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PROJECTIVE BUNDLES, MONOIDAL TRANSFORMATIONS, AND DERIVED CATEGORIES OF COHERENT SHEAVES

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ABSTRACT. This paper studies derived categories of coherent sheaves on varieties that are obtained by projectivization of vector bundles and by monoidal transformations. Conditions for the existence of complete exceptional sets in such categories are derived; they give new examples of varieties on which exceptional sets exist.

INTRODUCTION

The aim of this paper is to study bounded derived categories of coherent sheaves on projective varieties of a special form. The apparatus that we use was developed in papers by Beĭlinson, Gorodentsev, Kapranov and Bondal. In [1] Beĭlinson described the derived category of coherent sheaves on the projective variety $\mathbf{P}^n$. Gorodentsev and Rudakov [6] introduced the concept of exceptional set and constructed a series of exceptional bundles on $\mathbf{P}^n$, which are obtained by successive modifications (for definitions see §1 of this paper). Rudakov described all the exceptional bundles on a two-dimensional quadric [11]. In a series of papers [7]–[9], Kapranov described the derived categories of coherent sheaves on quadrics, Grassmannians, and flag varieties. Bondal [2] established a relationship with representations of finite-dimensional associative algebras and studied the question of when the exceptional set generates the derived category. In addition, Bondal and Kapranov [3] developed a set of tools needed for working in triangulated categories.

The present paper, in the context of the work mentioned above, is devoted to the description of derived categories of coherent sheaves on varieties that are obtained by projectivizing vector bundles and by monoidal transformations. We give sufficient conditions for the existence of exceptional sets on such varieties, thereby essentially extending the class of varieties on which exceptional sets exist.

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§1. FUNDAMENTAL CONCEPTS

All of our varieties will be smooth and projective, defined over the field $\mathbf{C}$. For convenience, we shall denote by $\mathscr{D}(X)$ the bounded category of coherent sheaves on X, which is usually denoted $D^b_{\text{coh}}(\text{Sh}\, X)$. This will not lead to any confusion, since we do not consider any other derived categories.

Let $\mathscr{B}$ be a full subcategory of an additive category. The *right orthogonal* to $\mathscr{B}$ is the full subcategory $\mathscr{B}^\perp \subset \mathscr{A}$ consisting of the objects C such that $\text{Hom}(B, C) =$

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0 for all $B \in \mathscr{B}$. The left orthogonal ${}^{\perp}\mathscr{B}$ is defined analogously. If $\mathscr{B}$ is a triangulated subcategory of a triangulated category $\mathscr{A}$, then $\mathscr{B}^{\perp}$ and ${}^{\perp}\mathscr{B}$ are also triangulated subcategories.

Definition 1.1. Let $\mathscr{B}$ be a strictly full triangulated subcategory of a triangulated category $\mathscr{A}$. We say that $\mathscr{B}$ is *right admissible* (resp. *left admissible*) if for each $X \in \mathscr{A}$ there is a distinguished triangle $B \to X \to C$, where $B \in \mathscr{B}$ and $C \in \mathscr{B}^{\perp}$ (resp. $D \to X \to B$, where $D \in {}^{\perp}\mathscr{B}$ and $B \in \mathscr{B}$). A subcategory is said to be *admissible* if it is left and right admissible [3].

In our case all triangulated categories will be derived from abelian categories, and hence have the functor $R^{\cdot}\operatorname{Hom}$.

Definition 1.2. An *exceptional object* in a derived category $\mathscr{A}$ is an object E satisfying the conditions $R^i\operatorname{Hom}(E, E) = 0$ when $i \neq 0$ and $\operatorname{Hom}(E, E) = \mathbf{C}$.

Definition 1.3. *A complete exceptional set* in $\mathscr{A}$ is an ordered set of exceptional objects $(E_0, \dots, E_n)$, satisfying the semiorthogonality condition $R^{\cdot}\operatorname{Hom}(E_i, E_j) = 0$ when $i > j$, and generating the category $\mathscr{A}$.

The concept of an exceptional set is a (very important) special case of the concept of a semiorthogonal set of subcategories:

Definition 1.4. A set of admissible subcategories $(\mathscr{B}_0, \dots, \mathscr{B}_n)$ in a derived category $\mathscr{A}$ is said to be *semiorthogonal* if the condition $\mathscr{B}_j \subset \mathscr{B}_i^{+}$ holds when $j < i$ for any $0 \leq i \leq n$, and $\mathscr{B}_j \subset {}^{\perp}\mathscr{B}_i$ for $j > i$. In addition, a semiorthogonal set is said to be *complete* if it generates the category $\mathscr{A}$.

The definitions of modifications and spirals can be found in many of the abovementioned papers. We do not need them, so we omit them.

