# Derived Categories of Coherent Sheaves

Alexei Bondal[*]        Dmitri Orlov [†]

## 1   Introduction

This paper is devoted to studying the derived categories $\mathcal{D}^b(X)$ of coherent sheaves on smooth algebraic varieties $X$ and on their noncommutative counterparts. Derived categories of coherent sheaves proved to contain the complete geometric information about varieties (in the sense of the classical Italian school of algebraic geometry) as well as the related homological algebra.

The situation when there exists a functor $\mathcal{D}^b(M) \longrightarrow \mathcal{D}^b(X)$ which is fully faithful is of special interest. We are convinced that any example of such a functor is both algebraically and geometrically meaningful.

A particular case of a fully faithful functor is an equivalence of derived categories $\mathcal{D}^b(M) \stackrel{\sim}{\longrightarrow} \mathcal{D}^b(X)$ .

We show that smooth projective varieties with ample canonical or anticanonical bundles are uniquely determined by their derived categories. Hence the derived equivalences between them boil down to autoequivalences. We prove that for such a variety the group of exact autoequivalences is the semidirect product of the group of automorphisms of the variety and the Picard group plus translations.

Equivalences and autoequivalences for the case of varieties with non-ample (anti) canonical sheaf are now intensively studied. The group of autoequivalences is believed to be closely related to the holonomy group of the mirror-symmetric family.

We give a criterion for a functor between derived categories of coherent sheaves on two algebraic varieties to be fully faithful. Roughly speaking, it is in orthogonality of the images under the functor of the structure sheaves of distinct closed points of the variety. If a functor $\Phi : \mathcal{D}^b(M) \longrightarrow \mathcal{D}^b(X)$ is fully faithful, then it induces a so-called semiorthogonal decomposition of $\mathcal{D}^b(X)$ into $\mathcal{D}^b(M)$ and its right orthogonal category.

[*]Algebra Section, Steklov Mathematical Institute, Russian Academy of Sciences, 8 Gubkin str., GSP-1, Moscow 117966, Russia. E-mail: bondal@mi.ras.ru

[†]Algebra Section, Steklov Mathematical Institute, Russian Academy of Sciences, 8 Gubkin str., GSP-1, Moscow 117966, Russia. E-mail: orlov@mi.ras.ru

It turned out that derived categories have nice behavior under special birational transformations like blow ups, flips and flops. We describe a semiorthogonal decomposition of the derived category of the blow-up of a smooth variety $X$ in a smooth center $Y \subset X$. It contains one component isomorphic to $\mathcal{D}^b(X)$ and several components isomorphic to $\mathcal{D}^b(Y)$.

We also consider some flips and flops. Examples support the conjecture that for any generalized flip $X \dashrightarrow X^+$ there exists a fully faithful functor $\mathcal{D}^b(X^+) \to \mathcal{D}^b(X)$ and it must be an equivalence for generalized flops. This suggests the idea that the minimal model program of the birational geometry can be viewed as a 'minimization' of the derived category $\mathcal{D}^b(X)$ in a given birational class of $X$.

Then we widen the categorical approach to birational geometry by including in the picture some noncommutative varieties. We propose to consider noncommutative desingularizations and formulate a conjecture generalizing the derived McKay correspondence.

We construct a semiorthogonal decomposition for the derived category of the complete intersections of quadrics. It is related to classical questions of algebraic geometry, like 'quadratic complexes of lines', and to noncommutative geometric version of Koszul quadratic duality.

## 2   Equivalences between derived categories

The first question that arises in studying algebraic varieties from the point of view of derived categories is when varieties have equivalent derived categories of coherent sheaves. Examples of such equivalences for abelian varieties and K3 surfaces were constructed by Mukai [Mu1], [Mu2], Polishchuk [Po] and the second author in [Or2], [Or3]. See below on derived equivalences for birational maps.

