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# EXCEPTIONAL SHEAVES ON DEL PEZZO SURFACES

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ABSTRACT. In the present paper exceptional sheaves on del Pezzo surfaces are studied, and a description of rigid bundles on these surfaces is given. It is proved that each exceptional sheaf can be included in a complete exceptional collection. Furthermore, it is shown that all such collections can be obtained from each other by means of a sequence of standard operations called transformations.

## INTRODUCTION

The goal of the present paper is to study exceptional sheaves and exceptional collections of sheaves on del Pezzo surfaces. An exceptional sheaf $E$ (or, more generally, an object of derived category) is a simple sheaf satisfying the conditions $\mathrm{Ext}^i(E, E) = 0$ for $i \neq 0$, and an exceptional collection is an ordered collection of exceptional sheaves satisfying the conditions $\mathrm{Ext}^i(E_\alpha, E_\beta) = 0$ for all $\alpha > \beta$ and all $i$.

The existence of exceptional sheaves and collections imposes heavy restrictions on the variety. Of special interest are varieties on which there exist complete exceptional collections, that is, collections generating the derived category of coherent sheaves. Examples of such varieties are given by the projective space $\mathbb{P}^n$, the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, the Grassmann and flag varieties (cf. [1], [5] and [6]), and the blowups of varieties carrying a complete exceptional collection at subvarieties having the same property (cf. [8]).

It is easy to see that each del Pezzo surface carries a complete exceptional collection. In the present paper we give a description of exceptional sheaves and collections of sheaves on del Pezzo surfaces. Descriptions of these objects on $\mathbb{P}^2$ and the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ can be found in [5] and [9], respectively. Our main results are as follows.

In §2 we show that each exceptional object in the derived category of coherent sheaves on an arbitrary del Pezzo surface is a sheaf and that each exceptional sheaf either is locally free or is a torsion sheaf of the form $\mathcal{O}_e(n)$, where $e$ is a $(-1)$-curve.

In §§4 and 5 we consider rigid bundles, that is, bundles satisfying the condition $\mathrm{Ext}^1(E, E) = 0$. We show that on del Pezzo surfaces rigid bundles split into a direct sum of exceptional bundles.

Using these facts, in §6 we prove that each exceptional collection is a part of a complete exceptional collection; in particular, each exceptional sheaf can be included in a complete exceptional collection.

The last section is devoted to transformations of exceptional collections. Transformation is an operation allowing to construct exceptional collections starting from a given one. Its definition is given in §1. Using transformations we can breed exceptional collections. Still more important, each complete exceptional collection can

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be obtained in this way starting from a fixed collection. This property, called *constructibility*, is proven in § 7.

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## § 1. Basic notions and definitions

1.0. Throughout this paper $S$ will denote a smooth projective surface over $\mathbb{C}$. A surface $S$ is called a *del Pezzo surface* if its anticanonical sheaf $\omega_S^*$ is ample.

1.1. The rank of a coherent sheaf $F$ will be denoted by $r(F)$; $c_1(F)$ and $c_2(F)$ will denote the first and second Chern classes of the sheaf $F$.

Let $F$ be a torsion free sheaf, and let $A$ be a divisor. The rational number $(c_1(F)\cdot A)/r(F)$ is called the *slope* of $F$ with respect to $A$ and is denoted by $\mu_A(F)$. If $A \in |-K_S|$, then we simply write $\mu(F)$.

For arbitrary coherent sheaves $E$ and $F$ on $S$ we define $\chi(E, F)$ as the alternating sum

$$\chi(E, F) = \sum_i (-1)^i \dim \operatorname{Ext}^i(E, F).$$

This formula defines a bilinear form on the space $K_0(S)$. According to the Riemann-Roch theorem,

$$\chi(E, F) = r(E)r(F)\left[\chi(\mathscr{O}_S) + \frac{\mu(F)-\mu(E)}{2} + q(F) + q(E) - \frac{1}{r(E)r(F)}(c_1(E)c_1(F))\right], \tag{1.1}$$

where $q(E) = (c_1^2(E) - 2c_2(E))/(2r(E))$. The bilinear form $\chi(E, F)$ can be decomposed into a sum of symmetric and antisymmetric parts:

$$\chi(E, F) = \chi_+(E, F) + \chi_-(E, F).$$

Furthermore, the antisymmetric part is given by the following simple formula:

$$\chi_-(E, F) = \frac{1}{2} r(E)r(F)(\mu(F) - \mu(E)).$$

For convenience, we will denote $\dim \operatorname{Ext}^i(E, F)$ by $h^i(E, F)$.

1.2. **Definition.** A coherent sheaf $E$ is called

a) *simple* if $\operatorname{Hom}(E, E) = \mathbb{C}$;
b) *rigid* if $\operatorname{Ext}^1(E, E) = 0$;
c) *superrigid* if $\operatorname{Ext}^1(E, E) = \operatorname{Ext}^2(E, E) = 0$;
d) *exceptional* if $\operatorname{Hom}(E, E) = \mathbb{C}$ and $\operatorname{Ext}^i(E, E) = 0$ for all $i \neq 0$.

1.3. **Definition.** An ordered collection of sheaves $(E_1, \dots, E_n)$ is called an *exceptional collection* if all sheaves in this collection are exceptional and for $1 \le \alpha < \beta \le n$ we have

$$\operatorname{Ext}^i(E_\beta, E_\alpha) = 0 \quad \text{for all} \quad i.$$

The notions of exceptional objects and collections can also be introduced for the bounded derived category of coherent sheaves on $S$, which will be denoted by $\mathscr{D}^b(S)$.

1.4. **Definition.** An object $X \in \mathscr{D}^b(S)$ is called *exceptional* if $\operatorname{Hom}(X, X) = \mathbb{C}$ and $\operatorname{Ext}^i(X, X) = 0$ for all $i \neq 0$.

In a similar way one can define exceptional collections of objects of $\mathscr{D}^b(S)$.

1.5. **Definition.** An exceptional collection $(X_1, \dots, X_n)$ of objects of $\mathscr{D}^b(S)$ is called *complete* if it generates the derived category $\mathscr{D}^b(S)$ (i.e., the minimal complete triangulated subcategory in $\mathscr{D}^b(S)$ containing the objects $X_i$ coincides with $\mathscr{D}^b(S)$).

It is known that there exist complete exceptional collections of sheaves on all del Pezzo surfaces.

For example, $(\mathscr{O}_{\mathbb{P}^2}, \mathscr{O}_{\mathbb{P}^2}(1), \mathscr{O}_{\mathbb{P}^2}(2))$ is a complete exceptional collection on $\mathbb{P}^2$. If $S$ is obtained by blowing up $\mathbb{P}^2$ at $d$ points, and $l_1, \dots, l_d$ are the exceptional curves, then the collection $(\mathscr{O}_{l_1}(-1), \dots, \mathscr{O}_{l_d}(-1), \mathscr{O}_S, \mathscr{O}_S(h), \mathscr{O}_S(2h))$, where $\mathscr{O}_S(h)$ is the inverse image of the sheaf $\mathscr{O}_{\mathbb{P}^2}(1)$, is exceptional and complete.

We need some definitions and notation from the theory of triangulated categories. Our main sources here are the papers [2] and [3].

Let $\mathscr{A}$ be an additive category, and $\mathscr{B} \subset \mathscr{A}$ a full subcategory. The full subcategory $\mathscr{B}^\perp \subset \mathscr{A}$ consisting of all objects $C$ such that $\operatorname{Hom}(B, C) = 0$ for all $B \in \mathscr{B}$ is called the *right orthogonal* to $\mathscr{B}$. In a similar way we define the *left orthogonal* ${}^\perp\mathscr{B}$. If $\mathscr{A}$ is a triangulated category and $\mathscr{B}$ is a triangulated subcategory, then $\mathscr{B}^\perp$ and ${}^\perp\mathscr{B}$ also are triangulated subcategories.

1.6. **Definition.** Let $\mathscr{A}$ be a triangulated category, and $\mathscr{B} \subset \mathscr{A}$ a strictly full triangulated subcategory. We call $\mathscr{B}$ *right admissible* (resp., *left admissible*) if for each $X \in \mathscr{A}$ there exists a distinguished triangle $B \to X \to C$ with $B \in \mathscr{B}$, $C \in \mathscr{B}^\perp$ (resp., a distinguished triangle $D \to X \to B$ with $D \in {}^\perp\mathscr{B}$, $B \in \mathscr{B}$). A subcategory is called *admissible* if it is both right and left admissible (cf. [2], [3]).

1.7. **Proposition.** *Let $\mathscr{B}$ be a strictly full triangulated subcategory of $\mathscr{A}$. The following conditions are equivalent*:

a) *$\mathscr{B}$ is right admissible (resp., left admissible)*;

b) *the inclusion functor $\mathscr{B} \to \mathscr{A}$ has a right conjugate (resp., left conjugate)*;

c) *$\mathscr{A}$ is generated as a triangulated category by $\mathscr{B}$ and $\mathscr{B}^\perp$ (resp., by $\mathscr{B}$ and ${}^\perp\mathscr{B}$).*

The proof can be found in [2].

1.8. **Proposition.** *Let $\mathscr{B} = \langle E_0, \dots, E_n\rangle$ be a subcategory in $\mathscr{A}$ generated by an exceptional collection. Then $\mathscr{B}$ is an admissible subcategory of $\mathscr{A}$.*

The proof can be found in [2].

In what follows we need definitions of transformations and helixes.

1.9. **Definition.** Let $(E, F)$ be an exceptional pair. We define objects called the *left* and *right* transformations of the pair $(E, F)$ and denoted by $L_E F$ and $R_F E$ using the following distinguished triangles in the derived category $\mathscr{D}^b(X)$:

$$L_E F \to R^{\bullet}\operatorname{Hom}(E, F) \otimes E \to F,$$
$$E \to R^{\bullet}\operatorname{Hom}(E, F)^* \otimes F \to R_F E.$$

A transformation of an exceptional collection $(E_0, \dots, E_n)$ is defined as a transformation of a pair of neighboring objects in this collection.

Let $(E_0, \dots, E_n)$ be an exceptional collection. We extend it to an (infinite in both directions) sequence of objects of $\mathscr{D}^b(X)$ putting by induction

$$E_{n+i} = R^n E_{i-1}, \qquad E_{-i} = L^n E_{n-i+1}, \qquad i > 0.$$

1.10. **Definition.** An infinite (in both directions) sequence $E_i$ of objects of the derived category $\mathscr{D}^b(X)$ of coherent sheaves on a variety $X$ of dimension $m$ is called a *helix of period* $n$ if

$$E_i = E_{i+n} \otimes K[m - n + 1]$$

(here $K$ is the canonical class, and the number in brackets measures the shift of an object in $\mathscr{D}^b(X)$).

1.11. **Definition.** An exceptional collection is called a *coil of a helix* if the corresponding sequence is a helix of period $n+1$.

It turns out that the notion of a coil can be used to find out whether or not the derived category is generated by an exceptional collection.

1.12. **Proposition.** *Let $(E_0, \dots, E_n)$ be an exceptional collection on a variety $X$ with ample anticanonical class. Then the following assertions are equivalent*:

1) *the collection $\{E_i\}$ generates the derived category $\mathscr{D}^b(X)$*;
2) *the collection $\{E_i\}$ is a coil of a helix.*

For the proof we refer to [2]. We remark that 1) implies 2) for any variety.

## § 2. Exceptional sheaves

2.0. We recall that a smooth projective surface $S$ is called a *del Pezzo surface* if its anticanonical class $\omega_S^*$ is ample. This class contains two minimal surfaces, viz. $\mathbb{P}^2$ and the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. All other surfaces are obtained by blowing up $\mathbb{P}^2$ at $d$ points in general position, where $d$ does not exceed $8$.

2.1. The main problem dealt with in this section is to give a description of exceptional sheaves (more generally, exceptional objects in the derived category) on del Pezzo surfaces. The description we give is neither complete nor constructive, but it allows us to determine which sheaves cannot be exceptional.

The following lemma proved by Mukai in [7] for surfaces of type K3 can be restated in a form in which it holds for arbitrary smooth projective surfaces.

2.2. **Lemma.** *Let $S$ be a smooth projective surface.*

1) *For each coherent torsion-free sheaf $E$ on $S$ the following inequality holds*:

$$h^1(E, E) \geq h^1(E^{**}, E^{**}) + 2\,\mathrm{length}\,(E^{**}/E).$$

2) a) *For each exact triple*

$$0 \to G_2 \to E \to G_1 \to 0$$

*of coherent sheaves on $S$ such that* $\mathrm{Hom}\,(G_2, G_1) = \mathrm{Ext}^2\,(G_1, G_2) = 0$ *the following inequality holds*:

$$h^1(E, E) \geq h^1(G_1, G_1) + h^1(G_2, G_2).$$

b) *If moreover $E$ is a rigid sheaf, i.e., $h^1(E, E) = 0$, then the following equalities hold*:

$$h^0(E, E) = h^0(G_1, G_1) + h^0(G_2, G_2) + \chi(G_1, G_2),$$
$$h^2(E, E) = h^2(G_1, G_1) + h^2(G_2, G_2) + \chi(G_2, G_1).$$

*Proof*. Here we do not prove 1). A proof can be found in [7, Proposition 2.14].

2) Consider the exact triple $0 \to G_2 \to E \to G_1 \to 0$. It can be interpreted as a filtration of the sheaf $E$ with quotients $G_1$ and $G_2$.

There is a spectral sequence for the filtered object with $E_1$ term

$$E_1^{pq} = \bigoplus_j \operatorname{Ext}^{p+q}(G_j, G_{j+p}),$$

converging to $\operatorname{Ext}^{p+q}(E, E)$.

Taking into consideration that $\operatorname{Hom}(G_2, G_1) = \operatorname{Ext}^2(G_1, G_2) = 0$, we see that in our case the $E_1$ term of the spectral sequence has the following form:

$$\begin{array}{ccccc}
 & & \uparrow q & & \\
0 & * & 0 & 0 & \\
0 & * & * & 0 & \xrightarrow{d} \\
0 & 0 & * & 0 & \\
0 & 0 & * & * & \xrightarrow[p]{} \\
0 & 0 & 0 & * &
\end{array}$$

The differential acts horizontally. The sequence degenerates at the $E_2$ term, i.e., $E_2^{pq} = E_\infty^{pq}$. Furthermore, $E_1^{01} = E_\infty^{01}$, from which it follows that $\dim E_1^{01} \le \dim \operatorname{Ext}^1(E, E)$, that is,

$$h^1(E, E) \ge h^1(G_1, G_1) + h^1(G_2, G_2),$$

as required.

Now, if $h^1(E, E) = 0$, then the map $d_1 \colon E_1^{00} \to E_1^{10}$ is surjective, and therefore

$$\begin{aligned}
\dim \operatorname{Ext}^0(E, E) &= \dim E_1^{00} - \dim E_1^{10} + \dim E_1^{1\,-1} \\
&= h^0(G_1, G_1) + h^0(G_2, G_2) - h^1(G_1, G_2) + h^0(G_1, G_2) \\
&= h^0(G_1, G_1) + h^0(G_2, G_2) + \chi(G_1, G_2).
\end{aligned}$$

The second equality is proved in a similar way.

2.3. **Corollary.** *A rigid torsion-free sheaf on a smooth projective surface is locally free.*

This follows immediately from assertion 1) of the preceding lemma.

The inequalities in the following lemma were first proved in [4] under more restrictive assumptions.