§2. Projective bundles

Let E be a vector bundle of rank r over a smooth projective variety M. Then there exists a projective bundle $\mathbf{P}(E)$ with projection $p\colon \mathbf{P}(E) \to M$. A canonical surjection $p^{\cdot}E^{\cdot} \to O_E(1)$ is defined, fibered over $\mathbf{P}(E)$, which gives a universal exact sequence

$$0 \to O_E(-1) \to p^{\cdot}E \to Q \to 0.$$

We denote by $\mathscr{D}(M)$ and $\mathscr{D}(E)$ the bounded derived categories of coherent sheaves on M and $\mathbf{P}(E)$ respectively. There exists a functor $Lp^{\cdot}\colon \mathscr{D}(M) \to \mathscr{D}(E)$ which is derived from the inverse image functor for coherent sheaves on M under the projection $p\colon \mathbf{P}(E) \to M$. Since p is a flat morphism, there are no higher derived functors of the functor $p^{\cdot}$, and hence $Lp^{\cdot}$ is simply $p^{\cdot}$.

Lemma 2.1. *Let $X, Y \in \mathscr{D}(M)$; then there exist isomorphisms*

$$R^i\operatorname{Hom}(p^{\cdot}X, p^{\cdot}Y) = R^i\operatorname{Hom}(X, Y)$$

and hence the functor $p^{\cdot}\colon \mathscr{D}(M) \to \mathscr{D}(E)$ is a full and faithful embedding.

Proof. The functor $p^{\cdot}$ is left conjugate to the functor $R^{\cdot}p_{\cdot}$, so that there is a natural isomorphism

$$R^i\operatorname{Hom}(p^{\cdot}X, Z) = R^i\operatorname{Hom}(X, R^{\cdot}p_{\cdot}Z),$$

i.e.,

$$R^i\operatorname{Hom}(p^{\cdot}X, p^{\cdot}Y) = R^i\operatorname{Hom}(X, R^{\cdot}p_{\cdot}p^{\cdot}Y).$$

But for any $Y \in \mathscr{D}(M)$ there exists a finite resolution by locally free sheaves, i.e., a finite complex of locally free sheaves quasi-isomorphic to Y. And for a locally free sheaf $\mathscr{F}$ on M the projection formula is true:

$$R^i p_\cdot p^\cdot \mathscr{F} = R^i p_\cdot O_{\mathbf{P}(E)} \otimes \mathscr{F}.$$

Since $R^i p^\cdot O_{\mathbf{P}(E)} = 0$ for $i > 0$ and $p_\cdot O_{\mathbf{P}(E)} = O_M$, we obtain the isomorphism $R^i p_\cdot p^\cdot Y \simeq Y$ for any $Y \in \mathscr{D}(M)$. This completes the proof of the lemma.

We denote by $\mathscr{D}(M)_0$ the subcategory of $\mathscr{D}(E)$ that is the image of $\mathscr{D}(M)$ under the functor $p^\cdot : \mathscr{D}(M) \to \mathscr{D}(E)$. By $\mathscr{D}(M)_k$ we denote the subcategory of $\mathscr{D}(E)$ whose objects are all objects of the form $p^\cdot X \otimes O_E(k)$, where $X \in \mathscr{D}(M)$. All the $\mathscr{D}(M)_k$ are equivalent to $\mathscr{D}(M)$ and are faithful full subcategories of $\mathscr{D}(E)$.

Definition 2.2. Let $\mathscr{A}$ be a triangulated category of finite type (i.e., for any $X, Y \in \mathscr{A}$ each $\operatorname{Ext}^i(X, Y)$ is finite-dimensional and almost all $\operatorname{Ext}^i = 0$). We shall call $\mathscr{A}$ *right saturated* (resp. *left saturated*) if any contravariant (resp. covariant) cohomological functor of finite type $\mathscr{A} \to \mathrm{Vect}$ is representable.

The following assertions are proved in [3].

Assertion 2.3. *Let $\mathscr{B}$ be right saturated (resp. left saturated). Assume that $\mathscr{B}$ is embedded in a triangulated category $\mathscr{A}$ as a full triangulated subcategory. Then $\mathscr{B}$ is right admissible (resp. left admissible).*

Assertion 2.4. *Let M be a smooth projective variety. Then the bounded derived category of coherent sheaves $D^b_{\mathrm{coh}}(\mathrm{Sh}\, M)$ is right and left saturated.*

In our cases it immediately follows from these assertions that all the subcategories $\mathscr{D}(M)_k$ are admissible in $\mathscr{D}(E)$.