Yet we prove that a variety $X$ is uniquely determined by its category $\mathcal{D}^b(X)$, if its anticanonical (Fano case) or canonical (general type case) sheaf is ample. To this end, we use only the graded (not triangulated) structure of the category. By definition a graded category is a pair $(\mathcal{D}, T_{\mathcal{D}})$ consisting of a category $\mathcal{D}$ (which we always assume to be $k$-linear over a field $k$) and a fixed equivalence $T_D : \mathcal{D} \longrightarrow \mathcal{D}$, called translation functor. For derived categories the translation functor is defined to be the shift of grading in complexes.

Of crucial importance for exploring derived categories are existence and properties of the Serre functor, defined in [BK].

**Definition 2.1** [BK] [BO2] *Let $\mathcal{D}$ be a $k$-linear category with finite–dimensional* $\mathrm{Hom}'s$. *A covariant additive functor $S : \mathcal{D} \to \mathcal{D}$ is called a* Serre functor *if it is an equivalence and there are given bi–functorial isomorphisms for any $A, B \in \mathcal{D}$:*

$$\varphi_{A,B} : \mathrm{Hom}_{\mathcal{D}}(A \, , \, B) \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(B \, , \, SA)^*.$$

A Serre functor in a category $\mathcal{D}$, if it exists, is unique up to a graded natural isomorphism.

If $X$ is a smooth algebraic variety, $n = \dim X$, then the functor $(\cdot) \otimes \omega_X[n]$ is the Serre functor in $\mathcal{D}^b(X)$. Thus, the Serre functor in $\mathcal{D}^b(X)$ can be viewed as a categorical incarnation of the canonical sheaf $\omega_X$.

**Theorem 2.2** [BO2] *Let $X$ be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. If $\mathcal{D}^b(X)$ is equivalent as a graded category to $\mathcal{D}^b(X')$ for some other smooth algebraic variety $X'$, then $X$ is isomorphic to $X'$.*

The idea of the proof is that for varieties with ample canonical or anticanonical sheaf we can recognize the skyscraper sheaves of closed points in $\mathcal{D}^b(X)$ by means of the Serre functor. In this way we find the variety as a set. Then we reconstruct one by one the set of line bundles, Zariski topology and the structural sheaf of rings.

This theorem has a generalization to smooth orbifolds related to projective varieties with mild singularities, as it was shown by Y. Kawamata [Kaw].

Now consider the problem of computing the group $\mathrm{Aut}\mathcal{D}^b(X)$ of exact (i.e. preserving triangulated structure) autoequivalences of $\mathcal{D}^b(X)$ for an individual $X$.

**Theorem 2.3** [BO2] *Let $X$ be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. Then the group of isomorphism classes of exact autoequivalences $\mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is generated by the automorphisms of the variety, twists by all invertible sheaves and translations.*

In the hypothesis of Theorem 2.3 the group $\mathrm{Aut}\mathcal{D}^b(X)$ is the semi-direct product of its subgroups $\mathrm{Pic}X \oplus \mathbf{Z}$ and $\mathrm{Aut}X$, $\mathbf{Z}$ being generated by the translation functor:

$$\mathrm{Aut}\mathcal{D}^b(X) \cong \mathrm{Aut}X \ltimes (\mathrm{Pic}X \oplus \mathbf{Z}).$$

# 3 Fully faithful functors and semiorthogonal decompositions

An equivalence is a particular instance of a fully faithful functor. This is a functor $F : \mathcal{C} \to \mathcal{D}$ which for any pair of objects $X, Y \in \mathcal{C}$ induces an isomorphism $\mathrm{Hom}(X \, , \, Y) \simeq \mathrm{Hom}(FX \, , \, FY)$. This notion is especially useful in the context of triangulated categories.

If a functor $\Phi : \mathcal{D}^b(M) \longrightarrow \mathcal{D}^b(X)$ is fully faithful, then it induces a so-called semiorthogonal decomposition of $\mathcal{D}^b(X)$ into $\mathcal{D}^b(M)$ and its right orthogonal.

Let $\mathcal{B}$ be a full triangulated subcategory of a triangulated category $\mathcal{D}$. The right orthogonal to $\mathcal{B}$ is the full subcategory $\mathcal{B}^\perp \subset \mathcal{D}$ consisting of the objects $C$ such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathcal{B}$. The left orthogonal $^\perp\mathcal{B}$ is defined analogously. The categories $^\perp\mathcal{B}$ and $\mathcal{B}^\perp$ are also triangulated.