2.4. **Lemma.** *Let $S$ be a surface whose anticanonical class $\omega_S^*$ is generated by global sections.*

a) *The following inequality holds for arbitrary two sheaves $F$ and $G$:*

$$h^0(F, G) \ge h^2(G, F).$$

b) *If, moreover, $\omega_S^*$ is ample and* $\dim \operatorname{supp} F > 0$, *then one has the following strict inequality:*

$$h^0(F, F) > h^2(F, F).$$

*Proof.* a) Consider the exact sequence

$$0 \to \mathcal{O}_S \to \omega_S^* \to \omega_S^*|_D \to 0$$

corresponding to a section $\varphi \in H^0(S, \omega_S^*)$. Applying to this sequence the functor of local $\mathcal{H}om$ with $G$ as the second argument, we get the following sequence:

$$0 \longrightarrow \mathcal{H}om(\omega_S^*|_D, G) \longrightarrow G \otimes \omega_S \longrightarrow G \longrightarrow \mathcal{E}xt^1(\omega_S^*|_D, G) \longrightarrow 0.$$

We denote the torsion subsheaf of $G$ by $TG$, and the torsion-free quotient sheaf by $G^1$. Thus, $G$ is included in the exact sequence

$$O \to TG \to G \to G^1 \to O.$$

From this sequence it follows that

$$\mathcal{H}om(\omega_S^*|_D, G) \simeq \mathcal{H}om(\omega_S^*|_D, TG).$$

The sheaf $TG$ fits into the exact sequence

$$0 \to T^0G \to TG \to T^1G \to 0,$$

in which $T^0G$ is the torsion subsheaf with zero-dimensional support, and $T^1G$ is the quotient sheaf without the subsheaf with zero-dimensional support. The support of $T^1G$ is a divisor, and if $D$ does not contain components of this divisor, then $\mathcal{H}om(\omega_S^*|_D, T^1G) = 0$. If, moreover, $D$ does not intersect the support of $T^0G$, then $\mathcal{H}om(\omega_S^*|_D, T^0G) = 0$. It is easy to reduce to this situation since $\omega_S^*$ is generated by global sections, and therefore its set of base points is empty. Hence one can choose a section $\varphi \in H^0(S, \omega_S^*)$ such that

$$\mathcal{H}om(\omega_S^*|_D, TG) = 0.$$

In this case we obtain the following exact sequence:

$$0 \to G \otimes \omega_S \to G \to \mathcal{E}xt^1(\omega_S^*|_D, G) \to 0.$$

Applying the functor $\operatorname{Hom}(F, \cdot)$ to this exact sequence, we obtain an inclusion

$$\operatorname{Hom}(F, G \otimes \omega_S) \hookrightarrow \operatorname{Hom}(F, G).$$

Now the inequality

$$h^0(F, G) \geq h^2(G, F)$$

follows from the Serre duality.

b) We observe that $\mathcal{E}xt^1(\omega_S^*|_D, G) \simeq G \otimes \mathcal{O}_D$, and, replacing $G$ by $F$ and arguing as above, for some $\varphi \in H^0(S, \omega_S^*)$ we obtain an exact sequence of the form

$$0 \to F \otimes \omega_S \to F \to F \otimes \mathcal{O}_D \to 0.$$

By assumption, $\dim \operatorname{supp} F > 0$ and the sheaf $\omega_S^*$ is ample. Hence, by the ampleness criterion,

$$D \cap \operatorname{supp} F \neq \varnothing.$$

Thus $F \otimes O_D$ is a nontrivial sheaf. From this it follows that the identity map $F \xrightarrow{\text{id}} F$ does not factor through $F \otimes \omega_S$. Therefore we obtain the strict inequality

$$h^0(F, F) > h^2(F, F).$$

The lemma is proved.

2.5. *Remark.* It is worthwhile to note that all del Pezzo surfaces with the exception of the blowup of the plane at eight points satisfy the conditions of the preceding lemma. In the exceptional case the anticanonical linear system has a single fundamental point. But it is not hard to see that the assertion is also true in this case provided that $T^0G = 0$.

2.6. **Corollary.** *Let $G$ be a sheaf on a del Pezzo surface such that $T^0G = 0$. Then*

a) $h^0(G, G) > h^2(G, G)$; *and*

b) *an arbitrary sheaf $F$ satisfies the inequality*

$$h^0(F, G) \geq h^2(G, F).$$

2.7. **Corollary.** *Let $S$ be a del Pezzo surface, and let $G$ be a rigid sheaf. Then the torsion subsheaf $TG$ and the torsion-free quotient sheaf $G'$ are rigid sheaves, and the sheaf $T^0G$ is trivial.*

*Proof.* Consider the exact triple

$$0 \to TG \to G \to G' \to 0.$$

We know that $\operatorname{Hom}(TG, G') = 0$. By Corollary 2.6,

$$h^2(G', TG) \leq h^0(TG, G') = 0.$$

Hence by Lemma 2.2

$$h^1(G, G) \geq h^1(TG, TG) + h^1(G', G').$$

But $h^1(G, G) = 0$, and therefore $TG$ and $G'$ are rigid sheaves. The sheaf $TG$ fits into the exact sequence

$$0 \to T^0G \to TG \to T^1G \to 0.$$

As above, it is easy to see that $h^0(T^0G, T^1G) = 0$, and, by virtue of the inequality of Lemma 2.4 a), $h^2(T^1G, T^0G) = 0$. Hence the sheaves $T^0G$ and $T^1G$ are also rigid. But the sheaf $T^0G$ cannot be rigid since

$$h^1(T^0G, T^0G) = 2\operatorname{length}(T^0G).$$

The corollary is proved.

We proceed to a description of exceptional sheaves on del Pezzo surfaces. Exceptional sheaves are rigid and simple (the converse is also true on del Pezzo surfaces). Hence, as we have already shown, torsion-free exceptional sheaves are locally free (cf. Corollary 2.3). Now we consider exceptional torsion sheaves. They admit a very simple description.

2.8. **Lemma.** *Let $F$ be an exceptional torsion sheaf on a del Pezzo surface $S$. Then $F$ has the form $\mathcal{O}_C(D)$, where $C$ is a $(-1)$-curve and $d$ is an integer.*

*Proof.* First we compute $\chi(F, F)$. By formula (1.1) we have

$$\chi(F, F) = r^2 + (r-1)c_1^2 - 2rc_2.$$

Taking into consideration that $F$ is an exceptional sheaf of rank zero, we conclude that

$$c_1^2 = -1.$$

The support of the sheaf $F$ lies in the curve $C$. Furthermore, since $F$ is rigid, it does not have zero-dimensional torsion subsheaves (cf. Corollary 2.7). Suppose that the curve $C$ is not irreducible. Consider its irreducible component $C_0$ and the exact sequence

$$0 \to F_1 \to F \to F_0 \to 0$$

given by restriction to $C_0$ (here $\operatorname{supp} F_0 = C_0$, $\operatorname{supp} F_1 = C \setminus C_0$, and $F_1$ and $F_0$ do not have zero-dimensional torsion subsheaves). From this it follows that

$\operatorname{Hom}(F_1, F_0) = 0$, and by the above inequality $\operatorname{Ext}^2(F_0, F_1)$ also vanishes. Applying Lemma 2.2 b), we get the equalities

$$h^0(F, F) = h^0(F_0, F_0) + h^0(F_1, F_1) + \chi(F_0, F_1),$$
$$h^2(F, F) = h^2(F_0, F_0) + h^2(F_1, F_1) + \chi(F_1, F_0).$$

Since $F$ is an exceptional sheaf, $h^0(F, F) - h^2(F, F) = 1$. On the other hand,

$$\begin{aligned} h^0(F, F) - h^2(F, F) &= h^0(F_0, F_0) - h^2(F_0, F_0) + h^0(F_1, F_1) \\ &\quad - h^2(F_1, F_1) + \chi(F_0, F_1) - \chi(F_1, F_0) \\ &\geq 2 + \chi(F_0, F_1) - \chi(F_1, F_0) = 2. \end{aligned}$$

The last equality follows from the equalities

$$\chi(F_0, F_1) - \chi(F_1, F_0) = \big(r(F_0)c_1(F_1) - r(F_1)c_1(F_0)\big)(-K_S) = 0$$

(we used that $r(F_0) = r(F_1) = 0$). Thus the support of the sheaf $F$ is an irreducible curve $C$ with $C^2 = -1$, and therefore $F$ is a locally free sheaf of rank 1 on some $(-1)$-curve $C$. The lemma is proved.

We end our description of exceptional sheaves on del Pezzo surfaces with the following claim.

2.9. **Proposition.** *Let $F$ be an exceptional sheaf on a del Pezzo surface $S$. Then $F$ is either locally free or is a torsion sheaf of the form $\mathscr{O}_C(d)$, where $C$ is a $(-1)$-curve.*

*Proof.* Assume the contrary. Then by Lemma 2.8 $F$ is not a torsion sheaf and there exists an exact sequence

$$0 \to TF \to F \to F' \to 0,$$

where $TF$ is the torsion subsheaf and $F'$ is the torsion-free quotient sheaf. Since $F$ is rigid, Corollary 2.7 shows that $TF$ and $F'$ also have this property. By Lemma 2.2 b) we have the following equalities:

$$h^0(F, F) = h^0(TF, TF) + h^0(F', F') + \chi(F', TF),$$
$$h^2(F, F) = h^2(TF, TF) + h^2(F', F') + \chi(TF, F').$$

As in the proof of the preceding lemma, we see that, since $F$ is exceptional,

$$\begin{aligned} 1 = h^0(F, F) - h^2(F, F) &\geq 2 + \chi(F', TF) - \chi(TF, F') \\ &= 2 + \big(r(F') \cdot c_1(TF) - r(TF) \cdot c_1(F')\big) \cdot (-K_S) \\ &= 2 + r(F') \cdot c_1(TF) \cdot (-K_S). \end{aligned}$$

But the linear system $|-K_S|$ is ample, and $c_1(TF)$ is an effective divisor. Hence $(-K_S) \cdot c_1(TF) > 0$, which yields a contradiction. Thus, if $F$ is not a torsion sheaf, then it is torsion free, and Corollary 2.3 shows that then it is locally free. The proposition is proved.

In conclusion of this section we prove a result concerning description of exceptional objects in the bounded derived category of coherent sheaves on a del Pezzo surface $S$. This category will be denoted by $\mathscr{D}^b(S)$. By exceptional object we mean an object $X$ in $\mathscr{D}^b(S)$ satisfying the following conditions:

a) $\operatorname{Hom}^0(X, X) = \mathbb{C}$;
b) $\operatorname{Ext}^i(X, X) = 0$ for $i \neq 0$.

This is a natural generalization of the notion of exceptional sheaf to arbitrary objects of the category $\mathscr{D}^b(S)$. It is clear that any exceptional sheaf is an exceptional object. It turns out that on the del Pezzo surface the converse is also true, that is, all exceptional objects are sheaves. This last assertion is wrong for many other surfaces. The simplest example is given by the scroll $\mathbb{F}_2$. More precisely, the following is true.

2.10. **Proposition.** *An object $A$ of the derived category $\mathscr{D}^b(S)$ is exceptional if and only if it is isomorphic to $\delta E[i]$ for some exceptional sheaf $E$ on $S$ (here $\delta$ is the canonical inclusion of the category of coherent sheaves in the derived category).*

*Remark.* In other words, in this case $A$ is a complex with only one nontrivial cohomology sheaf, and this sheaf is isomorphic to an exceptional sheaf $E$.

*Proof.* We need to verify that only one cohomology sheaf is nontrivial. Put $H^i = H^i(A)$. Consider the spectral sequence converging to $\operatorname{Hom}^{p+q}(A, A)$ whose $E_1$ term is

$$E_1^{pq} = \bigoplus_i \operatorname{Ext}^{2p+q}(H^i, H^{i-p})$$

(cf. [3], [4]). In our case the nonzero terms of the spectral sequence lie in the strip $0 \le 2p+q \le 2$:

$$\begin{array}{ccccccc}
* & * & \uparrow q & & & & \\
0 & * & 0 & & \xrightarrow{d} & & \\
0 & * & * & 0 & & & \\
 & 0 & * & 0 & & & \\
 & 0 & * & * & 0 & \longrightarrow & \\
 & & 0 & * & 0 & p & \\
 & & 0 & * & * & & 
\end{array}$$

Since $A$ is a rigid object, from this it is clear that $E_1^{01} = E_\infty^{01} = 0$. Therefore, all the sheaves $H^i$ are rigid, i.e.,

$$h^1(H^i, H^i) = 0.$$

Since $H^i$ are rigid, by 2.6 and 2.7 we have the following inequalities:

$$h^2(H^{i+1}, H^i) \le h^0(H^i, H^{i+1}), \qquad h^0(H^i, H^i) > h^2(H^i, H^i).$$

Taking into consideration that the spectral sequence degenerates at the term $E_2$, i.e., $E_2 = E_\infty$, and that $A$ is an exceptional object, we conclude that the differential

$$d^{-1,2}\colon \bigoplus_i \operatorname{Hom}(H^i, H^{i+1}) \to \bigoplus_i \operatorname{Ext}^2(H^i, H^i)$$

is an isomorphism and the differential

$$d^{0,0}\colon \bigoplus_i \operatorname{Hom}(H^i, H^i) \to \bigoplus_i \operatorname{Ext}^2(H^{i+1}, H^i)$$

is an epimorphism whose kernel is at most one-dimensional. But these conditions are compatible with the inequalities only if at most one $H^i$ is nontrivial. This completes the proof of the proposition.

2.11. **Corollary.** *If $(E, F)$ is an exceptional pair of sheaves on a del Pezzo surface $S$, then at most one of the spaces* $\operatorname{Ext}^i(E, F)$ *is nontrivial; furthermore, for this space $i \ne 2$.*

*Proof.* Since the pair $(E, F)$ is exceptional, we have $h^0(F, E) = 0$. On the other hand, we know that $h^0(F, E) \ge h^2(E, F)$; hence only $h^0(E, F)$ and $h^1(E, F)$ can be nontrivial. Then the left transformation in the derived category is given by the following five-term sequence:

$$0 \to \lambda_E^0 F \to \operatorname{Hom}(E, F) \otimes E \to F \to \lambda_E^1 F \to \operatorname{Ext}^1(E, F) \otimes E \to 0.$$

By the above proposition, only one of the sheaves $\lambda_E^i F$ is nontrivial. If $\lambda_E^1 F = 0$, then $\operatorname{Ext}^1(E, F) = 0$ and everything is proved.

Suppose that $\lambda_E^0 F = 0$. We split the above sequence into two triples

$$0 \to \operatorname{Hom}(E, F) \otimes E \to F \to Q \to 0,$$
$$0 \to Q \to L_E F \to \operatorname{Ext}^1(E, F) \otimes E \to 0.$$

We apply the functor $\operatorname{Hom}(F, *)$ to the first of these triples. Since $\operatorname{Ext}^{\cdot}(F, E)$ is trivial, we get

$$\operatorname{Ext}^i(F, Q) = \operatorname{Ext}^i(F, F) = \begin{cases} \mathbb{C} & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases}$$

Next we apply the functor $\operatorname{Hom}(E, *)$ to the second triple. Then we get

$$\operatorname{Ext}^i(E, Q) = \begin{cases} 0 & \text{for } i = 0, \\ \operatorname{Ext}^1(E, F) & \text{for } i = 1, \\ 0 & \text{for } i = 2. \end{cases}$$

Finally, applying the functor $\operatorname{Hom}(*, Q)$ to the first triple and using the above equalities, we see that

$$\operatorname{Ext}^i(Q, Q) = \begin{cases} \mathbb{C} & \text{for } i = 0, \\ 0 & \text{for } i = 1, \\ \operatorname{Ext}^1(E, F) \otimes \operatorname{Hom}(E, F) & \text{for } i = 2. \end{cases}$$

But, as we already know, $h^0(Q, Q) > h^2(Q, Q)$, and if both $\operatorname{Hom}(E, F)$ and $\operatorname{Ext}^1(E, F)$ are not trivial, we arrive at a contradiction. This completes the proof of the corollary.