Lemma 2.5. *For any $X \in \mathscr{D}(M)_k$ and $Y \in \mathscr{D}(M)_n$ we have $R^\cdot \operatorname{Hom}(X, Y) = 0$ when $r - 1 \geq k - n > 0$.*

Proof. It suffices to prove the lemma for $k = 0$ and $-r + 1 \leq n < 0$. Let $X = p^\cdot X'$, where $X' \in \mathscr{D}(M)$. We have an isomorphism

$$R^\cdot \operatorname{Hom}(X, Y) = R^\cdot \operatorname{Hom}(X', p^\cdot p_\cdot Y).$$

But $Y \in \mathscr{D}(M)_n$, and hence there is a $Y' \in \mathscr{D}(M)$ such that $Y = p^\cdot Y' \otimes O_E(n)$. Moreover, it follows from the proof of Lemma 1.1 that there is an isomorphism

$$R^\cdot p_\cdot Y \simeq Y' \otimes R^\cdot p_\cdot O_E(n);$$

but $R^\cdot p_\cdot O_E(n) = 0$ when $-r + 1 \leq n < 0$. This completes the proof.

Thus, the ordered set of admissible subcategories $(\mathscr{D}(M)_{-r+1}, \dots, \mathscr{D}(M)_0)$ turns out to be semiorthogonal. Now if the set $(\mathscr{D}(M)_{-r+1}, \dots, \mathscr{D}(M)_0)$ were also complete, i.e., generates the category $\mathscr{D}(E)$, then the category $\mathscr{D}(E)$ could be rather simply constructed modulo the category $\mathscr{D}(M)$. In our case this is in fact so.

Theorem 2.6. *The set of admissible subcategories $(\mathscr{D}(M)_{-r+1}, \dots, \mathscr{D}(M)_0)$ is a complete set, i.e., it generates the category $\mathscr{D}(E)$.*

Proof. We consider the fiber square over M

$$\begin{array}{ccc} & \mathbf{P}(E) \times_M \mathbf{P}(E) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathbf{P}(E) & & \mathbf{P}(E) \\ {}_{p'}\searrow & & \swarrow_{p''} \\ & M & \end{array}$$

The morphisms p' and p'' correspond to the morphism $p\colon \mathbf{P}(E) \to M$. On $\mathbf{P}(E)$ there exists a universal exact sequence

$$0 \to O_E(-1) \to p^{\cdot}(E) \to Q \to 0.$$

Moreover, the bundle

$$O_E(1) \boxtimes Q := p_1{}^{\cdot} O_E(1) \otimes p_2{}^{\cdot} Q$$

is defined over $\mathbf{P}(E) \times_M \mathbf{P}(E)$. This bundle has a canonical section s. Since the morphism $p\colon \mathbf{P}(E) \to M$ is flat, for any coherent sheaf $\mathscr{F}$ on $\mathbf{P}(E)$ there exists a natural isomorphism

$$R^i p_{1\cdot} p_2^{\cdot} \mathscr{F} \simeq p^{\cdot} R^i p''_{\cdot} \mathscr{F}$$

(see [12], Chapter III, Proposition 9.3). Under the canonical identification

$$H^0(\mathbf{P}(E) \times_M \mathbf{P}(E), O_E(1) \boxtimes Q) \simeq H^0(\mathbf{P}(E), O_E(1) \otimes p_{1\cdot} p_2^{\cdot} Q) \simeq H^0(M, E \otimes E^{\cdot})$$

the section s corresponds to the identity map $E \xrightarrow{\text{id}} E$. Just as in [10] (Chapter II, §3.1), one can show that the set of zeros of the section s coincides with the diagonal $\Delta \subset \mathbf{P}(E) \times_M \mathbf{P}(E)$. Hence, s defines a resolution of the sheaf O_Δ, which is a Koszul complex:

$$\begin{aligned} 0 &\to \Lambda^{r-1}(O_E(-1) \boxtimes Q^*) \to \Lambda^{r-2}(O_E(-1) \boxtimes Q^*) \to \cdots \to O_E(-1) \boxtimes Q^* \\ &\to O_{\mathbf{P}(E)} \boxtimes O_{\mathbf{P}(E)} \to O_\Delta \to 0. \end{aligned}$$

Tensoring this sequence by $p_2^{\cdot}\mathscr{F}$, where $\mathscr{F}$ is a coherent sheaf on $\mathbf{P}(E)$, we obtain a compact of sheaves:

$$\begin{aligned} 0 &\to O_E(-r+1) \boxtimes (\Lambda^{r-1} Q^* \otimes \mathscr{F}) \to O_E(-r+2) \boxtimes (\Lambda^{r-2} Q^* \otimes \mathscr{F}) \\ &\to \cdots \to O_E(-1) \boxtimes (Q^* \otimes \mathscr{F}) \to O_{\mathbf{P}(E)} \boxtimes \mathscr{F} \to p_2^{\cdot}\mathscr{F}|_\Delta \to 0. \end{aligned}$$