**Definition 3.1** [BK] *A sequence of triangulated subcategories* $(\mathcal{B}_0, ..., \mathcal{B}_n)$ *in a triangulated category* $\mathcal{D}$ *is said to be* semiorthogonal *if* $\mathcal{B}_j \subset \mathcal{B}_i^\perp$ *whenever* $0 \le j < i \le n$. *If a semiorthogonal sequence generates* $\mathcal{D}$ *as a triangulated category, then we call it by* semiorthogonal decomposition *of the category* $\mathcal{D}$ *and denote this as follows:*

$$\mathcal{D} = \Big\langle \mathcal{B}_0, ...., \mathcal{B}_n \Big\rangle.$$

Examples of semiorthogonal decompositions are provided by exceptional sequences of objects [Bo]. These arise when all $\mathcal{B}_i$'s are equivalent to the derived categories of finite dimensional vector spaces $\mathcal{D}^b(k - \mathrm{mod})$. Objects which correspond to the 1-dimensional vector space under a fully faithful functor $F : \mathcal{D}^b(k - \mathrm{mod}) \to \mathcal{D}$ can be intrinsically defined as *exceptional*, i.e. those satisfying the conditions $\mathrm{Hom}^i(E, E) = 0$, when $i \ne 0$, and $\mathrm{Hom}^0(E, E) = k$. There is a natural action of the braid group on exceptional sequences [Bo] and, under some conditions, on semiorthogonal sequences of subcategories in a triangulated category [BK].

We propose to consider the derived category of coherent sheaves as an analogue of the motive of a variety, and semiorthogonal decompositions as a tool for simplification of this 'motive' similar to splitting by projectors in Grothendieck motivic theory.

Let $X$ and $M$ be smooth algebraic varieties of dimension $n$ and $m$ respectively and $E$ an object in $\mathcal{D}^b(X \times M)$. Denote by $p$ and $\pi$ the projections of $M \times X$ to $M$ and $X$ respectively. With $E$ one can associate the functor $\Phi_E : \mathcal{D}^b(M) \longrightarrow \mathcal{D}^b(X)$ defined by the formula:

$$\Phi_E(\cdot) := \mathbf{R}\pi_*(E \overset{\mathbf{L}}{\otimes} p^*(\cdot)).$$

It happens that any fully faithful functor is of this form.

**Theorem 3.2** [Or2] *Let* $F : \mathcal{D}^b(M) \to \mathcal{D}^b(X)$ *be an exact fully faithful functor, where* $M$ *and* $X$ *are smooth projective varieties. Then there exists a unique up to isomorphism object* $E \in \mathcal{D}^b(M \times X)$ *such that* $F$ *is isomorphic to the functor* $\Phi_E$.

The assumption on existence of the right adjoint to $F$, which was originally in [Or2], can be removed in view of saturatedness of $\mathcal{D}^b(M)$ due to [BK], [BVdB].

This theorem is in conjunction with the following criterion.

**Theorem 3.3** [BO1] *Let $M$ and $X$ be smooth algebraic varieties and $E \in \mathcal{D}^b(M \times X)$. Then $\Phi_E$ is fully faithful functor if and only if the following orthogonality conditions are verified:*

$$i) \quad \mathrm{Hom}_X^i(\Phi_E(\mathcal{O}_{t_1}), \, \Phi_E(\mathcal{O}_{t_2})) = 0 \qquad \textit{for every } i \textit{ and } t_1 \neq t_2.$$

$$ii) \quad \mathrm{Hom}_X^0(\Phi_E(\mathcal{O}_t), \, \Phi_E(\mathcal{O}_t)) = k,$$

$$\mathrm{Hom}_X^i(\Phi_E(\mathcal{O}_t), \, \Phi_E(\mathcal{O}_t)) = 0, \qquad \textit{for } i \notin [0, dimM].$$

*Here $t$, $t_1$, $t_2$ stand for closed points in $M$ and $\mathcal{O}_{t_i}$ for the skyscraper sheaves.*

The criterion is a particular manifestation of the following important principle: suppose $M$ is realized as an appropriate moduli space of pairwise homologically orthogonal objects in a triangulated category $\mathcal{D}$ taken 'from the real life', then one can expect a sheaf of finite (non-commutative) algebras $\mathcal{A}_M$ over $\mathcal{O}_M$ and a fully faithful functor from the derived category $\mathcal{D}^b(coh(\mathcal{A}_M))$ of coherent modules over $\mathcal{A}_M$ to $\mathcal{D}$.