## § 3. Restriction of exceptional sheaves to rational and elliptic curves

3.0. In this section we prove technical lemmas on restrictions of exceptional bundles on del Pezzo surfaces to rational and elliptic curves. These lemmas are used in the proof of our main results.

It is known [4] that an exceptional bundle on an arbitrary del Pezzo surface $S$ is stable with respect to $|-K_S|$ in the sense of Mumford-Takemoto. Let $\mu(E)$ denote the slope of the bundle $E$,

$$\mu(E) = -\frac{(c_1(E) \cdot K)}{r(E)} \qquad (K = K_S).$$

3.1. **Lemma.** *Let $R$ be a rational curve on a del Pezzo surface $S$ satisfying the inequality $-R \cdot K \leq K^2$ (e.g., a $(-1)$-curve), and let $E$ be an exceptional bundle on $S$. Then the restriction of the bundle $E$ to $R$ has the form*

$$E' = E\big|_R = n\mathcal{O}_R(s) \oplus m\mathcal{O}_R(s+1).$$

*Proof.* Consider the tensor product of the restriction sequence with $E^* \otimes E$:

$$0 \to E^* \otimes E(-R) \to E^* \otimes E \to E^* \otimes E\big|_R \to 0.$$

Corresponding to it is the long exact cohomology sequence

$$\cdots \to \operatorname{Ext}^1(E, E) \to \operatorname{Ext}^1(E', E') \to \operatorname{Ext}^2(E, E(-R)) \to \cdots.$$

The group $\operatorname{Ext}^1(E, E)$ vanishes since $E$ is an exceptional bundle. By the Serre duality,

$$\operatorname{Ext}^2(E, E(-R))^* \cong \operatorname{Hom}(E, E(R+K)).$$

Furthermore,

$$\mu(E(R+K)) = \mu(E) - R \cdot K - K^2 \leq \mu(E)$$

by the hypothesis of the lemma. If $\mu(E(R+K)) < \mu(E)$, then $\operatorname{Hom}(E, E(R+K))$ is trivial since the exceptional bundle $E$ is stable.

Suppose that $\mu(E(R+K)) = \mu(E)$ and there exists a nonzero map $\varphi: E \to E(R+K)$. Then, since $E$ is locally free, the stability and equality of slopes imply that $\varphi$ is an isomorphism, which is clearly impossible.

Thus we have shown that

$$\operatorname{Ext}^1(E', E') = \operatorname{Ext}^2(E, E(-R)) = 0.$$

Hence the restriction of our exceptional bundle to the curve $R$ is rigid. On the other hand, by the Grothendieck theorem, each bundle on a rational curve is a direct sum of line bundles, viz. $E' = \bigoplus_i n_i \mathcal{O}_R(s_i)$. Since $E'$ is rigid, we conclude that $|s_i - s_j| \leq 1$. The lemma is proved.

3.2. **Corollary.** *Let $e$ be a $(-1)$-curve on a del Pezzo surface $S$, and let $S'$ be the surface obtained by blowing down this curve ($S \xrightarrow{\sigma} S'$). Let $E$ be an exceptional bundle on $S$ such that $c_1(E) \cdot e = 0$. Then there exists an exceptional bundle $F$ on the surface $S'$ such that $E = \sigma^* F$.*

*Proof.* From the preceding lemma and the equality $c_1(E) \cdot e = 0$ it follows that the restriction of the bundle $E$ to the curve $e$ is trivial. Hence there exists a bundle $F$ on $S'$ such that $E = \sigma^* F$. The fact that $F$ is exceptional follows from the equality $\operatorname{Ext}^i(F, F) = \operatorname{Ext}^i(\sigma^* F, \sigma^* F)$.

Next we determine the nature of splitting of bundles making up an exceptional pair under the restriction to a $(-1)$-curve.

3.3. **Lemma.** *Let $(E, F)$ be an exceptional pair of bundles on $S$ whose slopes satisfy the inequalities*

$$\mu(F) - K^2 < \mu(E) < \mu(F),$$

*and let $e$ be a $(-1)$-curve. Then there exists an integer $s$ such that*

$$E \oplus F\big|_e = n_1 \mathcal{O}_e(s) \oplus n_2 \mathcal{O}_e(s+1) \oplus n_3 \mathcal{O}_e(s+2)$$

*and*

$$E\big|_e = m_1 \mathcal{O}_e(s) \oplus m_2 \mathcal{O}_e(s+1),$$

*where $n_i$ and $m_j$ are nonnegative integers.*

*Proof.* Denote by $E'$ and $F'$ the restrictions to the curve $e$ of the bundles $E$ and $F$, respectively. We recall that, since the pair $(E, F)$ is exceptional, the inequalities for the slopes show that the groups $\operatorname{Ext}^i(F, E)$, $i = 0, 1, 2$, and $\operatorname{Ext}^j(E, F)$, $j = 1, 2$, are trivial and $\operatorname{Hom}(E, F) \neq 0$.

We claim that $\operatorname{Ext}^1(E', F') = 0$ and $\operatorname{Ext}^1(F'(-1), E') = 0$. The sequence

$$0 \to E^* \otimes F(-e) \to E^* \otimes F \to E^* \otimes F|_e \to 0$$

yields the following exact sequence:

$$\operatorname{Ext}^1(E, F) \to \operatorname{Ext}^1(E', F') \to \operatorname{Ext}^2(E, F(-e)).$$

Here $\operatorname{Ext}^1(E, F) = 0$ by our assumptions. By Serre's duality, we have

$$\operatorname{Ext}^2(E, F(-e))^* \cong \operatorname{Hom}(F, E(e+K)).$$

But $\mu(E(e+K)) = \mu(E) - e \cdot K - K^2 < \mu(F)$. Hence, by stability of exceptional bundles, $\operatorname{Hom}(F, E(e+K)) = 0$. Therefore

$$\operatorname{Ext}^1(E', F') = 0.$$

Using the Serre duality it is easy to show that the pair $(F(K), E)$ is also exceptional. Moreover, the slopes of bundles in this pair satisfy the inequalities from the statement of the lemma. Hence

$$\operatorname{Ext}^1(F'(-1), E') = \operatorname{Ext}^1(F(K)|_e, E|_e) = 0.$$

Since by Lemma 3.1 we know how exceptional bundles split under the restriction to the curve $e$, the assertion of the lemma follows immediately.

3.4. **Lemma.** *Let $(E, F)$ be an exceptional null-pair on a del Pezzo surface, that is, $\mu(E) = \mu(F)$ and $\operatorname{Ext}^i(E, F) = \operatorname{Ext}^i(F, E) = 0$ for $i = 0, 1, 2$, and let $e$ be a $(-1)$-curve. Then either*

$$E' \oplus F' = (E \oplus F)|_e = n_1\mathcal{O}_e(s) \oplus n_2\mathcal{O}_e(s+1)$$

*or*

$$(E, F) = (\mathcal{O}(D), \mathcal{O}(D+K+e))$$

*for some divisor $D$.*

*Proof.* We start with computing $\operatorname{Ext}^2(E, F(-e))$. By the Serre duality

$$\operatorname{Ext}^2(E, F(-e))^* \cong \operatorname{Hom}(F, E(e+K)).$$

Since $\mu(E(e+K)) = \mu(E) - K^2 + 1 \le \mu(F)$, we have $\operatorname{Ext}^2(E, F(-e)) = 0$ if $K^2 > 1$ or $K^2 = 1$ but $F \not\cong E(e+K)$.

In a similar way one can verify that under these conditions $\operatorname{Ext}^2(F, E(-e)) = 0$. Arguing as in Lemma 3.3, we deduce from the adjunction sequences

$$0 \to E^* \otimes F(-e) \to E^* \otimes F \to E^* \otimes F|_e \to 0,$$
$$0 \to F^* \otimes E(-e) \to F^* \otimes E \to F^* \otimes E|_e \to 0$$

that

$$\operatorname{Ext}^1(E', F') = \operatorname{Ext}^1(F', E') = 0,$$

i.e.,

$$E' \oplus F' = n_1\mathcal{O}_e(s) \oplus n_2\mathcal{O}_e(s+1).$$

Suppose now that $K^2 = 1$ and $F \cong E(e+K)$, so that the pair $(E, F)$ coincides with the pair $(F, E(e+K))$. By the Riemann-Roch theorem we have

$$\chi(E, E(e+K)) = \frac{r^2}{2}\left(\frac{2}{r^2} + \mu(E(e+K)) - \mu(E) + \left[\frac{c_1(E)}{r} - \frac{c_1(E) + r\cdot(e+K)}{r}\right]^2\right),$$

where $r = r(E)$. Furthermore, since $K^2 = 1$, we have $\mu(E(e+K)) = \mu(E)$. Hence

$$\chi(E, E(e+K)) = 1 + \frac{r^2}{2}(e+K)^2 = 1 - r^2.$$

On the other hand, by our assumption $\chi(E, E(e+K)) = 0$, i.e., $r(E) = 1$ and $(E, F) = (\mathcal{O}(D), \mathcal{O}(D+e+K))$ for some divisor $D$. The lemma is proved.

We remark that the order in an exceptional null-pair can be chosen arbitrarily. In what follows we will always put the pair $(\mathcal{O}(D), \mathcal{O}(D+e+K))$ in the reverse order, viz. $(\mathcal{O}(D+e+K), \mathcal{O}(D))$.

3.5. **Corollary.** *Let $(E, F)$ be an exceptional pair on a del Pezzo surface $S$ whose slopes satisfy the inequalities*

$$\mu(F) - K^2 < \mu(E) \leq \mu(F).$$

*Then either*

$$E \oplus F\big|_e = n_1\mathcal{O}_e(s) \oplus n_2\mathcal{O}_e(s+1)$$

*or*

$$E \oplus F(K)\big|_e = m_1\mathcal{O}_e(s) \oplus m_2\mathcal{O}_e(s+1).$$

*Proof.* If $\mu(E) < \mu(F)$, then this is an immediate consequence of Lemma 3.3. If $\mu(E) = \mu(F)$ and $(E, F) = (\mathcal{O}(D+e+K), \mathcal{O}(D))$, then $(F(K), e) = (\mathcal{O}(D+K), \mathcal{O}(D+e+K))$ has the required restriction to the curve.

The last two lemmas of the present section deal with restrictions of exceptional bundles on a del Pezzo surface $S$ to elliptic curves from the linear series $|-K_S|$.

3.6. **Lemma.** *Let $C \in |-K_S|$, and let $E$ be an exceptional bundle on $S$. Then $E' = E|_C$ is a simple bundle, i.e.,*

$$\operatorname{Hom}(E', E') = \mathbb{C}.$$

*Proof.* Consider the exact sequence

$$0 \to E^* \otimes E(K) \to E^* \otimes E \to E^* \otimes E|_C \to 0.$$

The corresponding long exact cohomology sequence has the form

$$0 \to \operatorname{Hom}(E, E(K)) \to \operatorname{Hom}(E, E) \to \operatorname{Hom}(E', E') \to \operatorname{Ext}^1(E, E(K)).$$

Since all exceptional bundles on $S$ are stable, we have $\operatorname{Hom}(E, E(K)) = 0$. Furthermore, from the definition of exceptional bundles it follows that

$$\operatorname{Hom}(E, E) = \mathbb{C}, \qquad \operatorname{Ext}^1(E, E(K))^* \cong \operatorname{Ext}^1(E, E) = 0.$$

Therefore, $\operatorname{Hom}(E', E') = \mathbb{C}$.

3.7. **Lemma.** *If the slopes of exceptional bundles $E_1$ and $E_2$ on a surface $S$ satisfy the inequality $\mu(E_2) \geq \mu(E_1)$ and $\operatorname{Ext}^1(E_2, E_1)$ is trivial, then*

$$\operatorname{Ext}^1(E_1, E_2) = 0.$$

*Proof.* Suppose first that $\mu(E_2) > \mu(E_1)$. The sequence

$$0 \to E_2^* \otimes E_1(K) \to E_2^* \otimes E_1 \to E_2^* \otimes E_1|_C \to 0$$

gives rise to a sequence

$$\to \operatorname{Hom}(E_2', E_1') \to \operatorname{Ext}^1(E_2, E_1(K)) \to \operatorname{Ext}^1(E_2, E_1),$$

where, as usual, $E_i'$ denotes the restriction of $E_i$ to the curve $C$.

By our assumption $\operatorname{Ext}^1(E_2, E_1) = 0$. We compute the slopes of $E_2'$ and $E_1'$. Since $r(E_i') = r(E_i)$ and $\deg(E_i') = c_1(E_i) \cdot C$, we have

$$\mu(E_i') = \frac{\deg(E_i')}{r(E_i')} = \mu(E_i).$$

By our hypothesis $\mu(E_2') > \mu(E_1')$. From the preceding lemma it follows that the bundles $E_i'$ are simple, and simple bundles on elliptic curves are stable. Hence $\operatorname{Hom}(E_2', E_1') = 0$. Therefore,

$$\operatorname{Ext}^1(E_1, E_2) \cong \operatorname{Ext}^1(E_2, E_1(K))^* = 0.$$

Suppose now that $\mu(E_2) = \mu(E_1)$. If $E_2 \cong E_1$, then it is clear that $\operatorname{Ext}^1(E_1, E_2)$ is trivial. Suppose that $E_1 \ncong E_2$. Then, since these bundles are stable and their slopes are equal, we have

$$\operatorname{Hom}(E_1, E_2) = \operatorname{Hom}(E_2, E_1) = 0,$$
$$0 = \operatorname{Ext}^2(E_1, E_2) = \operatorname{Ext}^2(E_2, E_1),$$

so that

$$\chi(E_2, E_1) = -h^1(E_2, E_1) = 0, \qquad \chi(E_1, , E_2) = -h^1(E_1, E_2).$$

On the other hand, since $\mu(E_1) = \mu(E_2)$, from the Riemann-Roch theorem it follows that the Euler characteristic is symmetric, that is,

$$\chi(E_1, E_2) = \chi(E_2, E_1).$$

The lemma is proved.

## § 4. Destabilizing filtrations

4.0. In this section we recall two destabilizing filtrations of sheaves. The first is a filtration of semistable sheaves with isotypic quotients, and the second is the canonical filtration of Harder-Narasimhan.

By stability in this section we understand stability in the sense of Gieseker. In this case the *slope of a sheaf* $E$ is a polynomial in the positive integral variable $n$

$$\gamma(E, n) = \frac{\chi(E \otimes (\omega_S^*)^{\otimes n})}{r(E)} = a_1(S)n^2 + (a_2(S) + \mu(E))n + a_3(S, E), \tag{4.1}$$

where $a_1(S)$ and $a_2(S)$ are constants depending only on the surface $S$.

We write $\gamma(E, n) > \gamma(F, n)$ if this inequality is satisfied for all sufficiently large $n$. Furthermore, for the sake of brevity we write $\gamma(E)$ instead of $\gamma(E, n)$.