There is a spectral sequence in which the E_1-term is constructed as

$$\begin{aligned} E_1^{ij} &= R^i p_{1\cdot}(O_E(j) \boxtimes (\Lambda^{-j} Q^* \otimes \mathscr{F})) = O_E(j) \otimes R^i p_{1\cdot} p_2^{\cdot}(\Lambda^{-j} Q^* \otimes \mathscr{F}) \\ &= O_E(j) \otimes p'^{\cdot} R^i p''_{\cdot}(\Lambda^{-j} Q^* \otimes \mathscr{F}). \end{aligned}$$

This spectral sequence converges to $R^{i+j} p_{1\cdot}(p_2^{\cdot}\mathscr{F}|_\Delta)$. When $i = j = 0$ we have $R^{i+j} p_{1\cdot}(p_2^{\cdot}\mathscr{F}|_\Delta) = \mathscr{F}$, and for $i + j \neq 0$ we have $R^{i+j} p_{1\cdot}(p_2^{\cdot}\mathscr{F}|_\Delta) = 0$. It remains to note that

$$E_1^{ij} = O_E(j) \otimes p'^{\cdot} R^i p''_{\cdot}(\Lambda^{-j} Q^* \otimes \mathscr{F})$$

belongs to the subcategory $\mathscr{D}(M)$ and j can take values from $-r+1$ to 0. Thus we have proved that any coherent sheaf $\mathscr{F}$ on $\mathbf{P}(E)$, as an object of the derived category $\mathscr{D}(E)$, belongs to the subcategory generated by the set $(\mathscr{D}(M)_{-r+1}, \dots, \mathscr{D}(M)_0)$. Hence, this set generates the category $\mathscr{D}(E)$, which proves the theorem.

Corollary 2.7. *If there exists a complete exceptional set in the derived category $\mathscr{D}(M)$, then the derived category $\mathscr{D}(E)$ also possesses a complete exceptional set.*

Proof. Let $(E_0, \dots, E_n)$ be a complete exceptional set in $\mathscr{D}(M)$. Then the set

$$(p^{\cdot} E_0 \otimes O_E(-r+1), \dots, p^{\cdot} E_n \otimes O_E(-r+1), \dots, p^{\cdot} E_0, \dots, p^{\cdot} E_n)$$

is obviously a complete exceptional set in $\mathscr{D}(E)$.

§3. Grassmann bundles and flag bundles

In this section we sketch the description of the derived category of coherent sheaves on a Grassmann bundle. A rigorous proof could be presented by mechanically combining the previous section and the results of Kapranov [9] (see also [7]).

Let E be a vector bundle of rank r over a smooth projective variety M. We denote by $G_d(E)$, $0 < d < r$, the Grassmann bundle of d-planes in E with projection $p\colon G_d(E) \to M$ and universal subbundle S of rank d of the bundle $p^{\cdot}E$. There exists an exact sequence

$$0 \to S \to p^{\cdot}E \to Q \to 0.$$

Let $\mathscr{D}(M)$ and $\mathscr{D}(G_dE)$ denote the bounded derived categories of coherent sheaves on M and G_dE respectively. The functor $p^{\cdot}\colon \mathscr{D}(M) \to \mathscr{D}(G_dE)$ is full and faithful. Let $\mathscr{D}(M)_0$ denote the image of the category $\mathscr{D}(M)$ in $\mathscr{D}(G_dE)$ under the functor $p^{\cdot}$. On the variety G_dE there exist vector bundles $\Sigma^\alpha S$, where the α range over Young diagrams. We denote by $\mathscr{D}(M)_\alpha$ the full subcategory of $\mathscr{D}(G_dE)$ whose objects are objects of the form $p^{\cdot}X \otimes \Sigma^\alpha S$, where $X \in \mathscr{D}(M)$. Since the bundle S has rank d, then $\alpha = (\alpha_1, \dots, \alpha_d)$, where $\{\alpha_i\}$ is a nonincreasing sequence of integers, which are the lengths of the rows of the Young diagram α. Kapranov [9] constructed a complete exceptional set on a Grassmann variety. From his work and §2 of the paper it is easy to deduce that if we consider the set $N(d, r) = (\mathscr{D}(M)_\alpha)$, where α ranges over all Young diagrams with at most d rows and at most $r - d$ columns, i.e., $\alpha = (\alpha_1, \dots, \alpha_d)$ and $0 \le \alpha_d \le \dots \le \alpha_1 \le r - d$, then this set is semiorthogonal in the following sense.

For any $X \in \mathscr{D}(M)_\alpha$ and $Y \in \mathscr{D}(M)_\beta$, where $\mathscr{D}(M)_\alpha$ and $\mathscr{D}(M)_\beta$ belong to the set $N(d, r)$, we have that $R^{\cdot}\operatorname{Hom}(X, Y) = 0$ if there exists an i such that $\alpha_i < \beta_i$. It follows from this that the set $N(d, r)$ can be totally ordered, as in the case of a projectivization of a vector bundle, but this ordering is far from being unique.