There are also strong indications that this principle should have a generalization, at the price of considering noncommutative DG moduli spaces, to the case when the orthogonality condition is dropped.

## 4 Derived categories and birational geometry

Behavior of derived categories under birational transformations shows that they can serve as a useful tool in comprehending various phenomena of birational geometry and play possibly the key role in realizing the minimal model program.

First, we give a description of the derived category of the blow-up of a variety with smooth center in terms of the categories of the variety and of the center. Let $Y$ be a smooth subvariety of codimension $r$ in a smooth algebraic variety $X$. Denote $\widetilde{X}$ the smooth algebraic variety obtained by blowing up $X$ along the center $Y$. There exists a fibred square:

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{j} & \widetilde{X} \\ p \downarrow & & \pi \downarrow \\ Y & \xrightarrow{i} & X \end{array}$$

where $i$ and $j$ are smooth embeddings, and $p : \widetilde{Y} \to Y$ is the projective fibration of the exceptional divisor $\widetilde{Y}$ in $\widetilde{X}$ over the center $Y$. Recall that $\widetilde{Y} \cong \mathbb{P}(N_{X/Y})$ is the projective normal bundle. Denote by $\mathcal{O}_{\widetilde{Y}}(1)$ the relative Grothendieck sheaf.

**Proposition 4.1** [Or1] *Let $L$ be any invertible sheaf on $\widetilde{Y}$. The functors*

$$\mathbf{L}\pi^* : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(\widetilde{X}),$$

$$\mathbf{R}j_*(L \otimes p^*(\cdot)) : \mathcal{D}^b(Y) \longrightarrow \mathcal{D}^b(\widetilde{X})$$

*are fully faithful.*

Denote by $D(X)$ the full subcategory of $\mathcal{D}^b(\widetilde{X})$ which is the image of $\mathcal{D}^b(X)$ with respect to the functor $\mathbf{L}\pi^*$ and by $D(Y)_k$ the full subcategories of $\mathcal{D}^b(\widetilde{X})$ which are the images of $\mathcal{D}^b(Y)$ with respect to the functors $\mathbf{R}j_*(\mathcal{O}_{\widetilde{Y}}(k) \otimes p^*(\cdot))$.
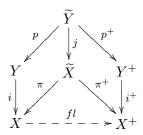
**Theorem 4.2** [Or1][BO1] *We have the semiorthogonal decomposition of the category of the blow-up:*

$$\mathcal{D}^b(\widetilde{X}) = \Big\langle D(Y)_{-r+1}, ..., D(Y)_{-1}, D(X) \Big\rangle.$$

Now we consider the behavior of the derived categories of coherent sheaves with respect to the special birational transformations called flips and flops.

Let $Y$ be a smooth subvariety of a smooth algebraic variety $X$ such that $Y \cong \mathbb{P}^k$ and $N_{X/Y} \cong \mathcal{O}(-1)^{\oplus(l+1)}$ with $l \leq k$.

If $\widetilde{X}$ is the blow-up of $X$ along $Y$, then the exceptional divisor $\widetilde{Y} \cong \mathbb{P}^k \times \mathbb{P}^l$ is the product of projective spaces. We can blow down $\widetilde{X}$ in such a way that $\widetilde{Y}$ projects to the second component $\mathbb{P}^l$ of the product. As a result we obtain a smooth variety $X^+$, which for simplicity we assume to be algebraic, with subvariety $Y^+ \cong \mathbb{P}^l$. This is depicted in the following diagram:
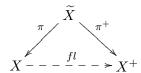


The birational map $X \dashrightarrow X^+$ is the simplest instance of flip, for $l \leq k$. If $l = k$, this is a flop.

