QUASI-COHERENT SHEAVES IN COMMUTATIVE AND NONCOMMUTATIVE GEOMETRY.

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On occasion 80-th birthday of Igor Rostislavovich Shafarevich

ABSTRACT. In this paper we give a definition of quasi-coherent modules on any presheaves of sets on the categories of affine commutative and noncommutative schemes. This definition generalizes the usual definition of quasi-coherent sheaves on schemes. The property of a quasi-coherent module to be a sheaf in various topologies is investigated. Using presheaves of groupoids, an embedding of commutative geometry to noncomutative one is constructed.

INTRODUCTION

In this paper we show how quasi-coherent sheaves appear in commutative and noncommutative geometry in a natural way. Considering presheaves of sets on the categories of affine commutative and noncommutative schemes as very wide generalization of the notion of scheme (or algebraic space), we show that quasi-coherent sheaves (modules) can be defined on such objects as well. Further, we study basic properties of quasi-coherent modules on presheaves of sets and, more generally, on presheaves of groupoids.

The notion of a scheme in commutative algebraic geometry is introduced in two steps. The first step is defining of affine schemes; the second is to prescribe how to glue a scheme from affine pieces. In this way we make use of topological properties of commutative schemes. In particular, the functor *Spec* from the category of commutative algebras to the category of topological spaces is used. On the other hand, any scheme can be considered as a presheaf of sets on the category of affine schemes. We will exploit just this simple fact to introduce and treat noncommutative geometry.

As in commutative algebraic geometry, the category of *affine* noncommutative schemes is the category which is opposite to the category of algebras (we work over a basic ring k). Any object which can be regarded as a noncommutative scheme has to give rise to a presheaf of sets on the category of affine noncommutative schemes.

In this paper we consider all presheaves of sets on the category of affine schemes, both in the commutative and in the noncommutative cases. With any presheaf of sets X we associate a category of (right) quasi-coherent modules $\mathrm{Qcoh}_r(X)$ on it (Definition 2.2).

This work is done under partial financial support of the Russian Foundation for Basic Research (grant no. 02-01-00468), grant in support of leading scientific schools (no. 00-15-96085) and grant INTAS-OPEN-2000-269. The research described in this publication was made possible in part by Award No RM1-2405-MO-02 of the U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union (CRDF)..

This definition does not depend on topology. Further, we prove that any quasi-coherent module is a sheaf in the canonical topology on X (Corollary 3.2 and Theorem 4.3). In the commutative case we define a new topology on the category of affine schemes that is directly related to quasi-coherent modules and will be called effective descent topology (3.7). There is an equivalence between the category of quasi-coherent modules on a presheaf and the category of quasi-coherent module on its accosiated sheaf with respect to the given topology (Theorem 3.6).

Furthermore, we describe "good" embedding of the commutative geometry into the noncommutative one. To do it we need to enlarge our class of objects and to consider not only presheaves of sets but presheaves of groupoids on the category of noncommutative affine schemes (4.4). This embedding satisfies the property that the category of quasi-coherent modules on any object X does not depend on whether it is considered as living in the commutative or noncommutative world (Theorem 4.2).

I would like to thank Alexei Bondal and Alexander Rosenberg for useful discussions. My special thank is to Andrei Suslin who answered to many my questions.

The main part of this work was done while the author was visiting the Max-Plank-Institute für Mathematik in Bonn.

1. Some categorical preliminaries. Sites.

1.1. In this section we collect some known facts related to categories and sites which will be needed in the sequel. All they can be found in [1] (see also [5]). Let $\mathcal{C}$ be a category. Denote by $\mathcal{C}^\frown$ the category of contravariant functors from $\mathcal{C}$ to the category of sets. An object of $\mathcal{C}^\frown$ is called a presheaf of sets on $\mathcal{C}$. There is a natural functor $h : \mathcal{C} \longrightarrow \mathcal{C}^\frown$ that sends an object $R \in \mathcal{C}$ to the functor $h_R(-) := \mathrm{Hom}(-\,,\ R)$. The presheaf h_R is called representable. Henceforth, we will write R instead h_R . By Ioneda Lemma the functor h is a full embedding, moreover, for any presheaf $X \in C^\frown$ there is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}^\frown}(R, X) = X(R)$.