We have to note that all the subcategories $\mathscr{D}(M)_\alpha$ belonging to the set $N(d, n)$ are equivalent to the category $\mathscr{D}(M)$, since

$$\begin{aligned} R^i\operatorname{Hom}(p^{\cdot}X \otimes \Sigma^\alpha S, p^{\cdot}Y \otimes \Sigma^\alpha S) &\simeq R^i\operatorname{Hom}(p^{\cdot}X, p^{\cdot}Y \otimes \Sigma^\alpha S \otimes \Sigma^\alpha S^{\cdot}) \\ &\simeq R^i\operatorname{Hom}(X, Y \otimes R^{\cdot}p_{\cdot}(\Sigma^\alpha S \otimes \Sigma^\alpha S^{\cdot})) \simeq R^i\operatorname{Hom}(X, Y). \end{aligned}$$

The last of these follows from the fact that $R^ip_{\cdot}(\Sigma^\alpha S \otimes \Sigma^\alpha S^{\cdot}) = 0$ when $i \neq 0$, and $R^0p_{\cdot}(\Sigma^\alpha S \otimes \Sigma^\alpha S^{\cdot}) = O_M$, which is deduced from [9].

Just as in §2 above, one proves that the set $N(d, n)$ of admissible subcategories of $\mathscr{D}(G_dE)$ is complete, and that the existence of a complete exceptional set in $\mathscr{D}(M)$ implies the existence of a complete exceptional set in $\mathscr{D}(G_dE)$.

Remark. One can consider flag bundles over a variety M. We see immediately that by induction all the assertions are proved in exactly the same way, since a flag bundle can be decomposed into a chain of Grassmann bundles.

§4. Monoidal transformations

In this section we shall consider monoidal transformations from the point of view of derived categories of coherent sheaves and the presence of complete exceptional sets in them.

Let Y be a smooth subvariety of a smooth projective variety X. Then we denote by $\widetilde{X}$ the smooth projective variety which is obtained by a monoidal transformation (blowing-up) of X with center Y. There exists a fiber square

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\ j\ } & \widetilde{X} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\ i\ } & X \end{array}$$

where i and j are embeddings of smooth varieties, and $p\colon \widetilde{Y} \to Y$ is the projective fibration of the exceptional divisor $\widetilde{Y}$ in $\widetilde{X}$ over the center Y and $\widetilde{Y} = \mathbf{P}(N_Y X)$ is the projectivization of the normal bundle to Y in X. We denote by $\mathscr{D}(X), \mathscr{D}(\widetilde{X}), \mathscr{D}(Y)$, and $\mathscr{D}(\widetilde{Y})$ the bounded derived categories of coherent sheaves on the respective varieties. There exists a functor $L^{\cdot}\pi^*\colon \mathscr{D}(X) \to \mathscr{D}(\widetilde{X})$ which is derived from the inverse image functor for coherent sheaves.

Lemma 4.1. *For any objects $A, B \in \mathscr{D}(X)$ there exists an isomorphism*

$$R^i \operatorname{Hom}(L^{\cdot}\pi^* A, L^{\cdot}\pi^* B) = R^i \operatorname{Hom}(A, B),$$

and hence the functor $L^{\cdot}\pi^*$ is a faithful and full embedding.

Proof. The proof is based on the fact that $R^0\pi_* O_{\widetilde{X}} = O_X$, and $R^i\pi_* O_{\widetilde{X}} = 0$ for all $i \neq 0$ (see, for example, [12]). Any object $B \in \mathscr{D}(X)$ has a finite resolution by locally free sheaves. But for every locally free sheaf $\mathscr{F}$ on X we have $L^{\cdot}\pi^*\mathscr{F} = \pi^*\mathscr{F}$, and by the projection formula we have an isomorphism $R^i\pi_*\pi^*\mathscr{F} = R^i\pi_* O_{\widetilde{X}} \otimes \mathscr{F}$. It follows from this that for any $B \in \mathscr{D}(X)$ we have $R^{\cdot}\pi_* L^{\cdot}\pi^* B = B$. Therefore we have isomorphisms

$$R^i \operatorname{Hom}(L^{\cdot}\pi^* A, L^{\cdot}\pi^* B) = R^i \operatorname{Hom}(A, R^{\cdot}\pi_* L^{\cdot}\pi^* B) \simeq R^i \operatorname{Hom}(A, B).$$

This proves the lemma.