**Theorem 4.3** [BO1] *In the above notations, the functor $\mathbf{R}\pi_*\mathbf{L}\pi^{+*} : \mathcal{D}^b(X^+) \longrightarrow \mathcal{D}^b(X)$ is fully faithful for $l \leq k$. If $l = k$, it is an equivalence.*

This theorem has an obvious generalization to the case when $Y$ is isomorphic to the projectivization of a vector bundle $E$ of rank $k$ on a smooth variety $W$, $q : Y \longrightarrow W$, and $N_{X/Y} = q^*F \otimes \mathcal{O}_E(-1)$ where $F$ is a vector bundle on $W$ of rank $l \leq k$. Then the blow-up with a smooth center can be viewed as the particular case of this flip when

$Y$ is a divisor in $X$. Kawamata [Kaw] generalized the theorem to those flips between smooth orbifolds which are elementary (Morse type) cobordisms in the theory of birational cobordisms due to Wlodarczyk *et al.* [Wl], [AKMW].

Let $X$ and $X^+$ be smooth projective varieties. A birational map $X \xrightarrow{fl} X^+$ will be called a generalized flip if for some (and consequently for any) its smooth resolution

$$
\begin{array}{ccc}
 & \widetilde{X} & \\
\pi \swarrow & & \searrow \pi^+ \\
X \dashedrightarrow{fl} & & X^+
\end{array}
$$

the difference $D = \pi^* K_X - \pi^{+*} K_{X^+}$ between the pull-backs of the canonical divisors is an effective divisor on $\widetilde{X}$. The particular case when $D = 0$ is called generalized flop.

Theorem 4.3 together with calculations of 3-dimensional flops with centers in $(-2)$ - curves [BO1] lead us to the following conjecture.

**Conjecture 4.4** *For any generalized flip $X \xrightarrow{fl} X^+$ there is an exact fully faithful functor $F : \mathcal{D}^b(X^+) \longrightarrow \mathcal{D}^b(X)$. It is an equivalence for generalized flops.*

This conjecture was recently proved in dimension 3 by T. Bridgeland [Br].

The functor $\mathbf{R}\pi_* \mathbf{L}\pi^{+*} : \mathcal{D}^b(X^+) \longrightarrow \mathcal{D}^b(X)$ is not always fully faithful under conditions of the conjecture, but we expect that it is such when $\widetilde{X}$ is replaced by the fibred product of $X$ and $X^+$ over some common singular contraction of $X$ and $X^+$. Namikawa proved that this is the case for Mukai symplectic flops [Na].

A fully faithful functor $\mathcal{D}^b(X^+) \longrightarrow \mathcal{D}^b(X)$ induces a semiorthogonal decomposition of $\mathcal{D}^b(X)$ into $\mathcal{D}^b(X^+)$ and its right orthogonal (which is trivial for flops). Hence, passing from $X$ to $X^+$ for generalized flips has the categorical meaning of breaking off semiorthogonal summands from the derived category. This suggests the idea that the minimal model program of birational geometry should be interpreted as a minimization of the derived category $\mathcal{D}^b(X)$ in a given birational class of $X$. Promisingly, chances are that the very existence of flips can be achieved by constructing $X^+$ as an appropriate moduli space of objects in $\mathcal{D}^b(X)$, in accordance with the principle of the previous section (this is done by T. Bridgeland for flops in dimension 3 [Br]).

## 5 Noncommutative resolutions of singularities

In this section we will give a perspective for categorical interpretation of the minimal model program by enriching the landscape with the derived categories of noncommutative varieties.

