X')$  and  $f^* : \mathbf{D}_Z^b(\mathrm{coh} X) \rightarrow \mathbf{D}_Z^b(\mathrm{coh} X')$  are equivalences.

**Proof.** The morphism  $f : X' \rightarrow X$  induces an isomorphism between the formal completions  $f : \mathfrak{X}'_{Z'} \xrightarrow{\sim} \mathfrak{X}_Z$  ([17], Lemma 2.6.2.2). Hence, by Corollary 2.9 we obtain that the derived categories  $\mathbf{D}_Z^b(\mathrm{coh} X)$  and  $\mathbf{D}_Z^b(\mathrm{coh} X')$  (resp.  $\mathbf{D}_Z(\mathrm{Qcoh} X)$  and  $\mathbf{D}_Z(\mathrm{Qcoh} X')$ ) are equivalent.  $\square$

**Proposition 3.4.** Let schemes  $X$  and  $X'$  satisfy (ELF) and let  $f : X' \rightarrow X$  be a morphism that is an isomorphism infinitely near  $Z$ . Suppose the complements  $X \setminus Z$  and  $X' \setminus Z$  are smooth. Then the functor  $\bar{f}^* : \mathbf{D}_{\mathrm{Sg}}(X) \rightarrow \mathbf{D}_{\mathrm{Sg}}(X')$  is fully faithful and, moreover, any object  $B \in \mathbf{D}_{\mathrm{Sg}}(X')$  is a direct summand of a some object of the form  $\bar{f}^* A$ .

**Proof.** By assumption  $\mathrm{Sing}(X) \cong \mathrm{Sing}(X') \subseteq Z$ , hence,  $f$  is an isomorphism infinitely near  $\mathrm{Sing}(X)$ . By Proposition 3.3 and Lemma 2.4 we obtain an equivalence

$$\mathbf{D}_{\mathrm{Sing}(X)}^b(\mathrm{coh} X) / \mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X) \xrightarrow{\sim} \mathbf{D}_{\mathrm{Sing}(X')}^b(\mathrm{coh} X') / \mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X')}(X').$$

By Proposition 2.7 their idempotent completions coincide with idempotent completions of triangulated categories of singularities. Hence, the natural functor  $\bar{f}^*$  is fully faithful and any object  $\mathbf{D}_{\mathrm{Sg}}(X')$  is a direct summand of an object from  $\mathbf{D}_{\mathrm{Sg}}(X)$ .  $\square$

**Corollary 3.5.** Let a scheme  $X$  satisfy (ELF) and the complement  $X \setminus Z$  is smooth. Then for any Nisnevich neighborhood  $\pi : Y \rightarrow X$  of  $Z$  the functor  $\bar{\pi}^* : \mathbf{D}_{\mathrm{Sg}}(X) \rightarrow \mathbf{D}_{\mathrm{Sg}}(Y)$  is fully faithful and, moreover, any object  $B \in \mathbf{D}_{\mathrm{Sg}}(Y)$  is a direct summand of a some object of the form  $\bar{\pi}^* A$ .

**Remark 3.6.** Let  $(A, p)$  be a pair consisting of a commutative  $k$ -algebra of finite type  $A$  and a prime ideal  $p$ . Consider the henselization  $(A_h, p_h)$  of this pair. By definition,  $A_h = \varinjlim B$ ,  $p_h = pA_h$ , where the limit is taking by the category of all Nisnevich neighborhoods  $\mathbf{Spec} B \rightarrow \mathbf{Spec} A$  of  $\mathbf{Spec} A/p$  in  $\mathbf{Spec} A$ . In particular, we have  $A/p \xrightarrow{\sim} A_h/p_h$ . Let  $\hat{A}$  be the  $p$ -adic completion of  $A$ . By one of application of Artin Approximation (Theorem 3.10, [1]) for any finitely generated  $\hat{A}$ -module  $\bar{M}$ , which is locally free on  $\mathbf{Spec} \hat{A}$  outside  $V(\hat{p})$ , there is an  $A_h$ -module  $M$  such that  $\hat{M} \cong \bar{M}$ . Assume now that  $\mathbf{Spec} A$  is regular outside  $V(p)$ . This implies that the natural functor from  $\mathbf{D}_{\mathrm{Sg}}(\mathbf{Spec} A_h)$  to  $\mathbf{D}_{\mathrm{Sg}}(\mathbf{Spec} \hat{A})$  is an equivalence, because any object of a triangulated category of singularities can be represented by a coherent sheaf which is locally free on the complement to the singularities. If moreover,  $K_{-1}(A_h) = 0$  then the triangulated category  $\mathbf{D}_{\mathrm{Sg}}(\mathbf{Spec} A_h)$  is idempotent complete (see next section), i.e. it coincides with  $\overline{\mathbf{D}_{\mathrm{Sg}}(\mathbf{Spec} A)}$ . For example, it is true when  $A$  is a normal local ring of dimension two of essentially finite type [19].