4.1. *Remark.* From formula (4.1) it follows that the inequality $\gamma(E) > \gamma(F)$ is possible when $\mu(E) > \mu(F)$ as well as when $\mu(E) = \mu(F)$. If the Gieseker slopes of the sheaves $E$ and $F$ coincide, then their Mumford-Takemoto slopes are also equal. Moreover, from the same formula it follows that stability in the sense of Mumford-Takemoto implies stability in the sense of Gieseker, and semistability in the sense of Gieseker implies semistability in the sense of Mumford-Takemoto.

It is more convenient for us to use stability in the sense of Gieseker in view of the following simple result.

4.2. **Lemma.** *If there exists a nontrivial map $\varphi: F \to E$, where $E$ and $F$ are two sheaves with equal slopes that are semistable in the sense of Gieseker, and $E$ is stable, then $\varphi$ is an epimorphism.*

We remark that if one considers stability in the sense of Takemoto-Mumford, then one can only claim that $\varphi$ is surjective at a general point.

We list some more standard properties of those sheaves on a del Pezzo surface $S$ that are semistable in the sense of Gieseker.

4.3. **Lemma.** 1) *Let*

$$0 \to F' \to F \to F'' \to 0$$

*be an exact sequence of coherent sheaves on* $S$. *Then*

$$\begin{aligned} \gamma(F') > \gamma(F) &\quad \text{if and only if} \quad \gamma(F) > \gamma(F''), \\ \gamma(F') < \gamma(F) &\quad \text{if and only if} \quad \gamma(F) < \gamma(F''), \\ \gamma(F') = \gamma(F) &\quad \text{if and only if} \quad \gamma(F) = \gamma(F''). \end{aligned}$$

2) *If the slopes of (semi)stable sheaves* $E$ *and* $F$ *on* $S$ *satisfy the inequality* $\gamma(E) > \gamma(F)$, *then* $\operatorname{Hom}(E, F) = 0$.

3) *If the slopes of (semi)stable sheaves* $E$ *and* $F$ *on* $S$ *satisfy the inequality* $\gamma(E) \leq \gamma(F)$, *then* $\operatorname{Ext}^2(E, F) = 0$.

4) *Any stable sheaf* $F$ *is simple, that is,* $\operatorname{Hom}(F, F) = \mathbb{C}$.

5) *Two stable sheaves with the same slopes are either isomorphic or do not have nontrivial maps to each other.*

Next we show that semistable sheaves have filtrations with isotypic quotients.

4.4. **Proposition.** *For each semistable sheaf* $\mathscr{F}$ *there exists a filtration*

$$0 = \mathscr{F}_{n+1} \subset \mathscr{F}_n \subset \cdots \subset \mathscr{F}_2 \subset \mathscr{F}_1 = \mathscr{F}$$

*such that the quotients* $G_i = \mathscr{F}_i/\mathscr{F}_{i+1}$ *are semistable, their slopes satisfy the equalities* $\gamma(G_i) = \gamma(\mathscr{F}_i) = \gamma(\mathscr{F})$, $i = 1, \dots, n$, *and* $\operatorname{Hom}(\mathscr{F}_{i+1}, G_i) = 0$ *for all* $i$.

*In turn, each quotient* $G_i$ *has a filtration with stable quotients isomorphic to* $E_i$.

*Proof*. If the sheaf $\mathscr{F}$ is stable, then the filtration is trivial. Otherwise there exists a surjection $\mathscr{F} \twoheadrightarrow E$, where the rank of $E$ is smaller than that of $\mathscr{F}$ and $\gamma(\mathscr{F}) = \gamma(E)$. Let $E_1$ be such a quotient sheaf with the smallest possible rank. It is clear that $E_1$ is stable. We denote the kernel of the epimorphism $\mathscr{F} \twoheadrightarrow E_1$ by $\mathscr{F}_2^{\,1}$.

Since an arbitrary subsheaf of the sheaf $\mathscr{F}_2^{\,1}$ is a subsheaf of $\mathscr{F}$ and the slopes of $\mathscr{F}_2^{\,1}$ and $\mathscr{F}$ coincide (cf. Lemma 4.3, 1)), the sheaf $\mathscr{F}_2^{\,1}$ is semistable.

Suppose that there exists a nontrivial map $\varphi\colon \mathscr{F}_2^{\,1} \to E_1$. By Lemma 4.2, this map is an epimorphism. We denote by $\mathscr{F}_2^{\,2}$ the kernel of $\varphi$. We proceed in the same way until we construct a sheaf $\mathscr{F}_2^{\,k_1} = \mathscr{F}_2$ from which there are no nontrivial maps to $E_1$. We remark that the sheaf $\mathscr{F}_2$ may be trivial.

If the sheaf $\mathscr{F}_2$ is nontrivial, then we apply to it the same procedure as to $\mathscr{F} = \mathscr{F}_1$. Thus we obtain a sheaf $\mathscr{F}_3$. We proceed like that until we get $\mathscr{F}_{n+1} = 0$.

We show that the resulting filtration has the desired properties.

The semistability of the quotients and the equalities $\gamma(G_i) = \gamma(\mathscr{F}_i) = \gamma(\mathscr{F})$ follow from the semistability of the elements of the filtration, the equalities $\gamma(\mathscr{F}_i) = \gamma(\mathscr{F})$, and the exact sequences

$$0 \to \mathscr{F}_{i+1} \to \mathscr{F}_i \to G_i \to 0.$$

Taking the quotient sheaf $G_1 = \mathscr{F}_1/\mathscr{F}_2$ as an example, we show that all $G_i$ have filtrations with stable quotients isomorphic to $E_i$.

Put $Q_j = \mathscr{F}_1/\mathscr{F}_2^{\,j}$. Then for each $j$ there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & E_1 & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & \mathscr{F}_2^{\,j} & \longrightarrow & \mathscr{F}_1 & \longrightarrow & Q_j & \longrightarrow & 0 \\
 & & \uparrow & & \| & & & & \\
0 & \longrightarrow & \mathscr{F}_2^{\,j+1} & \longrightarrow & \mathscr{F}_1 & \longrightarrow & Q_{j+1} & \longrightarrow & 0 \\
 & & \uparrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

with exact rows and columns. Hence the sheaves $Q_j$ fit into exact sequences

$$0 \to E_1 \to Q_{j+1} \to Q_j \to 0.$$

Next we observe that $Q_1 = E_1$ and $Q_{k_1} = G_1$. Hence for each $j$ we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & G_1^{j+1} & \longrightarrow & G_1 & \longrightarrow & Q_j & \longrightarrow & 0 \\
 & & & & \| & & \uparrow & & \\
0 & \longrightarrow & G_1^{j+2} & \longrightarrow & G_1 & \longrightarrow & Q_{j+1} & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & E_1 & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

From this it is clear that the sheaves $G_1^j$ form a filtration of the sheaf $G_1$ such that $G_1^j/G_1^{j+1} \cong E_1$.

It remains to verify that $\operatorname{Hom}(\mathscr{F}_{i+1}, G_i) = 0$.

By the construction of our filtration, $\operatorname{Hom}(\mathscr{F}_{i+1}, E_i) = 0$ for all $i$. Applying the functor $\operatorname{Hom}(\mathscr{F}_{i+1}, *)$ consecutively to the exact triples

$$0 \to E_i \to G_i^{k_i-1} \to E_i \to 0,$$
$$0 \to G_i^{k_i-1} \to G_i^{k_i-2} \to E_i \to 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$0 \to G_i^2 \to G_i \to E_i \to 0,$$

we conclude that $\operatorname{Hom}(\mathscr{F}_{i+1}, G_i) = 0$.

In a similar way we construct a canonical Harder-Narasimhan filtration for an arbitrary torsion-free sheaf. We shall not go into boring details but only give the statement of the corresponding result.

4.5. **Proposition.** *An arbitrary torsion-free sheaf $\mathcal{F}$ has a canonical filtration*

$$0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_2 \subset \mathcal{F}_1 = \mathcal{F}$$

*with semistable quotients $G_i = \mathcal{F}_i/\mathcal{F}_{i+1}$ whose slopes satisfy the inequalities*

$$\gamma(\mathcal{F}_i) > \gamma(\mathcal{F}_j), \quad \text{and} \quad \gamma(\mathcal{F}_i) > \gamma(G_i) > \gamma(G_j) \quad \text{for } i > j.$$

*Furthermore, for this filtration*

$$\operatorname{Hom}(\mathcal{F}_i, G_j) = 0 = \operatorname{Ext}^2(G_j, \mathcal{F}_i).$$

## § 5. Rigid bundles on a del Pezzo surface

5.0. The goal of this section is to show that an arbitrary torsion-free rigid sheaf on a del Pezzo surface (such a sheaf is necessarily locally free, so one can speak about rigid bundles) splits into a direct sum of exceptional sheaves.

The idea of the proof is to show that a rigid bundle is a direct sum of quotients of a destabilizing filtration. To this end, we compute the groups $\operatorname{Ext}^1$ for these quotients and apply the following result.

5.1. **Lemma.** *If a sheaf $\mathcal{F}$ has a filtration*

$$0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_2 \subset \mathcal{F}_1 = \mathcal{F}$$

*whose quotients $Q_i = \mathcal{F}_i/\mathcal{F}_{i+1}$ satisfy the condition $\operatorname{Ext}^1(Q_i, Q_j) = 0$ for $i < j$, then $\mathcal{F} = Q_1 \oplus \cdots \oplus Q_n$.*

We leave the proof of this lemma to the reader as an easy exercise.

In what follows by stability we mean stability in the sense of Gieseker.

5.2. **Theorem.** *An arbitrary rigid bundle $\mathcal{F}$ on a del Pezzo surface splits into a direct sum of exceptional bundles.*

*Proof.* We consider three cases:

1) $\mathcal{F}$ is a rigid semistable bundle possessing a filtration with stable quotients isomorphic to each other;
2) $\mathcal{F}$ is a rigid semistable bundle;
3) $\mathcal{F}$ is an arbitrary rigid bundle.

5.2.1. Case 1). *Suppose that the quotients $G_i$ of a rigid sheaf are isomorphic to a stable sheaf $E$. We show that $E$ is an exceptional sheaf and $\mathcal{F} = E \oplus \cdots \oplus E$.*

Consider the spectral sequence associated with the filtration, with $E_1$-term

$$E_1^{pq} = \bigoplus_i \operatorname{Ext}^{p+q}(G_i, G_{i+p}),$$

which sequence converges to $\operatorname{Ext}^{p+q}(\mathcal{F}, \mathcal{F})$. Since all $G_i = E$ are stable, $\operatorname{Ext}^2(G_i, G_j) = 0$ for all $i$ and $j$. Hence, the $E_1$-term of the spectral sequence has the form

$$
\begin{array}{ccccccc}
 & & & \uparrow q & & & \\
* & & 0 & 0 & & & \\
* & & * & 0 & & \xrightarrow{d} & \\
0 & & * & * & 0 & & \\
0 & & 0 & * & * & 0 & \longrightarrow p \\
 & & & 0 & * & * & \\
 & & & & 0 & * &
\end{array}
$$

It is easy to see that

$$E_1^{1-n,n} = E_\infty^{1-n,n} \subset \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) = 0.$$

Hence

$$\operatorname{Ext}^1(G_n, G_1) = \operatorname{Ext}^1(E, E) = 0.$$

Therefore, $\operatorname{Ext}^1(G_i, G_j) = 0 \ \forall i, j$. From this it follows that $E$ is rigid and $\mathcal{F} = E \oplus \cdots \oplus E$. Moreover, since $E$ is stable, it is simple. That means that $E$ is an exceptional sheaf.

5.2.2. Case 2). *Let $\mathcal{F}$ be a semistable rigid sheaf. Consider the filtration from Proposition* 4.4.

Step 1. *The members and quotients of this filtration are rigid sheaves.*

*Proof*. Consider the exact sequences

$$0 \to \mathcal{F}_{i+1} \to \mathcal{F}_i \to G_i \to 0.$$

By Proposition 4.4 we have $\operatorname{Hom}(\mathcal{F}_{i+1}, G_i) = 0$. Furthermore, since the sheaves $\mathcal{F}_{i+1}$ and $G_i$ are semistable and $\gamma(\mathcal{F}_{i+1}) = \gamma(G_i)$, from Lemma 4.3, 3) it follows that the group $\operatorname{Ext}^2(G_i, \mathcal{F}_{i+1})$ is also trivial. Hence one can apply Lemma 2.2, 2), from which it follows that

$$h^1(\mathcal{F}_i, \mathcal{F}_i) \geq h^1(\mathcal{F}_{i+1}, \mathcal{F}_{i+1}) + h^1(G_i, G_i).$$

Moreover, since $\mathcal{F}_1 = \mathcal{F}$ is a rigid sheaf, the sheaves $\mathcal{F}_i$ and $G_i$ are also rigid for all $i$.

Step 2. *The quotients of the filtration split into a direct sum of exceptional sheaves isomorphic to each other.*

In fact, at the preceding step we have shown that the sheaves $G_i$ are rigid. Moreover, by Proposition 4.4 these sheaves have the same filtration as in Case 1).

Step 3. $\operatorname{Ext}^1(G_n, G_1) = 0$, *where $n$ is the number of quotients in the filtration of the rigid semistable sheaf $\mathcal{F}$*.

*Proof*. Consider the spectral sequence associated with the filtration of the rigid sheaf $\mathscr{F}$, with $E_1$-term

$$E_1^{pq} = \bigoplus_i \operatorname{Ext}^{p+q}(G_i, G_{i+p}).$$

Since the sheaves $G_i$ are semistable and their slopes coincide, $\operatorname{Ext}^2(G_i, G_j) = 0$ for any pair of indices $i$ and $j$, i.e., the $E_1$-term of the spectral sequence has the same form as in Case 1). Hence, as above,

$$\operatorname{Ext}^1(G_n, G_1) = 0.$$

Step 4. $\operatorname{Ext}^1(G_1, G_n) = 0$.

*Proof*. By Step 2, we have the following decompositions:

$$G_n = E_n \oplus \cdots \oplus E_n = sE_n,$$
$$G_1 = E_1 \oplus \cdots \oplus E_1 = kE_1.$$

Since $\operatorname{Ext}^1(G_n, G_1) = 0$, we have $\operatorname{Ext}^1(E_n, E_1) = 0$. As was proved above, $E_n$ and $E_1$ are exceptional sheaves and their Gieseker slopes are the same. From this it follows that $\mu(E_n) = \mu(E_1)$. Now from Lemma 3.7 it follows that the space $\operatorname{Ext}^1(E_1, E_n)$ is trivial, and therefore $\operatorname{Ext}^1(G_1, G_n) = 0$.

Step 5. $\mathscr{F}$ *is a direct sum of exceptional sheaves.*

*Proof*. We prove the claim by induction on rank. The first induction step is obvious. As was shown in Step 1, the sheaf $\mathscr{F}_2$ (the second term of the filtration) is rigid and $r(\mathscr{F}_2) < r(\mathscr{F})$. By the induction hypothesis, this sheaf splits into a direct sum of exceptional sheaves. It is easy to see that the quotients $G_n, \dots, G_1$ of the filtration are its direct summands, i.e.,

$$\mathscr{F}_2 = G_2 \oplus \cdots \oplus G_n.$$

Consider the exact sequence

$$0 \to \mathscr{F}_2 \to \mathscr{F} \to G_1 \to 0.$$

Since $\operatorname{Ext}^1(G_1, G_n) = 0$, the sheaf $G_n$ is a direct summand of $\mathscr{F}$, that is, $\mathscr{F} = G_n \oplus \mathscr{F}'$, where $\mathscr{F}'$ is again a rigid semistable sheaf whose rank is less than that of $\mathscr{F}$. Applying the induction hypothesis to the sheaf $\mathscr{F}'$, we see that this sheaf, and therefore the sheaf $\mathscr{F}$, is a direct sum of exceptional sheaves.