Let $\pi : \widetilde{X} \to X$ be a proper birational morphism, where $X$ has rational singularities. Then $\mathbf{R}\pi_* : \mathcal{D}^b(\widetilde{X}) \to \mathcal{D}^b(X)$ identifies $\mathcal{D}^b(X)$ with the quotient of $\mathcal{D}^b(\widetilde{X})$ by the kernel

of $\mathbf{R}\pi_*$ . For this reason, let us call by a *categorical desingularization* of a triangulated category $\mathcal{D}$ a pair $(\mathcal{C}, \mathcal{K})$ consisting of an abelian category $\mathcal{C}$ of finite homological dimension and of $\mathcal{K}$ , a thick subcategory in $\mathcal{D}^b(\mathcal{C})$ such that $\mathcal{D} = \mathcal{D}^b(\mathcal{C})/\mathcal{K}$ . We expect that for $\mathcal{D} = \mathcal{D}^b(X)$ there exists a minimal desingularization, i.e. such one that $\mathcal{D}^b(\mathcal{C})$ has a fully faithful embedding in $\mathcal{D}^b(\mathcal{C}')$ for any other categorical desingularization $(\mathcal{C}', \mathcal{K}')$ of $\mathcal{D}$ . Such a desingularization is unique up to derived equivalence of $\mathcal{C}$ .

For the derived categories of singular varieties one may hope to find the minimal desingularizations in the spirit of noncommutative geometry.

Let $X$ be an algebraic variety. We call by noncommutative (birational) desingularization of $X$ a pair $(p, \mathcal{A})$ consisting of a proper birational morphism $p : Y \to X$ and an algebra $\mathcal{A} = \mathcal{E}nd(\mathcal{F})$ on $Y$ , the sheaf of local endomorphisms of a torsion free coherent $\mathcal{O}_Y$ - module $\mathcal{F}$ , such that the abelian category of coherent $\mathcal{A}$ -modules has finite homological dimension.

When $f : Y \to X$ is a morphism from a smooth $Y$ onto an affine $X$ with fibres of dimension $\leq 1$ and $\mathbf{R}f_*(\mathcal{O}_Y) = \mathcal{O}_X$ , M. Van den Bergh [VdB] has recently constructed a noncommutative desingularization of $X$ , which is derived equivalent to $\mathcal{D}^b(Y)$ .

**Conjecture 5.1** *Let $X$ has canonical singularities and $q : Y \to X$ a finite morphism with smooth $Y$ . Then the pair $(id_X, \mathcal{E}nd(q_*\mathcal{O}_Y))$ is a minimal desingularization of $X$ .*

In particular, we expect that $\mathcal{D}^b(coh(\mathcal{E}nd(q_*\mathcal{O}_Y)))$ has a fully faithful functor into $\mathcal{D}^b(\widetilde{X})$ for any smooth (commutative) resolution of $X$ . Moreover, if the resolution is crepant then the functor has to be an equivalence.

Let $X$ be the quotient of a smooth $Y$ by an action of a finite group $G$ . If the locus of the points in $Y$ with nontrivial stabilizer in $G$ has codimension $\geq 2$ , then the category of coherent $\mathcal{E}nd(q_*\mathcal{O}_Y)$ -modules is equivalent to the category of $G$ -equivariant coherent sheaves on $Y$ . Therefore the conjecture is a generalization of the derived McKay correspondence due to Bridgeland-King-Reid [BKR].

# 6 Complete intersection of quadrics and noncommutative geometry

This section is related to the previous one by Grothendieck slogan that projective geometry is a part of theory of singularities.

Let $X$ be a smooth intersection of two projective quadrics of even dimension $d$ over an algebraically closed field of characteristic zero. It appears that if we consider the hyperelliptic curve $C$ which is the double cover of $\mathbb{P}^1$ that parameterizes the pencil of quadrics,

with ramification in the points corresponding to degenerate quadrics, then $\mathcal{D}^b(C)$ is embedded in $\mathcal{D}^b(X)$ as a full subcategory [BO1]. This gives a categorical explanation for the classical description of moduli spaces of semistable bundles on the curve $C$ as moduli spaces of (complexes of) coherent sheaves on $X$ .

The orthogonal to $\mathcal{D}^b(C)$ in $\mathcal{D}^b(X)$ is decomposed into an exceptional sequence (of line bundles ). More precisely, we have a semiorthogonal decomposition

$$\mathcal{D}^b(X) \;=\; \Big\langle \mathcal{O}_X(-d+3), \ldots, \mathcal{O}_X, \mathcal{D}^b(C) \Big\rangle. \tag{1}$$

When a greater number of quadrics is intersected, objects of noncommutative geometry naturally show up: instead of coherent sheaves on hyperelliptic curves we must consider modules over a sheaf of noncommutative algebras. More about noncommutative geometry is in the talk of T. Stafford at this Congress.