4. THOMASON THEOREM AND GROUPS  $K_{-1}$ 

A full triangulated subcategory  $\mathcal{N}$  of a triangulated category  $\mathcal{T}$  is called **dense** in  $\mathcal{T}$  if each object of  $\mathcal{T}$  is a direct summand of an object isomorphic to an object in  $\mathcal{N}$ . There is a not so well-known but amazing theorem of R. Thomason which allows us to describe all strictly full dense subcategories in a triangulated category.

**Theorem 4.1.** (R. Thomason, [16]) *Let  $\mathcal{T}$  be an essentially small triangulated category. Then there is a one-to-one correspondence between the strictly full dense triangulated subcategories  $\mathcal{N}$  in  $\mathcal{T}$  and the subgroups  $H$  of the Grothendieck group  $K_0(\mathcal{T})$ .*

*To  $\mathcal{N}$  corresponds the subgroup which is the image of  $K_0(\mathcal{N})$  in  $K_0(\mathcal{T})$ . To  $H$  corresponds the full subcategory  $\mathcal{N}_H$  whose objects are those  $N$  in  $\mathcal{T}$  such that  $[N] \in H \subset K_0(\mathcal{T})$ .*

**Remark 4.2.** Recall that a full triangulated subcategory  $\mathcal{N}$  of  $\mathcal{T}$  is called **strictly full** if it contains every object of  $\mathcal{T}$  that is isomorphic to an object of  $\mathcal{N}$ .

Thus, to any scheme  $X$  we can attach the triangulated category  $\overline{\mathbf{D}}_{\text{Sg}}(X)$  and two subgroups in the Grothendieck group  $K_0(\overline{\mathbf{D}}_{\text{Sg}}(X))$  which are related to the natural dense subcategories  $\mathbf{D}_{\text{Sg}}(X)$  and  $\mathbf{D}_{\text{Sing}(X)}^b(\text{coh } X)/\mathfrak{P}\text{erf}_{\text{Sing}(X)}(X)$  and which by Thomason's theorem uniquely determine them.

The sequence of triangulated categories

$$(1) \quad \mathfrak{P}\text{erf}(X) \longrightarrow \mathbf{D}^b(\text{coh } X) \longrightarrow \overline{\mathbf{D}}_{\text{Sg}}(X)$$

is exact in Definition 1.1 of [14], i.e. the first functor is a full embedding and the quotient of this map is dense subcategory in the third category.

Following Amnon Neeman [11] this exact sequence can be considered as an exact sequence of triangulated categories of compact objects coming from a localizing sequence of compactly generated triangulated categories. As we know the category of perfect complexes  $\mathfrak{P}\text{erf}(X)$  is the category of compact objects in  $\mathbf{D}(\text{Qcoh } X)$ . It is proved by H. Krause [10] that the category  $\mathbf{D}^b(\text{coh } X)$  can be considered as the category of compact object in the homotopy category of injective quasi-coherent sheaves  $\mathbf{H}(\text{Inj } X)$  for a noetherian scheme  $X$ . On the other hand, the derived category  $\mathbf{D}(\text{Qcoh } X)$  is equivalent to the full subcategory  $\mathbf{H}_{\text{inj}}(\text{Qcoh } X) \subset \mathbf{H}(\text{Inj } X)$  of h-injective complexes, i.e. such complexes  $I$  that  $\text{Hom}_{\mathbf{H}(\text{Qcoh } X)}(A, I) = 0$  for all acyclic complexes  $A$  from homotopy category  $\mathbf{H}(\text{Qcoh } X)$  (see [15]). Since  $X$  is noetherian the category  $\mathbf{H}_{\text{inj}}(\text{Qcoh } X)$  closed with respect to formation of coproducts and, furthermore, the inclusion functor  $\mathbf{H}_{\text{inj}}(\text{Qcoh } X) \hookrightarrow \mathbf{H}(\text{Inj } X)$  respects coproducts. This means that  $\mathbf{H}_{\text{inj}}(\text{Qcoh } X)$  is localizing subcategory of  $\mathbf{H}(\text{Inj } X)$  and we have a localizing sequence

$$\mathbf{H}_{\text{inj}}(\text{Qcoh } X) \longrightarrow \mathbf{H}(\text{Inj } X) \xrightarrow{Q} \mathbf{H}(\text{Inj } X)/\mathbf{H}_{\text{inj}}(\text{Qcoh } X).$$

Moreover, the quotient functor  $Q$  has a right adjoint, which is called Bousfield localizing functor. It identifies the quotient category  $\mathbf{H}(\text{Inj } X)/\mathbf{H}_{\text{inj}}(\text{Qcoh } X)$  with the triangulated category of all acyclic complexes of injective objects  $\text{Inj } X \subset \text{Qcoh } X$ . The latter category is called stable derived

category and will be denoted by  $\mathbf{S}(\mathrm{Qcoh} X)$ . By Theorem 2.1 of [11] the idempotent completion  $\overline{\mathbf{D}_{\mathrm{Sg}}(X)}$  is equivalent to the category of all compact objects in the stable derived category  $\mathbf{S}(\mathrm{Qcoh} X)$  (for more details see [10]).