5.2.3. Case 3). $\mathscr{F}$ *is an arbitrary rigid sheaf on a del Pezzo surface.*

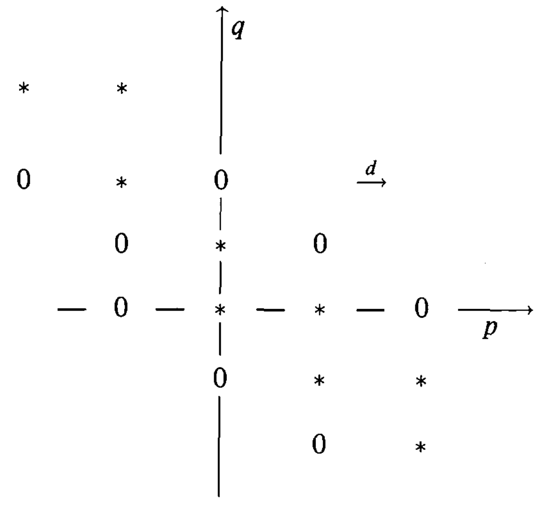
*Proof*. Consider the canonical destabilizing filtration of the sheaf $\mathscr{F}$ constructed in Proposition 4.5. The slopes of its semistable quotients $G_i = \mathscr{F}_i/\mathscr{F}_{i+1}$ satisfy the inequalities

$$\gamma(\mathscr{F}_1) > \gamma(G_i) > \gamma(G_{i-1}).$$

By Lemma 4.3, the sheaves $G_i$ satisfy the conditions

$$\operatorname{Hom}(G_i, G_j) = 0 \quad \text{for } i > j,$$
$$\operatorname{Ext}^2(G_i, G_j) = 0 \quad \text{for } i \leq j.$$

Hence the $E_1$-term of the spectral sequence associated with this filtration has the form



From this it follows that

$$E_{\infty}^{-1\,2} = E_1^{-1\,2} = \bigoplus_i \operatorname{Ext}^1(G_i, G_{i-1})$$

and

$$E_{\infty}^{0\,1} = E_1^{0\,1} = \bigoplus_i \operatorname{Ext}^1(G_i, G_i).$$

Since the spectral sequence converges to the groups $\operatorname{Ext}^i(\mathcal{F}, \mathcal{F})$ of the rigid sheaf $\mathcal{F}$, for each index we get the equalities

$$\operatorname{Ext}^1(G_i, G_{i-1}) = 0, \qquad \operatorname{Ext}^1(G_i, G_i) = 0.$$

This means that the sheaves $G_i$ are rigid, and since they are semistable, they can be represented as a direct sum of exceptional bundles, viz. $G_i = \bigoplus_s E_i^s$ (cf. Case 2)). Using the first equality, we get:

$$0 = \operatorname{Ext}^1(G_i, G_{i-1}) = \operatorname{Ext}^1\left(\bigoplus_s E_i^s, \bigoplus_k E_{i-1}^k\right).$$

Therefore, $\operatorname{Ext}^1(E_i^s, E_{i-1}^k) = 0$. Furthermore,

$$\gamma(E_i^s) = \gamma(G_i) > \gamma(G_{i-1}) = \gamma(E_{i-1}^k).$$

Hence the Mumford-Takemoto slopes of these sheaves satisfy the inequality

$$\mu(E_i^s) \geq \mu(E_{i-1}^k).$$

Now one can apply Lemma 3.7 to the exceptional sheaves $E_i^s$ and $E_{i-1}^k$ to show that the space $\operatorname{Ext}^1(E_{i-1}^k, E_i^s)$ is trivial. Hence

$$\operatorname{Ext}^1(G_{i-1}, G_i) = 0.$$

Now we proceed by induction on rank, as in the second case. From Proposition 4.5 it follows in particular that the sheaves $\mathcal{F}_1 = \mathcal{F}$ and $G_1$ fit into an exact sequence

$$0 \to \mathcal{F}_2 \to \mathcal{F} \to G_1 \to 0.$$

Since $G_1$ is semistable, from §4.5 and the inequalities for slopes it follows that the groups $\operatorname{Ext}^2(G_1, \mathcal{F}_2)$ and $\operatorname{Hom}(\mathcal{F}_2, G_1)$ are trivial. By Lemma 2.2, 2), rigidity of the sheaf $\mathscr{F}$ implies rigidity of the sheaf $\mathscr{F}_2$. By the induction hypothesis, $\mathscr{F}_2$ is a direct sum of exceptional sheaves. In particular, $\mathscr{F}_2 = G_n \oplus \cdots \oplus G_2$.

On the other hand, we have already shown that the group $\mathrm{Ext}^1(G_1, G_2)$ is trivial. Hence $\mathscr{F} = \mathscr{F}' \oplus G_2$, where $\mathscr{F}'$ is also a rigid torsion-free sheaf, which, by the induction hypothesis, splits into a direct sum of exceptional sheaves. This completes the proof of the theorem.

## § 6. Exceptional collections

6.0. In this section we show that each exceptional collection on a del Pezzo surface is a part of a complete exceptional collection. For the projective plane $\mathbb{P}^2$ and the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ this was shown in [5] and [9], respectively. Therefore, we will assume that our del Pezzo surface $S$ is the blowup of $\mathbb{P}^2$ at $d$ points, where $d \leq 8$. We will prove this assertion by induction on $d$, starting with $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

Consider an exceptional collection $\sigma = (E_1, \dots, E_k)$, where $E_i$ are exceptional sheaves on the surface $S$. We note that if the collection $\sigma$ is a part of a complete exceptional collection, then each collection $\sigma'$ obtained from $\sigma$ by a sequence of transformations also is a part of a complete exceptional collection.

6.1. **Definition.** Two exceptional collections $\sigma$ and $\sigma'$ of the same length are called *constructively equivalent* if they are obtained from each other by a sequence of transformations.

It is easy to check that this is actually an equivalence relation.

Moreover, it is clear that if a collection $\sigma$ is a part of a complete collection, then each collection $\sigma(H) = (E_1(H), \dots, E_k(H))$ obtained by twisting by an invertible sheaf $\mathscr{O}_S(H)$ also is a part of a complete collection.

The third type of operations to be considered is the following.

From a collection $\sigma = (E_1, \dots, E_k)$ we construct the collections

$$L\sigma = \big(E_k(K), E_1, \dots, E_{k-1}\big) \quad \text{and} \quad R\sigma = \big(E_2, \dots, E_k, E_1(-K)\big).$$

These collections are also exceptional. Moreover, if $\sigma$ is a part of a complete exceptional collection, then $L\sigma$ and $R\sigma$ also have this property. The last assertion follows from the fact that if $\sigma$ is a complete collection, then $L\sigma$ is obtained from $\sigma$ by a sequence of transformations, viz. by shifting $E_k$ through all $E_i$ (cf. [2]).

6.2. **Definition.** Two collections $\sigma$ and $\sigma'$ are called *equivalent* if they are obtained from each other by a sequence of operations of the three types described above.

Consider an exceptional collection $\sigma = (E_1, \dots, E_k)$. If some $E_i$ is a torsion sheaf, then there exists a collection $\sigma'$, equivalent to $\sigma$, such that $E_1' = \mathscr{O}_e(-1)$, where $e$ is a $(-1)$-curve. In this case all the sheaves $E_i'$ ($i > 1$) are lifted from the surface $S'$ obtained by blowing down the curve $e$, and we can proceed by induction. Hence in what follows we will assume that all the sheaves in the collection $\sigma$ are bundles, so that the slopes $\mu(E_i)$ are well defined for all $E_i$.

Denote by $\mu_0(\sigma)$ and $\mu_1(\sigma)$ the minimum and maximum value of the slopes $\mu(E_i)$, viz.

$$\mu_0(\sigma) = \big\{\min_i \mu(E_i) \mid E_i \in \sigma\big\},$$

$$\mu_1(\sigma) = \big\{\max_i \mu(E_i) \mid E_i \in \sigma\big\}.$$

We recall that if $(E, F)$ is an exceptional pair of bundles on a del Pezzo surface $S$, then there are three possibilities:

a) $\mu(E) < \mu(F)$, and so $(E, F)$ is a pair of type Hom;

b) $\mu(E) > \mu(F)$, and so $(E, F)$ is a pair of type Ext;
c) $\mu(E) = \mu(F)$, and so $(E, F)$ is a totally orthogonal pair, i.e., the pair $(F, E)$ is also exceptional (in what follows we will assume that this pair also has type Hom).

6.3. **Lemma.** *Let $(E, F)$ be an exceptional pair of type* Ext. *Then $(L_E F, E)$ is a pair of type* Hom *and*

$$\mu(F) < \mu(L_E F) < \mu(E),$$

*where $L_E F$ is the exceptional bundle obtained by transforming $F$ with the help of $E$, that is, we have an exact sequence*

$$0 \to F \to L_E F \to \operatorname{Ext}^1(E, F) \otimes E \to 0.$$

*Proof.* The above exact sequence is the definition of a transformation for the pair $(E, F)$. The inequalities for the slopes are verified by an elementary computation, which we leave to the reader.

6.4. **Lemma.** *Let $\sigma$ be an exceptional collection of bundles. Then there exists a collection $\sigma'$ of type* Hom *which is constructively equivalent to $\sigma$ and has the following properties*:

$$\mu_0(\sigma) \le \mu_0(\sigma'), \qquad \mu_1(\sigma') \le \mu_1(\sigma).$$

*Reminder.* A collection $\sigma$ is called a *collection of type* Hom if each pair $(E_i, E_j)$ is a pair of type Hom.

*Proof.* We prove the lemma by induction on the length of a collection. The first induction step is given by Lemma 6.3. Consider the subcollection $\tau = (E_1, \dots, E_{k-1})$. By the induction hypothesis there exists a subcollection $\tau' = (E'_1, \dots, E'_{k-1})$ of type Hom that is constructively equivalent to $\tau$. Then the collection $\tilde{\sigma} = (E'_1, \dots, E'_{k-1}, E_k)$ is constructively equivalent to $\sigma$, and by the induction hypothesis

$$\mu_0(\sigma) \le \mu_0(\tilde{\sigma}), \qquad \mu_1(\tilde{\sigma}) \le \mu_1(\sigma).$$

Furthermore, the slopes $\mu(E'_i)$ satisfy the inequalities

$$\mu(E'_1) \le \cdots \le \mu(E'_{k-1}).$$

If $\mu(E_k) \ge \mu(E'_{k-1})$, then we take $\sigma'$ to be $\tilde{\sigma}$, and if $\mu(E_k) < \mu(E'_{k-1})$, then we consider the transformation of $E_k$ with the help of $E'_{k-1}$. Then $(L_{E'_{k-1}} E_k, E'_{k-1})$ is a pair of type Hom and

$$\mu_0(\tilde{\sigma}) \le \mu(E_k) < \mu(L_{E'_{k-1}} E_k) < \mu(E'_{k-1}) = \mu_1(\tilde{\sigma}).$$

We continue shifting $E_k$ to the left in this way until we get a collection $\sigma' = (E'_1, \dots, E'_i, F, E'_{i+1}, \dots, E'_{k-1})$ such that $(E'_i, F)$ is a pair of type Hom. Then

$$\mu(E'_i) \le \mu(F) < \mu(E'_{i+1}),$$

and we get a collection $\sigma'$ of type Hom satisfying the conditions

$$\mu_0(\sigma) \le \mu_0(\sigma'), \qquad \mu_1(\sigma') \le \mu_1(\sigma).$$

The lemma is proved.

The following result shows that it is possible to find an equivalent collection of type Hom whose slopes are sufficiently close to each other.

6.5. **Claim.** *For each exceptional collection of bundles* $\sigma$ *there exists an equivalent collection* $\bar{\sigma}$ *of type* Hom *such that*

$$\mu_1(\bar{\sigma}) - K^2 < \mu_0(\bar{\sigma}).$$

*Proof.* By Lemma 6.4 we can reduce $\sigma$ to a collection of type Hom without increasing the difference $\mu_1(\sigma) - \mu_0(\sigma)$. Suppose that $\sigma$ itself is a collection of type Hom. Then we have

$$\mu_0(\sigma) = \mu(E_1) \leq \cdots \leq \mu(E_k) = \mu_1(\sigma).$$

If $\mu_1(\sigma) - \mu_0(\sigma) \geq K^2$, then we consider the collection $L\sigma = (E_k(K), E_1, \dots, E_{k-1})$. For this collection

$$\mu_0(L\sigma) = \mu_0(\sigma) = \mu(E_1), \qquad \mu_1(L\sigma) = \mu(E_{k-1}).$$

There exists a number $i$ such that

$$\mu(E_i) \leq \mu(E_k) - K^2 = \mu(E_k(K)) < \mu(E_{i+1}).$$

If $\mu_1(L\sigma) - \mu_0(L\sigma) \geq K^2$, then we consider the collection $L(L\sigma)$ and proceed like that until either the difference $(\mu_1 - \mu_0)$ becomes less than $K^2$ or we get a collection $\sigma_1 = (E_{i+1}(K), \dots, E_k(K), E_1, \dots, E_i)$ for which

$$\mu_0(\sigma_1) = \mu_0(\sigma) = \mu(E_1), \qquad \mu_1(\sigma_1) = \mu(E_k(K)) = \mu_1(\sigma) - K^2.$$

In the last case we get a collection $\sigma_1$ for which the difference $\mu_1(\sigma_1) - \mu_0(\sigma_1)$ is equal to $\mu_1(\sigma) - \mu_0(\sigma) - K^2$. By Lemma 6.4, one can construct a collection $\sigma_1'$ of type Hom that is constructively equivalent to $\sigma_1$. Repeating this procedure a sufficient number of times, we finally get a collection $\bar{\sigma}$ of type Hom such that

$$\mu_1(\bar{\sigma}) - K^2 < \mu_0(\bar{\sigma}),$$

which completes the proof of our claim.

6.6. **Claim.** *For each exceptional collection of bundles* $\sigma$ *there exists an equivalent collection* $\sigma'$ *of type* Hom *such that for a given* $(-1)$*-curve* $e$ *one has*

$$(E_1' \oplus \cdots \oplus E_k')\big|_e = n\mathscr{O}_e(-1) \oplus m\mathscr{O}_e,$$

*where* $n$ *and* $m$ *are nonnegative integers.*

*Proof.* By Claim 6.5 we can find an exceptional collection $\bar{\sigma}$ of type Hom that is equivalent to $\sigma$ and satisfies the inequality $\mu_1(\bar{\sigma}) - K^2 < \mu_0(\bar{\sigma})$. Twisting this collection, if necessary, we get a collection $\tau$ whose restriction to the curve $e$ satisfies an additional condition. To wit, let $\tau = (F_1, \dots, F_k)$. Then

$$(F_1 \oplus \cdots \oplus F_k)\big|_e = n_1\mathscr{O}_e(-1) \oplus n_2\mathscr{O}_e \oplus n_3\mathscr{O}_e(1), \qquad n_1 \neq 0.$$

Since each pair $(F_i, F_j)$ with $i < j$ satisfies the assumptions of either Lemma 3.3 or Lemma 3.5, there exists an $i$ such that the restriction of $F_1 \oplus \cdots \oplus F_i$ to the curve $e$ has the form

$$(F_1 \oplus \cdots \oplus F_i)\big|_e = n'\mathscr{O}_e(-1) \oplus m'\mathscr{O}_e$$

and the restriction of $F_{i+1} \oplus \cdots \oplus F_k$ to $e$ has the form

$$(F_{i+1} \oplus \cdots \oplus F_k)\big|_e = n''\mathscr{O}_e \oplus m''\mathscr{O}_e(1).$$

(We remark that $i$ may be equal to $k$, and then the collection $\tau$ itself is the required collection; moreover, the number $i$ satisfying the above conditions may not be unique.)