Consider a system of $m$ quadrics in $\mathbb{P}(V)$ , i.e. a linear embedding $U \overset{\phi}{\hookrightarrow} S^2V^*$ , where $\dim U = m$ , $\dim V = n$ , $2m \leq n$ . Let $X$ , the complete intersection of the quadrics, be a smooth subvariety in $\mathbb{P}(V)$ of dimension $n - m - 1$ . Let $A = \underset{i \geq 0}{\oplus} H^0(X, \mathcal{O}(i))$ be the coordinate ring of $X$ . This graded quadratic algebra is Koszul due to Tate [Ta]. The quadratic dual algebra $B = A^!$ is the generalized homogeneous Clifford algebra. It is generated in degree 1 by the space $V$ , the relations being given by the kernel of the dual to $\phi$ map $S^2V \to U^*$ , viewed as a subspace in $V \otimes V$ . The center of $B$ is generated by $U^*$ (a subspace of quadratic elements in $B$ ) and an element $d$ , which satisfies the equation $d^2 = f$ where $f$ is the equation of the locus of degenerate quadrics in $U$ . Algebra $B$ is finite over the central subalgebra $S = S^\bullet U^*$ . The Veronese subalgebra $B_{ev} = \oplus B_{2i}$ is finite over the Veronese subalgebra $S_{ev} = \oplus S^{2i}U^*$ . Since $\mathbf{Proj}\, S_{ev}$ is isomorphic to $\mathbb{P}(U)$ , the sheafification of $B_{ev}$ over $\mathbf{Proj}\, S_{ev}$ is a sheaf $\mathcal{B}$ of finite algebras over $\mathcal{O}_{\mathbb{P}(U)}$ . Consider the derived category $\mathcal{D}^b(coh(\mathcal{B}))$ of coherent right $\mathcal{B}$ -modules.

**Theorem 6.1** *Let $X$ be the smooth intersection of $m$ quadrics in $\mathbb{P}^{n-1}$ , $2m \leq n$ . Then there exists a fully faithful functor $\mathcal{D}^b(coh(\mathcal{B})) \to \mathcal{D}^b(X)$ . Moreover,*

(i) *if $2m < n$ , we have a semiorthogonal decomposition*

$$\mathcal{D}^b(X) \;=\; \Big\langle \mathcal{O}_X(-n+2m+1), \ldots, \mathcal{O}_X, \mathcal{D}^b(coh(\mathcal{B})) \Big\rangle,$$

(ii) *if $2m = n$ , there is an equivalence $\mathcal{D}^b(coh(\mathcal{B})) \overset{\sim}{\to} \mathcal{D}^b(X)$ .*

For $m = 0$ , i.e. when there is no quadrics, the theorem coincides with Beilinson's description of the derived category of the projective space [Be]. For $m = 1$ , this is Kapranov's description of the derived category of the quadric [Kap].

For odd $n$, the element $d$ generates the center of $\mathcal{B}$ over $\mathcal{O}_{\mathbb{P}(U)}$. Hence the spectrum of the center of $\mathcal{B}$ is a ramified double cover $Y$ over $\mathbb{P}(U)$. Also $\mathcal{B}$ yields a coherent sheaf of algebras $\mathcal{B}'$ over $Y$, such that $coh(\mathcal{B}')$ is equivalent to $coh(\mathcal{B})$. For the above case of two even dimensional quadrics, $\mathcal{B}'$ is an Azumaya algebra over $Y = C$. Since Brauer group of $Y$ (taken over an algebraically closed field of characteristic zero) is trivial, the category $coh(\mathcal{B}')$ is equivalent to $coh(\mathcal{O}_Y)$. Hence (1) follows from the theorem.

Furthermore, when $X$ is a K3 surface, the smooth intersection of 3 quadrics in $\mathbb{P}^5$, then the double cover $Y$ is also a K3 surface, but $\mathcal{B}'$ is in general a nontrivial Azumaya algebra over $Y$. The theorem states an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(coh(\mathcal{B}'))$.