By Theorem 11.10 of [14] the sequence (1) induces a long exact sequence for K-groups

$$K_0(\mathfrak{P}\mathrm{erf}(X)) \longrightarrow K_0(\mathbf{D}^b(\mathrm{coh} X)) \longrightarrow K_0(\overline{\mathbf{D}_{\mathrm{Sg}}(X)}) \longrightarrow K_{-1}(\mathfrak{P}\mathrm{erf}(X)) \longrightarrow 0.$$

Here we used Theorem 9.1 from [14] asserting that  $K_{-1}$  for a small abelian category is trivial. Therefore, we obtain a short exact sequence

$$0 \longrightarrow K_0(\mathbf{D}_{\mathrm{Sg}}(X)) \longrightarrow K_0(\overline{\mathbf{D}_{\mathrm{Sg}}(X)}) \longrightarrow K_{-1}(\mathfrak{P}\mathrm{erf}(X)) \longrightarrow 0,$$

which shows that  $K_{-1}(\mathfrak{P}\mathrm{erf}(X))$  is a measure of the difference between  $\mathbf{D}_{\mathrm{Sg}}(X)$  and its idempotent completion  $\overline{\mathbf{D}_{\mathrm{Sg}}(X)}$ . By the same reason, we have another short exact sequence

$$0 \longrightarrow K_0(\mathbf{D}_{\mathrm{Sing}(X)}^b(\mathrm{coh} X)/\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X)) \longrightarrow K_0(\overline{\mathbf{D}_{\mathrm{Sg}}(X)}) \longrightarrow K_{-1}(\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X)) \longrightarrow 0.$$

Now a long exact sequence for  $U = X \setminus \mathrm{Sing}(X)$

$$K_0(\mathfrak{P}\mathrm{erf}(X)) \rightarrow K_0(\mathfrak{P}\mathrm{erf}(U)) \rightarrow K_{-1}(\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X)) \rightarrow K_{-1}(\mathfrak{P}\mathrm{erf}(X)) \rightarrow 0,$$

which follows from the Thomason's Localization Theorem 7.4 [17], shows a difference between  $K_{-1}(\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X))$  and  $K_{-1}(\mathfrak{P}\mathrm{erf}(X))$ .

The negative K-groups (which is due to Bass) are defined from the following exact sequences

$$\begin{aligned} 0 \rightarrow K_i(\mathfrak{P}\mathrm{erf}(X)) \rightarrow K_i(\mathfrak{P}\mathrm{erf}(X[t])) \oplus K_i(\mathfrak{P}\mathrm{erf}(X[t^{-1}])) \rightarrow K_i(\mathfrak{P}\mathrm{erf}(X[t, t^{-1}])) \rightarrow \\ \rightarrow K_{i-1}(\mathfrak{P}\mathrm{erf}(X)) \rightarrow 0. \end{aligned}$$

In particular, the group  $K_{-1}(\mathfrak{P}\mathrm{erf}(X))$  is isomorphic to the cokernel of the canonical map  $K_0(\mathfrak{P}\mathrm{erf}(X[t])) \oplus K_0(\mathfrak{P}\mathrm{erf}(X[t^{-1}])) \rightarrow K_0(\mathfrak{P}\mathrm{erf}(X[t, t^{-1}]))$ .

By Theorem 2.10 we know that for any two schemes  $X$  and  $X'$ , the formal completions of which along singularities are isomorphic, we have

$$\overline{\mathbf{D}_{\mathrm{Sg}}(X)} \cong \overline{\mathbf{D}_{\mathrm{Sg}}(X')} \quad \text{and} \quad \mathbf{D}_{\mathrm{Sing}(X)}^b(\mathrm{coh} X)/\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X) \cong \mathbf{D}_{\mathrm{Sing}(X')}^b(\mathrm{coh} X')/\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X')}(X').$$

On the other hand, in this case the triangulated categories of singularities  $\mathbf{D}_{\mathrm{Sg}}(X)$  and  $\mathbf{D}_{\mathrm{Sg}}(X')$  are not necessary equivalent as we know.