Now the collection $\sigma'$ is obtained from $\tau$ by shifting $F_k, \dots, F_{i+1}$ to the left, i.e.,

$$\sigma' = (F_{i+1}(K), \dots, F_k(K), F_1, \dots, F_i).$$

It is clear that the restriction of this collection to $e$ satisfies the above condition. Furthermore, since the slopes satisfy the inequality

$$\mu(F_k(K)) = \mu(F_k) - K^2 = \mu_1(\tau) - K^2 < \mu_0(\tau) = \mu(F_1),$$

$\sigma'$ is a collection of type Hom. This completes the proof of our claim.

6.7. It will be convenient to introduce the following definitions.

**Definition.** A *sheaf* $F$ is called *superrigid* if $\operatorname{Ext}^i(F, F) = 0$ for all $i > 0$.

We remark that a superrigid sheaf is different from an exceptional sheaf in that we do not require it to be simple. For example, a sum of sheaves of an exceptional collection of type Hom is a superrigid sheaf. From the preceding section it follows that the converse is also true on del Pezzo surfaces.

One can also introduce the notion of superrigidity for objects in the derived category.

6.8. **Definition.** An object $A$ is called *superrigid* if $\operatorname{Hom}^i(A, A) = 0$ for each $i \neq 0$.

It is convenient to formulate the following lemmas using the language of derived categories.

6.9. **Lemma.** *Suppose that two superrigid objects $A$ and $B$ in the derived category satisfy the following conditions*:

a) $\operatorname{Hom}^i(A, B) = 0$ *for* $i \neq 0$;

b) $\operatorname{Hom}^i(B, A) = 0$ *for* $i \neq 1$.

*Consider the following distinguished triangle in the derived category*:

$$A \to \operatorname{Hom}^0(A, B)^* \otimes B \to C.$$

*Then $B \oplus C$ is a superrigid object. Moreover,* $\operatorname{Hom}^i(C, B) = 0$ *for all $i$ if $B$ is a simple object.*

*Proof*. 1) Consider the functor $\operatorname{Hom}(*, B)$ and apply it to the triangle

$$A \to \operatorname{Hom}^0(A, B)^* \otimes B \to C.$$

We get a long exact sequence

$$0 \to \operatorname{Hom}^0(C, B) \to \operatorname{Hom}^0(A, B) \otimes \operatorname{Hom}^0(B, B) \to \operatorname{Hom}^0(A, B) \to \operatorname{Hom}^1(C, B) \to 0.$$

From this it follows that $\operatorname{Hom}^1(C, B) = 0$, and if $B$ is simple, i.e., $\operatorname{Hom}^0(B, B) = \mathbb{C}$, then $\operatorname{Hom}^i(C, B) = 0$.

2) If we consider the functor $\operatorname{Hom}(B, *)$, then from the long exact sequence it follows that $\operatorname{Hom}^i(B, C)$ for $i \neq 0$.

3) Now, applying the functor $\operatorname{Hom}(*, A)$ to our triangle, we immediately see that $\operatorname{Hom}^i(C, A) = 0$ for $i \neq 1$.

By the above, considering the functor $\operatorname{Hom}(C, *)$, we get a long exact sequence

$$\to \operatorname{Hom}^0(A, B)^* \otimes \operatorname{Hom}^i(C, B) \to \operatorname{Hom}^i(C, C) \to \operatorname{Hom}^{i+1}(C, A) \to .$$

From this it follows that $H^i(C, C) = 0$ for $i \neq 0$, and therefore $C$ is a superrigid object. Combining these three results, we see that $B \oplus C$ is also superrigid. The lemma is proved.

is a direct sum of exceptional bundles that together with $\mathscr{O}_e(-1)$ form an exceptional collection, i.e.,

$$B = s_1 G_1 \oplus \cdots \oplus s_j G_j$$

and $(\mathscr{O}_e(-1), G_1, \dots, G_j)$ is an exceptional collection.

All the bundles $G_j$ are obtained by lifting exceptional bundles from the surface $S'$ obtained by blowing down the line $e$ on $S$. By the induction hypothesis, this collection is a part of a complete collection on $S'$, and so $(\mathscr{O}_e(-1), G_1, \dots, G_n)$ is a complete exceptional collection on $S$.

Consider the collection $\tau = (F_1, \dots, F_i, E_1, \dots, E_k, G_{j+1}, \dots, G_n)$.

1) We prove that this collection is exceptional. In fact, we know that the collection $(F_1, \dots, F_i, E_1, \dots, E_k)$ is exceptional. Consider the bundle $G_\alpha$, $j+1 \leq \alpha \leq n$. Since $\operatorname{Hom}^i(G_\alpha, B) = 0$ and $\operatorname{Hom}^i(G_\alpha, \mathscr{O}_e(-1)) = 0$ for all $i$, from the exact sequence

$$0 \to B \to A \to \operatorname{Hom}(A, \mathscr{O}_e(-1))^* \otimes \mathscr{O}_e(-1) \to 0$$

it follows that $\operatorname{Hom}^i(G_\alpha, A) = 0$. Furthermore, considering the sequence

$$0 \to C \to \operatorname{Hom}(A, \mathscr{O}_e(-1)) \otimes A \to \mathscr{O}_e(-1) \to 0$$

and using the same argument, we see that $\operatorname{Hom}^i(G_\alpha, C) = 0$. Hence the collection $\tau$ is exceptional.

2) To show that the collection $\tau$ is complete, we recall that the subcategory generated by an exceptional collection is admissible, i.e., this subcategory and its orthogonal generate the derived category. Hence to show that our collection is complete it suffices to verify that the left orthogonal to the subcategory $\mathscr{D}$ generated by the collection $\tau$ is trivial.

Consider an object $X$ from the left orthogonal to $\mathscr{D}$. Then $\operatorname{Hom}^i(X, A \oplus C) = 0$ for all $i$. Therefore, $\operatorname{Hom}^i(X, \mathscr{O}_e(-1)) = 0$ for all $i$. This follows from the first exact sequence.

From the second exact sequence it follows that $\operatorname{Hom}^i(X, B) = 0$ for all $i$. Hence $X$ lies in the left orthogonal to the collection $(\mathscr{O}_e(-1), G_1, \dots, G_n)$. Since this collection is complete, it follows that $X$ is equal to zero. This completes the proof of the theorem.

6.12. *Remark.* The first induction step in our proof of the theorem is furnished by the plane $\mathbb{P}^2$ and the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. We would like to point out that it suffices to consider only the projective plane $\mathbb{P}^2$. In fact, if our exceptional collection consists only of torsion sheaves, then there is no problem to supplement it to a complete exceptional collection. If our exceptional collection is not a collection of either bundles or torsion sheaves, we can apply transformations to replace it by a constructively equivalent collection consisting only of bundles. Next we fix a blowing down of our surface $S$ to $\mathbb{P}^2$ and proceed by induction on the number of blown up points on $S$, starting with $\mathbb{P}^2$.

Moreover, in the same way one can show that an arbitrary collection on $\mathbb{P}^1 \times \mathbb{P}^1$ is a part of a complete collection. To this end, it suffices to blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at a single point and to lift everything to the del Pezzo surface $X_2$ obtained by blowing up $\mathbb{P}^2$ at two points. On $X_2$ we consider the collection consisting of the collection lifted from $\mathbb{P}^1 \times \mathbb{P}^1$ and the torsion sheaf $\mathscr{O}_e$, where $e$ is the $(-1)$-line obtained by

Now we formulate another lemma. Since its proof is similar to that of the preceding lemma, we omit it.

6.10. **Lemma.** *Suppose that $A$ and $B$ are superrigid objects satisfying the following conditions*:

a) $\operatorname{Hom}^i(A, B) = 0$ *for* $i \neq 0$;
b) $\operatorname{Hom}^i(B, A) = 0$ *for* $i \neq 1$.

*Consider the distinguished triangle*

$$C \to \operatorname{Hom}^0(A, B) \otimes A \to B.$$

*Then the object $A \oplus C$ is also superrigid, and if in addition $A$ is simple, then* $\operatorname{Hom}^i(A, C) = 0$ *for all* $i$.

We now turn to the proof of the fact that each exceptional collection is a part of a complete exceptional collection.

6.11. **Theorem.** *On an arbitrary del Pezzo surface each exceptional collection is a part of a complete exceptional collection.*

*Proof.* As we already pointed out above, by the induction hypothesis we may assume that our collection consists of bundles. Furthermore, applying Claim 6.6 to a given exceptional curve $e$, we can find an equivalent exceptional collection of bundles $\sigma = (E_1, \dots, E_k)$ such that

$$(E_1 \oplus \cdots \oplus E_k)\big|_e = n\mathscr{O}_e(-1) \oplus m\mathscr{O}_e.$$

Denote $E_1 \oplus \cdots \oplus E_k$ by $A$. Then $A$ is a superrigid bundle and

a) $\operatorname{Hom}^i(A, \mathscr{O}_e(-1)) = 0$ for $i \neq 0$;
b) $\operatorname{Hom}^i(\mathscr{O}_e(-1), A) = 0$ for $i \neq 1$.

Consider the canonical short exact sequence

$$0 \to C \to \operatorname{Hom}\left(A, \mathscr{O}_e(-1)\right) \otimes A \to \mathscr{O}_e(-1) \to 0.$$

Then by Lemma 6.10 the bundle $A \oplus C$ is also superrigid. Hence from Theorem 5.2 it follows that this bundle is a sum of exceptional bundles, viz.

$$A \oplus C = n_1 F_1 \oplus \cdots \oplus n_i F_i \oplus m_1 E_1 \oplus \cdots \oplus m_k E_k.$$

Since $A \oplus C$ is not only rigid, but also superrigid, the collection

$$(F_1, \dots, F_i, E_1, \dots, E_k)$$

is exceptional (here we ordered the exceptional bundles in the decomposition of $A \oplus C$ by their slopes).

Consider now the canonical map

$$A \to \operatorname{Hom}\left(A, \mathscr{O}_e(-1)\right)^* \otimes \mathscr{O}_e(-1).$$

This map is surjective since it factors through the restriction of $A$ to the line $e$ and the map

$$A\big|_e \to \operatorname{Hom}\left(A, \mathscr{O}_e(-1)\right)^* \otimes \mathscr{O}_e(-1)$$

is surjective. Hence there is a short exact sequence

$$0 \to B \to A \to \operatorname{Hom}\left(A, \mathscr{O}_e(-1)\right)^* \otimes \mathscr{O}_e(-1) \to 0.$$

By Lemma 6.9, $B$ is a superrigid bundle, and since $\mathscr{O}_e(-1)$ is an exceptional sheaf, by the same lemma $\operatorname{Hom}^i(B, \mathscr{O}_e(-1)) = 0$ for all $i$. (We note that here $B$ corresponds to $B[-1]$ in the statement of the lemma.) Arguing as above, we see that the bundle $B$

blowing up $\mathbb{P}^1 \times \mathbb{P}^1$. Next we transform this collection into a collection of bundles and supplement it as above using only the results for $\mathbb{P}^2$. It is easy to see that, proceeding in this way, we prove the claim for the quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

## § 7. Constructibility of helixes

7.0. In this last section of our paper we show that an arbitrary complete exceptional collection of sheaves on a del Pezzo surface is equivalent to the collection $(\mathscr{O}_{e_1}(-1), \dots, \mathscr{O}_{e_d}(-1), \mathscr{O}_S, \mathscr{O}_S(1), \mathscr{O}_S(2))$. From this and Theorem 6.11 it follows that each exceptional sheaf is obtained by transformations from invertible sheaves and exceptional torsion sheaves.

As we have shown above, for an arbitrary exceptional and, in particular, complete collection there exists an equivalent collection $(F_1, \dots, F_n)$ such that the sheaf $\mathscr{F} = \bigoplus_{i=1}^n F_i$ fits into an exact sequence

$$0 \to x_2E_2 \oplus \cdots \oplus x_nE_n \to \mathscr{F} \to x_1\mathscr{O}_e(-1) \to 0 \tag{7.1}$$

and $(\mathscr{O}_e(-1), E_2, \dots, E_n)$ is an exceptional collection. Furthermore, $(E_2, \dots, E_n)$ is an exceptional collection of type Hom composed of sheaves lifted from a surface $S'$ under the map $S \to S'$ blowing down an exceptional curve $e$ to a point.

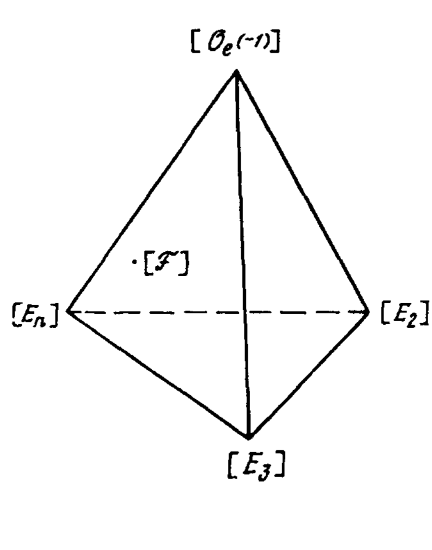
Moreover, using induction on the number of blown up points, one can assume that the sheaves $F_i$ are locally free (cf. 6.1). Then the sheaves $E_i$ are also locally free.

Since the collection $(F_1, \dots, F_n)$ is complete on $S$, the collection $(E_2, \dots, E_n)$ is complete on $S'$.

To clarify the idea of the proof that helixes are constructively equivalent, we consider the tensor product $K_0(S) \otimes Q = K$.

To each exceptional sheaf $F$ on $S$ there corresponds a vector $[F]$ in $K$. It is clear that the vectors in $K$ corresponding to an exceptional collection are linearly independent, and to a complete collection there corresponds a basis. The Euler characteristic $\chi(E, F)$ of sheaves is a bilinear form on $K$. Since all exceptional sheaves satisfy the equation $\chi(E, E) = 1$, the corresponding vectors cannot be proportional. Hence we can pass to the projectivization of $K$. Then vectors corresponding to sheaves of exceptional collections are projected into vertexes of certain simplexes.

From the exact sequence (7.1) it follows that the vector $[\mathscr{F}]$ lies inside the simplex with vertexes at the points corresponding to $[\mathscr{O}_e(-1)], [E_2], \dots, [E_n]$:



If we project the point $[\mathscr{F}]$ to the edge $([\mathscr{O}_e(-1)], [E_2])$, then we get a point corresponding to a superrigid sheaf that splits into a direct sum of a pair of exceptional sheaves $(E'_1, E'_2)$ equivalent to the pair $(\mathscr{O}_e(-1), E_2)$. Then we project $[\mathscr{F}]$ to the face $([E_2], \dots, [E_n])$. The image under this projection also corresponds to a superrigid sheaf, and so on.