This theorem illustrates the principle from section 3. The fully faithful functor is related to the moduli space of vector bundles on $X$, which are the restrictions to $X$ of the spinor bundles on the quadrics. The (commutative) moduli space involved is either $\mathbb{P}(U)$ or $Y$, depending on parity of $n$.

Algebraically, the fully faithful functor in the theorem is given by an appropriate version of Koszul duality. Theorem 6.1 has a generalization to a class of Koszul Gorenstein algebras, which includes the coordinate rings of superprojective spaces.

# References

[AKMW]  Abramovich D., Karu K., Matsuki K., Wlodarczyk J., Torification and Factorization of Birational Maps *preprint* math.AG/9904135.

[Be]  Beilinson A., Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra, *Funkcionalnyi analiz i ego pril.* 12 (1978), 68-69.

[Bo]  Bondal A., Representations of associative algebras and coherent sheaves, *Izv. Akad. Nauk SSSR, Ser.Mat.* 53 (1989), 25-44; English transl. in *Math. USSR Izv.* 34 (1990).

[BK]  Bondal A., Kapranov M., Representable functors, Serre functors, and mutations, *Izv. Akad. Nauk SSSR, Ser.Mat.*, 53 (1989), 1183-1205; English transl. in *Math. USSR Izv.*, 35 (1990), 519-541.

[BO1]  Bondal A., Orlov D., Semiorthogonal decomposition for algebraic varieties, *preprint MPIM* 95/15 (1995), *preprint* math.AG/9506012.

[BO2]  Bondal A., Orlov D., Reconstruction of a variety from the derived category and groups of autoequivalences, *Compositio Mathematica*, v.125 (2001) N.3, 327-344.

[BVdB]    Bondal A., Van den Bergh M., Generators and representability of functors in commutative and noncommutative geometry, *preprint* math.AG/0204218.

[Br]    Bridgeland T., Flops and derived categories, *preprint* math.AG/0009053.

[BKR]    Bridgeland T., King A., Reid M., Mukai implies McKay: the McKay correspondence as an equivalence of derived categories, *J. Amer. Math. Soc.*, 14 (2001), 535-554, *preprint* math.AG/9908027.

[Kap]    Kapranov M., On the derived categories of coherent sheaves on some homogeneous spaces, *Invent. Math.*, 92 (1988), 479-508.

[Kaw]    Kawamata Y., Francia's flip and derived categories, *preprint* math.AG/0111041.

[Mu1]    Mukai S., Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.* 81 (1981), 153–175.

[Mu2]    Mukai S., On the moduli space of bundles on a K3 surface I, *Vector bundles on algebraic varieties*, Tata Institute of Fundamental Research, Oxford University Press, Bombay and London, 1987.

[Na]    Namikawa Y., Mukai flops and derived categories, *preprint* math.AG/0203287.

[Or1]    Orlov D., Projective bundles, monoidal transformations and derived categories of coherent sheaves, *Izv. Akad. Nauk SSSR Ser.Mat.* 56 (1992), pp.852-862; English transl. in *Math. USSR Izv.* 38 (1993), 133-141.

[Or2]    Orlov D., Equivalences of derived categories and K3 surfaces, *J. of Math. Sciences, Alg. geom.-5*, 84, N5, (1997), 1361-1381, *preprint* math.AG/9606006.

[Or3]    Orlov D., On derived categories of coherent sheaves on abelian varieties and equivalences between them, *Izv. RAN, Ser. Mat.*, 66 (2002) N3, 131-158, (see also math.AG/9712017).

[Po]    Polishchuk A., Symplectic biextensions and a generalization of the Fourier-Mukai transform, *Math. Res. Let.*, v.3 (1996), 813-828.

[Ta]    Tate J., Homology of Noetherian rings and local rings, *Illinois J. Math.*, 1 (1957) N1, 14-27.

[VdB]    Van den Bergh M., Three-dimensional flops and non-commutative ring, *paper in preparation*.

[Wl]    Wlodarczyk J., Birational cobordisms and factorization of birational maps, *preprint* math.AG/9904074.