There is also another type of relations between schemes which give equivalences for triangulated categories of singularities but under which the quotient categories  $\mathbf{D}_{\mathrm{Sing}(X)}^b(\mathrm{coh} X)/\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X)$  are not necessary equivalent. It is described in [13].

Let  $S$  be a noetherian regular scheme. Let  $\mathcal{E}$  be a vector bundle on  $S$  of rank  $r$  and let  $s \in H^0(S, \mathcal{E})$  be a section. Denote by  $X \subset S$  the zero subscheme of  $s$ . Assume that the section  $s$  is regular, i.e. the codimension of the subscheme  $X$  in  $S$  coincides with the rank  $r$ .

Consider the projective bundles  $S' = \mathbb{P}(\mathcal{E}^\vee)$  and  $T = \mathbb{P}(\mathcal{E}^\vee|_X)$ , where  $\mathcal{E}^\vee$  is the dual bundle.

The section  $s$  induces a section  $s' \in H^0(S', \mathcal{O}_{\mathcal{E}}(1))$  of the Grothendieck line bundle  $\mathcal{O}_{\mathcal{E}}(1)$  on  $S'$ . Denote by  $Y$  the divisor on  $S'$  defined by the section  $s'$ . The natural closed embedding of  $T$  into  $S'$  goes through  $Y$ . All schemes defined above can be included in the following commutative diagram.

$$\begin{array}{ccccc} T & \xrightarrow{i} & Y & \xrightarrow{u} & S' \\ p \downarrow & & & \searrow \pi & \downarrow q \\ X & \xrightarrow{j} & & & S \end{array}$$

Consider the composition functor  $\mathbf{R}i_*p^* : \mathbf{D}^b(\mathrm{coh} X) \rightarrow \mathbf{D}^b(\mathrm{coh} Y)$  and denote it by  $\Phi_T$ .

**Theorem 4.3.** ([13]) *Let schemes  $X, Y$ , and  $T$  be as above. Then the functor*

$$\Phi_T : \mathbf{D}^b(\mathrm{coh}(X)) \longrightarrow \mathbf{D}^b(\mathrm{coh}(Y))$$

*defined by the formula  $\Phi_T(\cdot) = \mathbf{R}i_*p^*(\cdot)$  induces a functor*

$$\overline{\Phi}_T : \mathbf{D}_{\mathrm{Sg}}(X) \longrightarrow \mathbf{D}_{\mathrm{Sg}}(Y),$$

*which is an equivalence of triangulated categories.*

The functor  $\Phi_T = \mathbf{R}i_*p^*$  has a right adjoint functor which we denote by  $\Phi_{T*}$ . It can be represented as a composition  $\mathbf{R}p_*i^\flat$ , where  $i^\flat$  is right adjoint to  $\mathbf{R}i_*$ . Functor  $i^\flat$  has the form  $\mathbf{L}i^*(\cdot) \otimes \omega_{T/Y}[-r+1]$ , where  $\omega_{T/Y} \cong \Lambda^{r-1}\mathcal{N}_{T/Y}$  is the relative dualizing sheaf.

It is easy to see that all singularities of  $Y$  are concentrated over the singularities of  $X$ , hence the functor  $\Phi_{T*} = \mathbf{R}p_*i^\flat$  sends the subcategory  $\mathbf{D}_{\mathrm{Sing}(Y)}^b(\mathrm{coh} Y)$  to the subcategory  $\mathbf{D}_{\mathrm{Sing}(X)}^b(\mathrm{coh} X)$ . Therefore, we obtain the following corollary.

**Corollary 4.4.** *The functor  $\overline{\Phi}_{T*}$ , which realizes an equivalence between the triangulated categories of singularities of  $Y$  and  $X$ , gives also a functor*

$$\mathbf{D}_{\mathrm{Sing}(Y)}^b(\mathrm{coh} Y)/\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(Y)}(Y) \longrightarrow \mathbf{D}_{\mathrm{Sing}(X)}^b(\mathrm{coh} X)/\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X),$$

*and this functor is fully faithful.*

Note that the functor  $\overline{\Phi}_{T*} : \mathbf{D}_{\mathrm{Sing}(Y)}^b(\mathrm{coh} Y)/\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(Y)}(Y) \rightarrow \mathbf{D}_{\mathrm{Sing}(X)}^b(\mathrm{coh} X)/\mathfrak{P}\mathrm{erf}_{\mathrm{Sing}(X)}(X)$  is not an equivalence in general.

#### ACKNOWLEDGMENTS

I am grateful to Denis Auroux, Ludmil Katzarkov, Anton Kapustin, János Kollár, Amnon Neman, Tony Pantev, Leo Alonso Tarrío, and Ana Jeremías López for very useful discussions.

And, finally, I want to thank my PhD adviser Vasily Alexeevich Iskovskikh who always supported me during all my mathematical life and who passed away on January 4-th, 2009.

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