We use induction on the dimension of a category. The first induction step is based on the following two results.

7.1. **Lemma.** *Let $(E_1, E_2)$ be an exceptional pair. Consider the (infinite in both directions) sequence of exceptional sheaves defined by the recurrent formulae*

$$E_{i+1} = R_{E_i} E_{i-1}, \qquad E_{i-2} = L_{E_{i-1}} E_i.$$

*Then for each exceptional sheaf $E$ in the subcategory generated by the pair $(E_1, E_2)$ there exists a number $j \in \mathbb{Z}$ for which $E \cong E_j$. In other words, any exceptional sheaf belonging to the subcategory generated by an exceptional pair is obtained by transformations of this pair.*

*Proof.* Since any exceptional sheaf on a del Pezzo surface is determined by the corresponding vector in $K$, it suffices to prove the lemma in terms of $K_0(S)$.

Denote by $e_i$ the vector corresponding to the exceptional sheaf $E_i$, and put $h_i = \chi(e_i, e_{i+1})$. Then $e_{i-2} = \pm(he_{i-1} - e_i)$.

Suppose that the sequence of vectors $\{e_i\}_{i \in \mathbb{Z}}$ from $K_0(S)$ satisfies the conditions

$$\chi(e_i, e_i) = 1, \qquad \chi(e_i, e_{i+1}) = h, \qquad \chi(e_{i+1}, e_i) = 0, \qquad e_{i-2} = he_{i-1} - e_i$$

and that the square of the vector $e = x_1 e_1 + y_1 e_2$ is equal to one. Then we show that there exists an index $j$ such that $e_j = \pm e$.

Put $e = x_i e_i + y_i e_{i+1}$. We find out the relationship between the coordinates $(x_i, y_i)$ and $(x_{i-1}, y_{i-1})$. It is clear that

$$x_{i-1} = -y_i, \qquad x_i = hx_{i-1} + y_{i-1}.$$

We show that among the coordinates $(x_i, y_i)$ there is a pair $(\pm 1, 0)$.

Since $\chi(e, e) = 1$, the pairs $(x_i, y_i)$ satisfy the equation

$$(*) \qquad x^2 + y^2 + hxy - 1 = 0.$$

By the Vièta theorem, a pair $(x, y)$ is a solution of this equation if and only if the pairs $(y, -hy - x)$ and $(-hx - y, x)$ satisfy the same equation. Thus we get two transformations of a solution of equation $(*)$. We observe that up to sign these transformations coincide with the formulae for recomputation of coordinates. Hence it suffices to show that if $(x_0, y_0)$ is a solution of equation $(*)$ distinct from $(\pm 1, 0)$ and $(0, \pm 1)$, then one of the transformations decreases the sum of absolute values $|x| + |y|$.

Since $x_0$ and $x' = -hy_0 - x_0$ are two roots of the equation $(*)$ for a fixed $y_0$, by Viéte's theorem we have

$$x_0 x' = y_0^2 - 1 \geq 0.$$

Similarly, $y_0 y' = x_0^2 - 1 \geq 0$, from which it follows that $x_0$ and $x'$, as well as $y_0$ and $y'$, have the same sign.

Suppose that the following two inequalities are simultaneously satisfied:

$$|x'| \geq |x_0| = \frac{y_0^2 - 1}{|x'|}, \qquad |y'| \geq |y_0| = \frac{x_0^2 - 1}{|y'|}.$$

Then $x'^2 \geq y_0^2 - 1$ and $y'^2 \geq x_0^2 - 1$, which is impossible. The lemma is proved.

7.2. **Corollary.** *If the bundles making up an exceptional pair* $(F_1, F_2)$ *belong to the category generated by an* Ext-*pair* $(E_1, E_2)$, *then these pairs are constructively equivalent. If, moreover, the sheaves* $E_i$ *are locally free, then*

$$r(F_1) + r(F_2) \geq r(E_1) + r(E_2),$$

*where equality holds if and only if* $F_i = E_i$.

*Proof.* Constructive equivalence of these pairs follows from the preceding lemma. To prove the inequality we pass to $K_0(S)$. Let $f_1, f_2, e_1, e_2$ be the vectors in $K_0(S)$ corresponding to the sheaves $F_1, F_2, E_1, E_2$. Since $(E_1, E_2)$ is a pair of type Ext, $\chi(E_1, E_2) = -h < 0$. We claim that the coordinates $(x_i, y_i)$ of the vectors $f_i$ with respect to the basis $e_1, e_2$ are nonnegative. In fact, they satisfy the equation

$$x_i^2 + y_i^2 - hx_iy_i = 1,$$

from which it follows that $x_i$ and $y_i$ have the same sign. On the other hand, $r(F_i) = x_i r(E_1) + y_i r(E_2)$, and so $x_i$ and $y_i$ are nonnegative. The inequality now follows from the same relation.

Suppose that $r(E_i) > 0$. Then the equality $r(F_i) = r(E_j)$ is possible only if $(x_i, y_i) = (0, 1)$ or $(1, 0)$. This completes the proof of the corollary.

7.3. **Lemma.** *Suppose that the superrigid sheaves from an exact triple*

$$0 \to \mathscr{E}_3 \oplus \mathscr{E}_4 \to F \to \mathscr{E}_1 \oplus \mathscr{E}_2 \to 0 \tag{7.2}$$

*satisfy the following conditions:*

1) $\mathrm{Ext}^i(\mathscr{E}_j, \mathscr{E}_k) = 0$ *for* $k < j$ *and* $i = 0, 1, 2$;
2) $\mathrm{Ext}^2(\mathscr{E}_j, \mathscr{E}_k) = 0$ *for arbitrary* $j$ *and* $k$.

*Then*

a) $\mathrm{End}(\mathscr{E}_4) \simeq \mathrm{Hom}(\mathscr{E}_4, \mathscr{F})$;
b) $\mathrm{Ext}^i(\mathscr{E}_4, \mathscr{F}) = 0$ *for* $i = 1, 2$;
c) $\mathrm{Ext}^2(\mathscr{F}, \mathscr{E}_4) = 0$;
d) *there exists an exact sequence*

$$0 \to \mathscr{E}_4 \to \mathscr{F} \to \mathscr{G} \to 0, \tag{7.3}$$

*where* $\mathscr{G}$ *is a superrigid sheaf satisfying the condition* $\mathrm{Ext}^i(\mathscr{E}_4, \mathscr{G}) = 0$ *for* $i = 0, 1, 2$.

*We remark that in this case the sheaf* $\mathscr{E}_2$ *may be trivial.*

*Proof.* Apply the functor $\mathrm{Ext}(\mathscr{E}_4, *)$ to the exact sequence (7.2). By our assumptions, $\mathrm{Ext}^i(\mathscr{E}_4, \mathscr{E}_j) = 0$ for $j = 1, 2, 3$. Moreover, $\mathrm{Ext}^1(\mathscr{E}_4, \mathscr{E}_4) = \mathrm{Ext}^2(\mathscr{E}_4, \mathscr{E}_4) = 0$ since the sheaf $\mathscr{E}_4$ is superrigid. Therefore,

$$\mathrm{Hom}(\mathscr{E}_4, \mathscr{F}) \simeq \mathrm{End}(\mathscr{E}_4),$$
$$\mathrm{Ext}^i(\mathscr{E}_4, \mathscr{F}) = 0, \qquad i = 1, 2.$$

Applying the functor $\mathrm{Ext}(*, \mathscr{E}_4)$ to the exact sequence (7.2), we get an exact sequence

$$\to \mathrm{Ext}^2(\mathscr{E}_1 \oplus \mathscr{E}_2, \mathscr{E}_4) \to \mathrm{Ext}^2(\mathscr{F}, \mathscr{E}_4) \to \mathrm{Ext}^2(\mathscr{E}_3 \oplus \mathscr{E}_4, \mathscr{E}_4) \to 0.$$

Since by our assumption the groups $\mathrm{Ext}^2(\mathscr{E}_i, \mathscr{E}_j)$ are trivial for all $i$ and $j$, we have $\mathrm{Ext}^2(\mathscr{F}, \mathscr{E}_4) = 0$.

The standard inclusion of the sheaf $\mathscr{E}_4$ in a direct sum $\mathscr{E}_3 \oplus \mathscr{E}_4$ gives rise to a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & \mathscr{E}_3 & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & \mathscr{E}_3 \oplus \mathscr{E}_4 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{E}_1 \oplus \mathscr{E}_2 & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \longrightarrow & \mathscr{E}_4 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & \mathscr{E}_3 & &
\end{array}
\tag{7.4}
$$

with exact rows and columns. The second row in this diagram coincides with the sequence (7.3). Applying the functor $\operatorname{Ext}(\mathscr{E}_4, *)$ to this sequence, we get a long exact sequence

$$
\begin{aligned}
0 \longrightarrow \operatorname{Ext}^0(\mathscr{E}_4, \mathscr{E}_4) \xrightarrow{\alpha} \operatorname{Ext}^0(\mathscr{E}_4, \mathscr{F}) &\longrightarrow \operatorname{Ext}^0(\mathscr{E}_4, \mathscr{G}) \\
\longrightarrow \operatorname{Ext}^1(\mathscr{E}_4, \mathscr{E}_4) \longrightarrow \operatorname{Ext}^1(\mathscr{E}_4, \mathscr{F}) &\longrightarrow \operatorname{Ext}^1(\mathscr{E}_4, \mathscr{G}) \\
\longrightarrow \operatorname{Ext}^2(\mathscr{E}_4, \mathscr{E}_4) \longrightarrow \operatorname{Ext}^2(\mathscr{E}_4, \mathscr{F}) &\longrightarrow \operatorname{Ext}^2(\mathscr{E}_4, \mathscr{G}) \longrightarrow 0,
\end{aligned}
$$

from which it follows that the spaces $\operatorname{Ext}^i(\mathscr{E}_4, \mathscr{G})$ are trivial for $i = 0, 1, 2$. In fact, since $\operatorname{Ext}^i(\mathscr{E}_4, \mathscr{F}) = 0$ by the above and $\operatorname{Ext}^i(\mathscr{E}_4, \mathscr{E}_4) = 0$ for $i = 1, 2$ by the superrigidity of the sheaf $\mathscr{E}_4$, this follows immediately from the fact that $\alpha$ is an isomorphism.

Since the sheaf $\mathscr{F}$ is superrigid and $\operatorname{Ext}^2(\mathscr{F}, \mathscr{E}_4)$ is trivial, from the long exact sequence

$$
\begin{aligned}
\longrightarrow \operatorname{Ext}^1(\mathscr{F}, \mathscr{F}) \longrightarrow \operatorname{Ext}^1(\mathscr{F}, \mathscr{G}) &\longrightarrow \operatorname{Ext}^2(\mathscr{F}, \mathscr{E}_4) \\
\longrightarrow \operatorname{Ext}^2(\mathscr{F}, \mathscr{F}) &\longrightarrow \operatorname{Ext}^2(\mathscr{F}, \mathscr{G}) \longrightarrow 0
\end{aligned}
$$

it follows that

$$\operatorname{Ext}^1(\mathscr{F}, \mathscr{G}) = \operatorname{Ext}^2(\mathscr{F}, \mathscr{G}) = 0.$$

Applying the functor $\operatorname{Ext}(*, \mathscr{G})$ to the sequence (7.3), we conclude that $\mathscr{G}$ is superrigid since by the above $\operatorname{Ext}^i(\mathscr{E}_4, \mathscr{G}) = 0$ for $i = 0, 1, 2$ and $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G}) = 0$ for $i = 1, 2$.

7.4. **Lemma.** *In the assumptions of Lemma* 7.3 *we have*:

a) $\operatorname{End}(\mathscr{E}_1) \simeq \operatorname{Hom}(\mathscr{F}, \mathscr{E}_1)$;

b) $\operatorname{Ext}^i(\mathscr{F}, \mathscr{E}_1) = 0$ *for* $i = 1, 2$;

c) $\operatorname{Ext}^2(\mathscr{E}_1, \mathscr{F}) = 0$;

d) *there exists an exact sequence*

$$0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{E}_1 \to 0, \tag{7.5}$$

*where* $\mathscr{H}$ *is a superrigid sheaf satisfying the condition* $\operatorname{Ext}^i(\mathscr{H}, \mathscr{E}_1) = 0$ *for* $i = 0, 1, 2$.

*We remark that in this case the sheaf $\mathscr{E}_4$ may be trivial.*

7.5. **Proposition.** *Consider an exact sequence*

$$0 \to x_2E_2 \oplus \cdots \oplus x_mE_m \to y_1F_1 \oplus \cdots \oplus y_mF_m \to x_1E_1 \to 0 \tag{7.6}$$

*of locally free sheaves on $S$, where $(F_1, \dots, F_m)$ and $(E_2, \dots, E_m)$ are exceptional collections of type* Hom *and $(E_1, \dots, E_m)$ is an exceptional collection. Then*

a) *the collections $(F_1, \dots, F_m)$ and $(E_1, \dots, E_m)$ are constructively equivalent*;
b) $\sum_{i=1}^m r(F_i) \geq \sum_{i=1}^m r(E_i)$.

*Proof*. Denote the direct sum $y_1F_1 \oplus \cdots \oplus y_mF_m$ by $\mathscr{F}$ and prove the proposition by induction on $m$ (the number of bundles in the collections).

For $m = 2$ the exact sequence (7.6) has the form

$$0 \to x_2E_2 \to \mathscr{F} \to x_1E_1 \to 0.$$

If $\mathrm{Ext}^1(E_1, E_2)$ is trivial, then the sequence splits and all our assertions are true. Otherwise the exceptional bundles $F_i$ belong to the subcategory generated by the Ext-pair $(E_1, E_2)$ and the assertions of the proposition follow from Corollary 7.2.

The superrigid bundles $\mathscr{E}_1 = x_1E_1$, $\mathscr{E}_2 = x_2E_2$, $\mathscr{E}_4 = x_3E_3 \oplus \cdots \oplus x_mE_m$ and $\mathscr{F}$ satisfy the conditions of Lemma 7.3. Hence the bundle fits into an exact sequence

$$0 \to x_3E_3 \oplus \cdots \oplus x_mE_m \to \mathscr{F} \to \mathscr{G} \to 0, \tag{7.7}$$

and the superrigid sheaf $\mathscr{G}$ is obtained as an extension of $x_1E_1$ by means of $x_2E_2$:

$$0 \to x_2E_2 \to \mathscr{G} \to x_1E_1 \to 0. \tag{7.8}$$

Since $\mathscr{G}$ is a superrigid torsion-free sheaf, it splits into a direct sum of exceptional bundles, viz.

$$\mathscr{G} = x_1E_1' \oplus \cdots \oplus x_kE_k'',$$

where $(E_1', \dots, E_k'')$ is an exceptional collection of type Hom. Since the bundles $E_i'$ belong to the category generated by an exceptional pair, we have $k \leq 2$. On the other hand, from Lemma 7.3 it follows that the bundles $(E_1', E_2'', E_3, \dots, E_m)$ form an exceptional collection. If $k = 1$, then all the bundles $F_1, \dots, F_m$ belong to the subcategory generated by an exceptional collection consisting of $(m-1)$ bundles, which is impossible.