For any object $X \in \mathcal{C}^\frown$ denote by $\mathcal{C}/X$ the category of pairs (R, ϕ) , where $R \in \mathcal{C}$ and $\phi \in X(R)$. Morphisms from (S, ψ) to (R, ϕ) are morphisms $f : S \to R$ such that $X(f)(\phi) = \psi$. There is a canonical functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$ that takes a pair (R, ϕ) to the object R . Note that any presheaf $X \in \mathcal{C}^\frown$ is a colimit in $C^\frown$ of representable objects by the system $\mathcal{C}/X$.

1.2. Let $u : \mathcal{C} \to \mathcal{C}'$ be a functor. Denote by $\hat{u}^*$ the functor from $\mathcal{C}'^\frown$ to $\mathcal{C}^\frown$ that takes a presheaf $Y \in \mathcal{C}'^\frown$ to the composition Yu .

Since the category of sets has all small limits and colimits the functor $\hat{u}^*$ admits left and right adjoint functors $\hat{u}_!$ and $\hat{u}_*$ respectively (see [1], I, 5.1.). The functor $\hat{u}^*$ preserves all small limits and colimits. The functor $\hat{u}_*$ preserves all small limits. The functor $\hat{u}_!$ preserves all small colimits and the restriction of it on the subcategory $\mathcal{C} \subset \mathcal{C}^\frown$ coincides with u . Moreover, for any presheaf $X \in \mathcal{C}^\frown$ we have

$$\hat{u}_! X(S) \cong \varinjlim_{R \in \mathcal{C}/X} \mathrm{Hom}(S, uR).$$

It can be shown that the functor $\hat{u}_!$ is fully faithful if and only if the functor u is fully faithful (see [1], I.5.6). (We recall that a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is called fully faithful if it induces an isomorphism $\mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}'}(FY, FX)$ for any pair Y, X of objects from $\mathcal{C}$.)

1.3. A sieve in the category $\mathcal{C}$ is a full subcategory $\mathcal{D} \subset \mathcal{C}$ that satisfies the following condition: any object of $\mathcal{C}$ for which there exists a morphism from it to an object of $\mathcal{D}$ contains in $\mathcal{D}$. Let R be an object of $\mathcal{C}$, a sieve in the category $\mathcal{C}/R$ will be called a sieve on R .

It is easy to see that a sieve on R is nothing more than a subpresheaves of R in the category $\mathcal{C}^\frown$. By this reason, we will consider sieve on R as a presheaf. The notion of sieve is the main tool for the introduction of a topology on a category.

Definition 1.1. *A Grothendieck topology $\mathcal{T}$ on a category $\mathcal{C}$ is defined by giving for each object $R \in \mathcal{C}$ a set $J(R)$ of sieves on R which is called covering sieves. They have to satisfy the following conditions:*

- T 1) *For any R the maximal sieve $\mathcal{C}/R$ is in $J(R)$.*
- T 2) *If $T \in J(R)$ and $f : S \longrightarrow R$ is a morphism of $\mathcal{C}$, then the sieve $f^*(T) := \left\{U \xrightarrow{\alpha} S \mid f\alpha \in T\right\}$ is in $J(S)$.*
- T 3) *If $T \in J(R)$ is a covering sieve and U is a sieve on R such that $f^*(U) \in J(S)$ for each $S \xrightarrow{f} R$ in T , then $U \in J(R)$.*

Definition 1.2. *A category $\mathcal{C}$ endowed with a Grothendieck topology $\mathcal{T}$ is called by site and will be denotes as $\Phi = (\mathcal{C}, \mathcal{T})$.*

Let $\Phi = (\mathcal{C}, \mathcal{T})$ be a site, and R be an object of $\mathcal{C}$. A family of morphisms $F = (f_\alpha : R_\alpha \longrightarrow R)$, $\alpha \in I$ is called covering if the sieve $T_F \hookrightarrow R$ generated by the family F is a covering sieve on R .