We denote by $\mathscr{D}(X)_0$ the full subcategory of $\mathscr{D}(\widetilde{X})$ that is the image of $\mathscr{D}(X)$ relative to the functor $L^{\cdot}\pi^*$. Intuitively we see that in order to complete $\mathscr{D}(X)_0$ to $\mathscr{D}(\widetilde{X})$ the objects concentrated on the divisor $\widetilde{Y}$ do not suffice. We consider the category $\mathscr{D}(\widetilde{Y})$. Then by what was proved in §2 it has a complete semiorthogonal set of admissible subcategories $(\mathscr{D}(Y)_{-r+1}, \dots, \mathscr{D}(Y)_0)$, where r is the codimension of the subvariety Y in X, and $\mathscr{D}(Y)_0 = p^{\cdot}\mathscr{D}(Y)$. There exists an embedding functor $j_*\colon \mathscr{D}(\widetilde{Y}) \to \mathscr{D}(\widetilde{X})$, which is derived from taking the direct image for coherent sheaves on $\widetilde{Y}$ under the embedding $j\colon \widetilde{Y} \hookrightarrow \widetilde{X}$. However this functor will not be full, but we can consider the restriction of j_* to the subcategory $\mathscr{D}(Y)_k$. It turns out that in this case the functor j_{*k} will already be full and that it preserves the semiorthogonality condition in some truncated form. All these facts are given in the following assertion.

Assertion 4.2. *In the above notation the following assertions are true*:

(a) *The functor $j_{*k}\colon D(Y)_k \to D(\widetilde{X})$ is a full and faithful embedding.*

(b) *If $\widetilde{\mathscr{D}(Y)}_k$ denotes the subcategory of $\mathscr{D}(\widetilde{X})$ which is the image of $\mathscr{D}(Y)_k$ under the functor j_{*k}, then the set $(\widetilde{\mathscr{D}(Y)}_{-r+1}, \dots, \mathscr{D}(Y)_{-1}, \mathscr{D}(X)_0)$ will be semiorthogonal.*

Proof. Consider an object $A \in \mathscr{D}(Y)_k$ and let $C^{\cdot}$ be a finite resolution of locally free sheaves for the object $j_*A \in \mathscr{D}(\widetilde{X})$. Then the object $L^{\cdot}j^*j_*A$ is represented by the resolution $j^*C^{\cdot} \simeq C^{\cdot}|_{\widetilde{Y}}$. Since $\widetilde{Y}$ is the exceptional divisor of a blowing-up, there exists an isomorphism $j^*O_{\widetilde{X}}(-\widetilde{Y}) \simeq O_E(1)$. Furthermore, using the projection formula, we find that

$$j_*A \otimes O_{\widetilde{X}}(-\widetilde{Y}) = j_*(A \otimes j^*O_{\widetilde{X}}(-\widetilde{Y})) \simeq j_*(A \otimes O_E(1)).$$

Hence $C^{\cdot}(-\widetilde{Y})$ is a resolution for the object $j_*(A \otimes O_E(1))$. From the short exact sequence

$$0 \to C^{\cdot}(-\widetilde{Y}) \to C^{\cdot} \to C^{\cdot}|_{\widetilde{Y}} \to 0$$

we get the existence of a triangle $A \otimes O_E(1) \to A \to L^{\cdot} j^* j_* A$ in the category $\mathscr{D}(\widetilde{Y})$. Moreover, since the morphism $C^{\cdot}(-\widetilde{Y}) \to C^{\cdot}$ is zero on the divisor $\widetilde{Y}$, the morphism $A \otimes O_E(1) \to A$ is also zero. Thus we will obtain an isomorphism

$$L^{\cdot} j^* j_* A = A \oplus (O_E(1) \otimes A)[1].$$

Now it is easy to obtain the assertions.

(a) Let $A, B \in \mathscr{D}(Y)_k$; then we have an isomorphism

$$\begin{aligned} R^i \operatorname{Hom}(j_* A, j_* B) &\simeq R^i \operatorname{Hom}(L^{\cdot} j^* j_* A, B) \\ &\simeq R^i \operatorname{Hom}(A, B) \oplus R^i \operatorname{Hom}(A[1] \otimes O_E(1), B) \\ &\simeq R^i \operatorname{Hom}(A, B). \end{aligned}$$

The last statement follows from the fact that $A[1] \otimes O_E(1)$ is an object of $\mathscr{D}(Y)_{k+1}$, and from the semiorthogonality condition.

(b) Let $A \in \mathscr{D}(Y)_k$ and $B \in \mathscr{D}(Y)_n$. If $r - 2 \geq k - n > 0$, then $L^{\cdot} j^* j_* A$ is an object of the subcategory generated by $(\mathscr{D}(Y)_k, \mathscr{D}(Y)_{k+1})$, and hence

$$R^i \operatorname{Hom}(j_* A, j_* B) \simeq R^i \operatorname{Hom}(L^{\cdot} j^* j_* A, B) = 0.$$

From this we get the semiorthogonality of the set $(\widetilde{\mathscr{D}(Y)}_{-r+1}, \dots, \widetilde{\mathscr{D}(Y)}_{-1})$.