Thus we have shown that $\mathscr{G} = x_1'E_1' \oplus x_2''E_2''$, where $(E_1', E_2'')$ is a pair of type Hom. By the induction hypothesis the collection $(E_1', E_2'')$ is constructively equivalent to the collection $(E_1, E_2)$ and

$$r(E_1') + r(E_2'') \geq r(E_1) + r(E_2). \tag{7.9}$$

We rewrite the sequence (7.7) in the form

$$0 \to x_3E_3 \oplus \cdots \oplus x_mE_m \to \mathscr{F} \to x_1'E_1' \oplus x_2''E_2'' \to 0.$$

Since, as we have already observed, the collection $(E_1', E_2'', E_3, \dots, E_m)$ is exceptional and $(E_1', E_2'')$ and $(E_3, \dots, E_m)$ are collections of type Hom, the superrigid bundles $\mathscr{E}_1 = x_1'E_1'$, $\mathscr{E}_2 = x_2''E_2''$, $\mathscr{E}_3 = x_3E_3 \oplus \cdots \oplus x_mE_m$ and $\mathscr{F}$ satisfy the conditions of Lemma 7.4. Hence the sheaf $\mathscr{F}$ fits into an exact sequence

$$0 \to \mathscr{H} \to \mathscr{F} \to x_1'E_1' \to 0,$$

and the superrigid sheaf $\mathscr{H}$ fits into an exact triple

$$0 \to x_3E_3 \oplus \cdots \oplus x_mE_m \to \mathscr{H} \to x_2''E_2'' \to 0.$$

Since $\mathscr{H}$ is a superrigid bundle, it splits into a direct sum $\mathscr{H} = x_2'E_2' \oplus \cdots \oplus x_m'E_m'$. Arguing as above, it is easy to show that the collection $(E_2', \dots, E_m')$ has type Hom, and by the induction hypothesis it is constructively equivalent to a collection $(E_2'', E_3, \dots, E_m)$, where

$$r(E_2') + \cdots + r(E_m') \geq r(E_2'') + r(E_3) + \cdots + r(E_m).$$

Moreover, by Lemma 7.4 we have $\operatorname{Ext}^i(\mathscr{H}, E_1') = 0$ ($i = 0, 1, 2$), so that the collection $(E_1', E_2', \dots, E_m')$ is exceptional. Thus, starting from the sequence (7.6), we constructed a sequence

$$0 \to x_2'E_2' \oplus \cdots \oplus x_m'E_m' \to \mathscr{F} \to x_1'E_1' \to 0$$

of the same type with

$$r(E_1') + \cdots + r(E_m') \geq r(E_1) + \cdots + r(E_m). \tag{7.10}$$

We observe that the sum of the ranks of the bundles $E_i$ as well as that of the bundles $E_i'$ is bounded from above by the rank of the bundle $\mathscr{F}$. We transform the sequences of type (7.6) using the above procedure until the sum of the ranks of the bundles $E_i$ stops growing. Since this process cannot be infinite, the inequality (7.10) ultimately becomes an equality.

To avoid new notation, we assume that

$$r(E_1') + \cdots + r(E_m') = r(E_1) + \cdots + r(E_m).$$

Then the inequality (7.9) is also an equality, i.e., the exact sequence (7.8) is written in the form

$$0 \to x_2E_2 \to x_1'E_1' \oplus x_2''E_2'' \to x_1E_1 \to 0,$$

where $r(E_1) + r(E_2) = r(E_1') + r(E_2'')$. By Corollary 7.2, $E_1 = E_1'$ and $E_2 = E_2''$. But $(E_1', E_2'')$ is an exceptional pair of type Hom, i.e.,

$$\operatorname{Ext}^1(E_1', E_2'') = \operatorname{Ext}^1(E_1, E_2) = 0.$$

From the exact sequence (7.6) it follows that the bundle $E_2$ splits as a direct summand in the bundle $\mathscr{F}$, i.e.,

$$\mathscr{F} = x_2E_2 \oplus y_2'F_2' \oplus \cdots \oplus y_m'F_m',$$

where the superrigid bundle $\mathscr{F}' = y_2'F_2' \oplus \cdots \oplus y_m'F_m'$ fits into an exact sequence

$$0 \to x_3E_3 \oplus \cdots \oplus x_mE_m \to \mathscr{F}' \to x_1E_1 \to 0$$

and the exceptional collection $(F_2', \dots, F_m')$ is a subcollection of $(F_1, \dots, F_m)$. By the induction hypothesis, the collections $(F_2', \dots, F_m')$ and $(E_1, E_3, \dots, E_m)$ are constructively equivalent and

$$r(F_2') + \cdots + r(F_m') \geq r(E_1) + r(E_3) + \cdots + r(E_m).$$

This completes the induction argument and the proof of the proposition.

7.6. **Lemma.** *Suppose that a superrigid sheaf $\mathscr{G}$ fits into an exact sequence*

$$0 \to yE \to \mathscr{G} \to x\mathscr{O}_e(-1) \to 0, \tag{7.11}$$

*where $e$ is a $(-1)$-curve and $E$ is an exceptional bundle whose restriction to the curve $e$ is isomorphic to $r\mathscr{O}_e$. Then either $\mathscr{G}$ is a bundle or $\mathscr{G} = \mathscr{G}' \oplus x''\mathscr{O}_e(-1)$, where $\mathscr{G}'$ is a superrigid bundle isomorphic to $x_1'E_1'$ for some exceptional bundle $E_1'$ from the category generated by the pair $(\mathscr{O}_e(-1), E)$.*

*Proof.* Restricting the bundle $E$ to the $(-1)$-curve $e$, it is easy to compute the groups $\operatorname{Ext}^i(E, \mathscr{O}_e(-1))$ and $\operatorname{Ext}^i(\mathscr{O}_e(-1), E)$. From these computations it follows that $(\mathscr{O}_e(-1), E)$ is an exceptional pair of type Ext.

Suppose that $\mathscr{G}$ has a torsion subsheaf. If the sequence (7.11) splits, then the assertion of the lemma is obvious. Otherwise, let $T$ denote the torsion subsheaf of the sheaf $\mathscr{G}$, and put $\mathscr{G}' = \mathscr{G}/T$.

Since the sheaf $yE$ is locally free, $\operatorname{Hom}(T, E)$ is trivial and we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & T & \longrightarrow & x\mathscr{O}_e(-1) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & T & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{G}' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & yE & = & yE & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This commutative diagram yields an exact triple

$$0 \to yE \to \mathscr{G}' \to C \to 0. \tag{7.12}$$

The upper row of the commutative diagram gives an exact triple

$$0 \to T \to x\mathscr{O}_e(-1) \to C \to 0. \tag{7.13}$$

From this it follows that $T$ is a subsheaf of a locally free sheaf on the exceptional curve. Hence $T$ is also locally free on this curve, and since the curve is rational, we have $T = \bigoplus_i \mathscr{O}_e(d_i)$.

Since each sheaf on a curve is a sum of a locally free sheaf and a torsion sheaf, $C$ can be represented in the form $C' \oplus \gamma$, where $\gamma$ is a sheaf with support at points.

On surfaces, $\operatorname{Ext}^1$ from a sheaf with support at points to a locally free sheaf is trivial. Hence $\gamma$ splits as a direct summand in $\mathscr{G}'$ (cf. (7.12)). But the sheaf $\mathscr{G}'$ is torsion-free by construction. Hence $\gamma = 0$ and $C$ is locally free, i.e.,

$$C = \bigoplus_i \mathscr{O}_e(s_i).$$

From the short exact sequence (7.13) it follows that $s_i \geq -1$. Using the sequence (7.12), we show that $s_i \leq -1$.

We observe that for all $s_i$ we have $\operatorname{Ext}^1(\mathscr{O}_e(s_i), E) \neq 0$. Otherwise the torsion sheaf $\mathscr{O}_e(s_i)$ would split as a direct summand of the torsion-free sheaf $\mathscr{G}'$. By Serre's duality,

$$\operatorname{Ext}^1(\mathscr{O}_e(s_i), E)^* \cong \operatorname{Ext}^1(E, \mathscr{O}_e(s_i) \otimes K_S) = \operatorname{Ext}^1_e(E|_e, \mathscr{O}_e(s_i - 1)).$$

The last equality follows from the fact that the intersection number of the $(-1)$-curve $e$ with the canonical class of the surface $S$ is equal to $-1$. By assumption, $E|_e = r\mathscr{O}_e$; hence

$$\operatorname{Ext}^1_e(E|_e, \mathscr{O}_e(s_i - 1)) = \operatorname{Ext}^1_e(r\mathscr{O}_e, \mathscr{O}_e(s_i - 1)).$$

Since $\operatorname{Ext}^1(\mathcal{O}_e(s_i), E) \neq 0$, we have $s_i - 1 \leq -2$, i.e., $s_i \leq -1$.

Thus we have shown that $C = x'\mathcal{O}_e(-1)$, and therefore $T = x''\mathcal{O}_e(-1)$. Applying the functor $\operatorname{Ext}(*, \mathcal{O}_e(-1))$ to the exact sequence (7.12), we get the exact sequence

$$\to \operatorname{Ext}^1(x'\mathcal{O}_e(-1), \mathcal{O}_e(-1)) \to \operatorname{Ext}^1(\mathcal{G}', \mathcal{O}_e(-1)) \to \operatorname{Ext}^1(yE, \mathcal{O}_e(-1)) \to .$$

Since the pair $(\mathcal{O}_e(-1), E)$ is exceptional, we have $\operatorname{Ext}^1(x'\mathcal{O}_e(-1), \mathcal{O}_e(-1)) = 0$ and $\operatorname{Ext}^1(yE, \mathcal{O}_e(-1)) = 0$, and therefore $\operatorname{Ext}^1(\mathcal{G}', \mathcal{O}_e(-1)) = 0$. Hence the exact sequence $0 \to T \to \mathcal{G} \to \mathcal{G}' \to 0$ splits, i.e.,

$$\mathcal{G} = T \oplus \mathcal{G}' = x'\mathcal{O}_e(-1) \oplus \mathcal{G}'.$$

The fact that the sheaf $\mathcal{G}'$ is superrigid and therefore locally free easily follows from the last equality.

We decompose $\mathcal{G}'$ into a direct sum of exceptional bundles, viz. $\mathcal{G}' = x_1'E_1' \oplus x_2'E_2'$. The exceptional collection $(\mathcal{O}_e(-1), E_1', E_2')$ belongs to the category generated by the pair $(\mathcal{O}_e(-1), E)$. Hence $x_2' = 0$ and $\mathcal{G}' = x_1'E_1'$. The lemma is proved.

7.7. **Theorem.** *All exceptional sheaves and helixes on del Pezzo surfaces are constructible.*

*Proof*. This theorem was proved in the case of the plane $\mathbb{P}^2$ and the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ in [5] and [9], respectively. Hence it suffices to verify it for the plane with $d$ blown up points $(d \leq 8)$.

As was already proved in this paper, each exceptional collection, and in particular each exceptional sheaf can be included in a coil of a helix (a complete exceptional collection) $(F_1, \dots, F_n)$. Hence constructibility of sheaves follows from constructibility of helixes.

We can assume that all the sheaves $F_i$ are locally free, since this can be achieved by transformations of torsion sheaves using locally free sheaves, which are always present in a complete collection.

Moreover, passing if necessary to constructively equivalent collections, we can assume that the bundle $\mathcal{F} = F_1 \oplus \cdots \oplus F_n$ is superrigid and fits into an exact sequence

$$0 \to x_2E_2 \oplus \cdots \oplus x_nE_n \to \mathcal{F} \to x_1\mathcal{O}_e(-1) \to 0, \tag{7.14}$$

where the bundles $E_2, \dots, E_n$ are the inverse images of the bundles from the complete exceptional collection $(\tilde{E}_2, \dots, \tilde{E}_n)$ on $S'$ under the blowing up $S \xrightarrow{\sigma} S'$. Furthermore, the direct sum $x_2E_2 \oplus \cdots \oplus x_nE_n$ is a superrigid sheaf, and the collection $(\mathcal{O}_e(-1), E_2, \dots, E_n)$ is a loop of a helix on $S$.

We recall that by the induction hypothesis the complete exceptional collection $(\tilde{E}_2, \dots, \tilde{E}_n)$, and therefore $(\mathcal{O}_e(-1), E_2, \dots, E_n)$ are constructible.

As the transitional induction step we prove the following claim (by induction on $n$).

Suppose that a superrigid bundle $\mathcal{F} = y_1F_1 \oplus \cdots \oplus y_nF_n$ fits into an exact sequence (7.14), where the sheaves $\mathcal{O}_e(-1), E_2, \dots, E_n$ satisfy the above conditions. Then the collection $(F_1, \dots, F_n)$ is equivalent to $(\mathcal{O}_e(-1), E_2, \dots, E_n)$. The first induction step (the case $n = 2$) follows from Corollary 7.2. Using Lemma 7.3, we construct a superrigid sheaf $\mathcal{G}$ fitting into the exact sequences

$$0 \to x_3E_3 \oplus \cdots \oplus x_nE_n \to \mathcal{F} \to \mathcal{G} \to 0,$$
$$0 \to x_2E_2 \to \mathcal{G} \to x_1\mathcal{O}_e(-1) \to 0$$

and satisfying the condition $\operatorname{Ext}^i(E_k, \mathcal{G}) = 0$ ($i = 0, 1, 2$; $k \geq 3$). By Lemma 7.6, $\mathcal{G} = x_1'E_1' \oplus x_2''\mathcal{O}_e(-1)$. By Corollary 7.2, the pair $(E_1', \mathcal{O}_e(-1))$ is constructively equivalent to the pair $(\mathscr{O}_e(-1), E_2)$. Hence the collections $(\mathscr{O}_e(-1), E_2, \dots, E_n)$ and $(E_1', \mathscr{O}_e(-1), E_3, \dots, E_n)$ are constructively equivalent, and the sheaf $E_1'$ is locally free.

Applying once more Lemma 7.4, we construct a superrigid sheaf $\mathscr{H}$ fitting into the exact sequences

$$0 \to \mathscr{H} \to \mathscr{F} \to x_1' E_1' \to 0, \tag{7.15}$$

$$0 \to x_3 E_3 \oplus \cdots \oplus x_n E_n \to \mathscr{H} \to x'' \mathscr{O}_e(-1) \to 0 \tag{7.16}$$

and satisfying the condition $\operatorname{Ext}^i(\mathscr{H}, E_1') = 0$ for $i = 0, 1, 2$.

The sheaf $\mathscr{H}$ is superrigid, and therefore $\mathscr{H} = x_2' E_2' \oplus \cdots \oplus x_n' E_n'$, where $(\mathscr{O}_e(-1), E_2', \dots, E_n')$ is an exceptional collection. Furthermore, all the sheaves $E_i'$ are locally free. In view of the exact sequence (7.15), this follows from the fact that the sheaves $\mathscr{F}$ and $E_1'$ are locally free.

Using the induction hypothesis and the exact sequence (7.16), we conclude that the exceptional collections $(\mathscr{O}_e(-1), E_3, \dots, E_n)$ and $(E_2', \dots, E_n')$, and therefore $(E_1', \mathscr{O}_e(-1), E_3, \dots, E_n)$ and $(E_1', E_2', \dots, E_n')$ are constructively equivalent.

Thus, starting with the sequence (7.14), we obtain a sequence (7.15) of locally free sheaves, which can be rewritten in the form

$$0 \to x_2' E_2' \oplus \cdots \oplus x_n' E_n' \to \mathscr{F} \to x_1' E_1' \to 0.$$

Furthermore, the bundles $E_i'$ and $\mathscr{F}$ satisfy the conditions of Proposition 7.5. Hence the collections $(E_1', \dots, E_n')$ and $(F_1, \dots, F_n)$ are constructively equivalent. The theorem is proved.

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