Definition 1.3. *Let $\mathcal{C}$ be a category with fiber products. A pretopology on $\mathcal{C}$ is defined by specifying for each object R of $\mathcal{C}$ a set $Cov(R)$ of families of morphisms to R such that the following conditions hold:*

- P1) *If $(R_\alpha \to R)\ \alpha \in I$ is in $Cov(R)$ and $S \to R$ is a morphism of $\mathcal{C}$, then the family $(R_\alpha \times S \to S)$, $\alpha \in I$ is in $Cov(S)$.*
- P2) *If $(R_\alpha \to R)\ \alpha \in I$ is in $Cov(R)$ and $(R_{\beta_\alpha} \to R_\alpha)\ \beta_\alpha \in J_\alpha$ is in $Cov(R_\alpha)$ for each $\alpha \in I$, then the family $(R_\gamma \to R)\ \gamma \in \coprod_{\alpha \in I} J_\alpha$ is in $Cov(R)$.*
- P3) *The family $R \xrightarrow{id_R} R$ is in $Cov(R)$.*

Any pretopology P on $\mathcal{C}$ generates a topology $\mathcal{T}$ for which covering sieves are seaves contained some P -covering family.

1.4. A presheaf $X \in \mathcal{C}^\frown$ is called sheaf (resp. separate presheaf) if for any object $R \in \mathcal{C}$ and for any covering sieve $S \in J(R)$ the canonical morphism

$$\mathrm{Hom}_{\mathcal{C}^\frown}(R, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}^\frown}(S, X)$$

is a bijection (resp. an injection).

Denote by $Shv(\mathcal{C}, \mathcal{T})$ the category of all sheaves on the site $\Phi = (\mathcal{C}, \mathcal{T})$. The category $Shv(\mathcal{C}, \mathcal{T})$ is a full subcategory of $\mathcal{C}^\frown$.

The inclusion functor $i : Shv(\mathcal{C}, \mathcal{T}) \longrightarrow \mathcal{C}^\frown$ has a left adjoint functor $\underline{a} : \mathcal{C}^\frown \longrightarrow Shv(\mathcal{C}, \mathcal{T})$ which is known as the associated sheaf functor, or the sheafification functor. The functor $\underline{a}$ preserves all finite limits and colimits. For any presheaf X we can define a presheaf LX by the rule

$$LX(R) = \varinjlim_{T \in J(R)} \mathrm{Hom}_{\mathcal{C}^\frown}(T, X) \tag{1}$$

This correspondence gives a functor from $\mathcal{C}^\frown$ to itself which is denoted by L . The composition of the functor L with itself is isomorphic to the functor $i\underline{a}$, i.e. there is an isomorphism $\underline{a}X \cong LLX$ (see [1], III).

1.5. Let $\Phi = (\mathcal{C}, \mathcal{T})$ and $\Phi' = (\mathcal{C}', \mathcal{T}')$ be two sites.

A functor $u : \mathcal{C} \longrightarrow \mathcal{C}'$ is called continue if for any sheaf $Y \in Shv(\mathcal{C}', \mathcal{T}')$ the presheaf $\hat{u}^*(Y)$ is a sheaf on site Φ . In this case $\hat{u}^*$ induces a functor from $Shv(\mathcal{C}', \mathcal{T}')$ to $Shv(\mathcal{C}, \mathcal{T})$, which will be denoted by u^* . The functor u^* has a left adjoint functor $u_! = \underline{a}'\hat{u}_! i$.

A functor $u : \mathcal{C} \longrightarrow \mathcal{C}'$ is called cocontinue if for any object $R \in \mathcal{C}$ and for any covering sieve $S \hookrightarrow u(R)$ the sieve on R , generated by all morphisms $Z \longrightarrow R$ such that $u(Z) \longrightarrow u(R)$ factorized through S , is a covering sieve on R . Note that u is cocontinue iff the functor $\hat{u}_* : \mathcal{C}^\frown \longrightarrow \mathcal{C}'^\frown$ takes sheaves on Φ to sheaves on Φ' . Denote as u_* the functor from $Shv(\mathcal{C}, \mathcal{T})$ to $Shv(\mathcal{C}', \mathcal{T}')$ which is the restriction of $\hat{u}_*$ on the categories of sheaves. It has a left adjoint functor $u^* = \underline{a}\hat{u}^* i'$ (see [1], III).

Consider a functor $u : \mathcal{C}' \longrightarrow \mathcal{C}$ and a topology $\mathcal{T}$ on $\mathcal{C}$. The induced topology on $\mathcal{C}'$ is the largest topology such that the functor u is continue.