It remains to consider the case when $A \in \mathscr{D}(X)_0$ and $B \in \widetilde{\mathscr{D}(Y)}_k$ for $-r+1 \leq k \leq -1$. We consider $B' \in \mathscr{D}(Y)_k$ such that $B = j_* B'$. We also have the chain of isomorphisms

$$\begin{aligned} R^i \operatorname{Hom}(A, B) &\simeq R^i \operatorname{Hom}(L^{\cdot} \pi^* A', j_* B') \\ &\simeq R^i \operatorname{Hom}(A', R^{\cdot} \pi_* j_* B') \simeq R^i \operatorname{Hom}(A', i_* R^{\cdot} p_{\cdot} B') = 0. \end{aligned}$$

This is equal to zero because $i_* R^{\cdot} p_{\cdot} B' = 0$, which is true in turn because $R^{\cdot} p_{\cdot} B' = 0$. Thus the assertion has been completely proved.

Now we arrive at the proof of the main theorem of this section.

Theorem 4.3. *In the above notation, the semiorthogonal set of admissible subcategories* $(\widetilde{\mathscr{D}(Y)}_{-r+1}, \dots, \widetilde{\mathscr{D}(Y)}_{-1}, \mathscr{D}(X)_0)$ *is complete, i.e., it generates the category* $\mathscr{D}(\widetilde{X})$.

Proof. We divide the proof of the theorem into two parts.

1) In the first part we shall prove a simple assertion. Let $A \in \mathscr{D}(\widetilde{X})$ be such that $R^{\cdot} \pi_* A = 0$; then A belongs to the minimal full subcategory of $\mathscr{D}(\widetilde{X})$ which contains the image $j_* \mathscr{D}(\widetilde{Y})$. Let $C^{\cdot}$ be a finite locally free resolution for A. Then the assertion is equivalent to the fact that all the cohomology groups $H^k(C^{\cdot})$ of the complex $C^{\cdot}$ are coherent sheaves supported on $\widetilde{Y}$. Assume the contrary; hence there is a k such that $\operatorname{supp} H^k(C^{\cdot}) \not\subset \widetilde{Y}$. We note also that for any coherent sheaf $\mathscr{F}$ on $\widetilde{X}$ the support of $R^i \pi_* \mathscr{F}$ is concentrated on Y for $i > 0$. Now we use the spectral sequence for direct images with the term $E_2^{pq} = R^p \pi_*(H^q(C^{\cdot}))$, which converges to $E^n = \bigoplus_{p+q=n} R^{p+q} \pi_* A$, i.e., to zero, since $E^n = 0$ for all n. On the other hand, we have $\operatorname{supp} E_2^{0\,k} \not\subset Y$ and $\operatorname{supp} E_2^{2\,k-1} \subset Y$. From this it follows that $E_3^{0\,k} \neq 0$ and $\operatorname{supp} E_3^{0\,k} \not\subset Y$. Then by induction we will obtain that $E_m^{0\,k}$ vanishes for all $m \geq 2$. This gives us a contradiction.

2) We now proceed directly to the proof of the theorem. It will also be done "by contradiction". All the subcategories of the set

$$(\widetilde{\mathscr{D}(Y)}_{-r+1}, \dots, \widetilde{\mathscr{D}(Y)}_{-1}, \mathscr{D}(X)_0)$$

are admissible, and hence the full subcategory $\mathscr{D}$ generated by this set is admissible (this is proved in [3]). If $\mathscr{D}$ does not coincide with $\mathscr{D}(\widetilde{X})$, then the subcategory ${}^{\perp}\mathscr{D}$ is not empty. Let $A \in {}^{\perp}\mathscr{D}$; we consider a finite locally free resolution $C^{\cdot}$ for A.

Since $\mathscr{D}(X)_0 \subset \mathscr{D}$, we have

$$R^i \operatorname{Hom}(L^{\cdot}\pi^* B, FA) \simeq R^i \operatorname{Hom}(A, L^{\cdot}\pi^* B) = 0,$$

and hence $R^{\cdot}\pi_* FA = 0$, where F is the Serre functor in $\mathscr{D}(\widetilde{X})$ of tensoring by the canonical sheaf $K_{\widetilde{X}}[n]$, shifted in the derived category by the dimension of the variety $\widetilde{X}$. Therefore we will obtain from 1) that all the cohomology groups of $FC^{\cdot}$ are coherent sheaves with supports concentrated on $\widetilde{Y}$, and hence the same is also true for $C^{\cdot}$. It follows from this that A belongs to the minimal full subcategory containing $j_*\mathscr{D}(\widetilde{Y})$. Therefore, in order to prove that A is zero, it suffices to show that $L^{\cdot}j^*A$ is equal to zero, since

$$R^i \operatorname{Hom}(A, j_*B) \simeq R^i \operatorname{Hom}(L^{\cdot}j^*A, B).$$

But since $A \in {}^{\perp}\mathscr{D}$ and $\widetilde{\mathscr{D}(Y)_k} \subset \mathscr{D}$ for $-r+1 \geq k \geq -1$, we will obtain that $L^{\cdot}j^*A \in \mathscr{D}(Y)_0$. Therefore there is an object $B \in \mathscr{D}(Y)$ such that $L^{\cdot}j^*A = p^{\cdot}B$. If B is not zero, then there exists a coherent sheaf $\mathscr{F}$ on Y with the property $\operatorname{Hom}(B, \mathscr{F}) \neq 0$.

By the base change theorem ([4], p. 276) the object $\pi^{-1}R^{\cdot}i_*\mathscr{F}$ is isomorphic to the object $R^{\cdot}j_!p^{-1}\mathscr{F}$ in the derived category of abelian sheaves on $\widetilde{X}$. Since $R^{\cdot}f_! = R^{\cdot}f_*$ for a proper morphism f, we have an isomorphism

$$\pi^{-1}i_*\mathscr{F} = j_*p^{-1}\mathscr{F}.$$

Moreover, by definition,

$$L^{\cdot}\pi^* i_*\mathscr{F} \simeq O_{\widetilde{X}} \overset{L}{\underset{\pi^{-1}(O_X)}{\otimes}} \pi^{-1}i_*\mathscr{F} \simeq O_{\widetilde{X}} \overset{L}{\underset{\pi^{-1}(O_X)}{\otimes}} j_!p^{-1}\mathscr{F}.$$

By the projection formula ([4], p. 277) we obtain

$$O_{\widetilde{X}} \otimes^L_{\pi^{-1}(O_X)} j_!p^{-1}\mathscr{F} \simeq j_!(p^{-1}\mathscr{F} \otimes^L_{j^{-1}\pi^{-1}(O_X)} j^{-1}(O_{\widetilde{X}})).$$

The object $F = p^{-1}\mathscr{F} \otimes^L_{j^{-1}\pi^{-1}(O_X)} j^{-1}(O_{\widetilde{X}})$ is an object of the category $\mathscr{D}(\widetilde{Y})$, since

$$O_{\widetilde{Y}} = p^{-1}(O_Y) \underset{j^{-1}\pi^{-1}(O_X)}{\otimes} j^{-1}(O_{\widetilde{X}}),$$

and $\mathscr{F}$ is a coherent sheaf on Y. Hence, we have an isomorphism $j_*F \simeq L^{\cdot}\pi^*\mathscr{F}$; moreover, the direct image

$$\begin{aligned} p_{\cdot}F &\simeq p_{\cdot}(p^{-1}\mathscr{F} \overset{L}{\underset{j^{-1}\pi^{-1}(O_X)}{\otimes}} j^{-1}(O_{\widetilde{X}})) \\ &\simeq \mathscr{F} \overset{L}{\underset{i^{-1}(O_X)}{\otimes}} p_{\cdot}j^{-1}(O_{\widetilde{X}}) \simeq \mathscr{F} \overset{L}{\underset{i^{-1}(O_X)}{\otimes}} i^{-1}R^{\cdot}\pi_* O_{\widetilde{X}} \simeq \mathscr{F}. \end{aligned}$$

Applying these facts, we obtain

$$\begin{aligned} \operatorname{Hom}(B, \mathscr{F}) &= \operatorname{Hom}(p^{\cdot}B, F) = \operatorname{Hom}(L^{\cdot}j^*A, F) \\ &= \operatorname{Hom}(A, j_*F) = \operatorname{Hom}(A, L^{\cdot}\pi^* i_*\mathscr{F}) = 0. \end{aligned}$$

The last follows from the fact that $A \in {}^{\perp}\mathscr{D}$, and $L^{\cdot}\pi^* i_* \mathscr{F} \in \mathscr{D}(X)_0$ and $\mathscr{D}(X)_0$ is contained in $\mathscr{D}$. This contradicts the fact that $\mathrm{Hom}(B, \mathscr{F}) \neq 0$ and proves the theorem.

From the theorem we immediately get

Corollary 4.4. *If the categories $\mathscr{D}(X)$ and $\mathscr{D}(Y)$ have complete exceptional sets, then so does the category $\mathscr{D}(\widetilde{X})$.*

The proof of the corollary is obvious. It is necessary to consider a complete exceptional set in $\mathscr{D}(Y)$; then all the subcategories $\mathscr{D}(Y)_k$ in $\mathscr{D}(\widetilde{Y})$ also have complete exceptional sets, which are omitted in $\mathscr{D}(\widetilde{X})$ via the functor j_*. Moreover, it is necessary to lift a complete exceptional set in $\mathscr{D}(X)$ by the functor $L^{\cdot}\pi^*$. Together they give a complete exceptional set in $\mathscr{D}(\widetilde{X})$.

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