1.6. Fix an object $X \in \mathcal{C}^\frown$ and consider the category $\mathcal{C}/X$ with the canonical functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$. If $\mathcal{T}$ is a topology on $\mathcal{C}$, then it induces a topology $\mathcal{T}_X$ on $\mathcal{C}/X$. Denote by Φ_X the coming site $(\mathcal{C}/X, \mathcal{T}_X)$. It is proved in [1], III.5. that a sieve S on an object $(Z \longrightarrow X)$ is covering in the category $\mathcal{C}/X$ if and only if the sieve $\widehat{j_{X!}}(S) \hookrightarrow Z$ is covering in $\mathcal{C}$. Therefore, the functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$ is continue and cocontinue together. This gives us three adjoint functors:

$$\begin{array}{llcl} j_{X*}: & Shv(\mathcal{C}/X, \mathcal{T}_X) & \longrightarrow & Shv(\mathcal{C}, \mathcal{T}) \\ j_X^*: & Shv(\mathcal{C}, \mathcal{T}) & \longrightarrow & Shv(\mathcal{C}/X, \mathcal{T}_X) \\ j_{X!}: & Shv(\mathcal{C}/X, \mathcal{T}_X) & \longrightarrow & Shv(\mathcal{C}, \mathcal{T}) \end{array} \tag{2}$$

The functor j_{X*} is called the direct image functor. The functor j_X^* is called the inverse image functor or restriction functor to $\mathcal{C}/X$. The functor $j_{X!}$ is called extension by zero onto $\mathcal{C}$.

Let $m : Y \longrightarrow X$ be a morphism in $\mathcal{C}^\frown$. It induces a functor $j_m : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$. By the same reason as above, there are three functors

$$j_m^* : Shv(\mathcal{C}/X, \mathcal{T}_X) \longrightarrow Shv(\mathcal{C}/Y, \mathcal{T}_Y), \qquad j_{m*}, j_{m!} : Shv(\mathcal{C}/Y, \mathcal{T}_Y) \longrightarrow Shv(\mathcal{C}/X, \mathcal{T}_X),$$

where j_{m*} and $j_{m!}$ are right and left adjoint to j_m^* respectively.

Proposition 1.4. [1] *The functor* $j_{X!} : Shv(\mathcal{C}/X, \mathcal{T}_X) \longrightarrow Shv(\mathcal{C}, \mathcal{T})$ *can be represented as a composition*

$$Shv(\mathcal{C}/X, \mathcal{T}_X) \xrightarrow{e_X} Shv(\mathcal{C}, \mathcal{T})/\underline{a}X \longrightarrow Shv(\mathcal{C}, \mathcal{T}),$$

and the functor e_X *is an equivalence.*

Moreover, the canonical map $m : X \longrightarrow \underline{a}X$ *induces the equivalences* j_m^* *and* $j_{m*} = j_{m!}$ *between the categories* $Shv(\mathcal{C}/X, \mathcal{T}_X)$ *and* $Shv(\mathcal{C}/\underline{a}X, \mathcal{T}_{\underline{a}X})$ *for any presheaf* $X \in \mathcal{C}^\frown$*.*

1.7. A topology, for which every representable presheaf is a sheaf, is called subcanonical; the largest such topology is called canonical. The canonical topology will be denoted as can .

Let us describe covering families for the canonical topology.

We will need the definition of equalizer. If $f_i : X \longrightarrow Y, i = 1, 2$ are two morphisms of a category $\mathcal{C}$, then an equalizer is a morphism $k : K \to X$ which is final in a full subcategory of $\mathcal{C}/X$ consisting of morphisms $k' : K' \to X$ with $f_1 k' = f_2 k'$. If an equalizer of two morphisms exists, then it is unique up to isomorphism.

Suppose that all fiber products exist in the category $\mathcal{C}$. A family of maps $(X_\alpha \to X)_{\alpha \in I}$ in the category $\mathcal{C}$ is called a universal effectively epimorphic family if for any map $Z \longrightarrow X$ and for every object $Y \in \mathcal{C}$ the following diagram of sets

$$\mathrm{Hom}(Z, Y) \longrightarrow \prod_{\alpha \in I} \mathrm{Hom}(Z \times_X X_\alpha, Y) \rightrightarrows \prod_{\alpha, \beta \in I} \mathrm{Hom}(Z \times_X (X_\alpha \times_X X_\beta), Y)$$

is an equalizer.

It is clear that a family is covering for the canonical topology if and only if it is a universal effectively epimorphic family. Let X be an object of $\mathcal{C}$. A family $(W_\alpha \longrightarrow W)_{\alpha \in I}$ is covering for the object $(W \to X) \in \mathcal{C}/X$ in the canonical topology on $\mathcal{C}/X$ if and only if it is a universal effectively epimorphic family in $\mathcal{C}$.

Proposition 1.5. *Let* X *be a presheaf on the category* $\mathcal{C}$ *. Then the topology on* $\mathcal{C}/X$ *, induced by the canonical topology on* $\mathcal{C}$ *with respect to the functor* $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$ *, coincides with the canonical topology on* $\mathcal{C}/X$ *.*

It follows from Corollary 3.3. of [1] and from the description of covering families for the canonical topologies on $\mathcal{C}$ and on $\mathcal{C}/X$.

2. Ringed sites and quasi-coherent sheaves

2.1. A category $\mathcal{C}$ endowed with a presheaf of rings $\mathcal{A}$ is called a ringed category. A site $\Phi = (\mathcal{C}, \mathcal{T})$ endowed with a sheaf of rings $\mathcal{A}$ is called a ringed site.

Let $\Phi = (\mathcal{C}, \mathcal{T})$ be a ringed site with a sheaf of rings $\mathcal{A}$. A presheaf of (right) $\mathcal{A}$-modules is a presheaf $\mathcal{M}$ with a (right) module structure over ring object $\mathcal{A}$.

If $\mathcal{M}$ is a sheaf it will be called a sheaf of (right) $\mathcal{A}$-modules.

Denote by $\mathrm{Mod}_r(\mathcal{A}, \mathcal{T})$ the category of sheaves of (right) $\mathcal{A}$-modules on the site Φ. The category $\mathrm{Mod}_r(\mathcal{A}, \mathcal{T})$ is an abelian category satisfying the conditions AB5 and AB3* and has a set of generators (see [1], II).

2.2. Let X be an object of $\mathcal{C}^\frown$. The category $\mathcal{C}/X$ with the topology $\mathcal{T}_X$, induced by the functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$, forms the site $\Phi_X = (\mathcal{C}/X, \mathcal{T}_X)$. This site is ringed by the sheaf of rings $\mathcal{A}_X := j_X^* \mathcal{A}$. We will denote the category of sheaves of (right) $\mathcal{A}_X$-modules as $\mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$ (or as $\mathrm{Mod}_r(X, \mathcal{T}_X)$ when the sheaf of rings is fixed).

Let $\mathcal{M}$ be a sheaf of (right) $\mathcal{A}$-modules. The sheaf $j_X^* \mathcal{M}$ has the structure of (right) $\mathcal{A}_X$-module. If $\mathcal{N}$ is a sheaf of (right) $\mathcal{A}_X$-modules then $j_{X*}\mathcal{N}$ is a sheaf of (right) $j_{X*}\mathcal{A}_X$-modules. Using the canonical morphism $\mathcal{A} \longrightarrow j_{X*}\mathcal{A}$, we obtain that $j_{X*}\mathcal{N}$ has the structure of (right) $\mathcal{A}$-module. Thus, there is a couple of adjoint functors:

$$\mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X) \quad \underset{j_{X*}}{\overset{j_X^*}{\rightleftarrows}} \quad \mathrm{Mod}_r(\mathcal{A}, \mathcal{T})$$

(for functors between categories of sheaves of modules we use the same notation as in 1.6 for functors between categories of sheaves of sets.)

The functor $j_X^* : \mathrm{Mod}_r(\mathcal{A}, \mathcal{T}) \longrightarrow \mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$ has a left adjoint functor $j_{X!}$ that is called the extension by zero. Note that, unlike the functors j_X^* and j_{X*}, the composition of the functor $j_{X!}$ for sheaves of modules with the forgetful functor to the category of sheaves of sets does not coincide with the functor $j_{X!}$ for sheaves of sets defined in 1.6 (see [1], IV.11.).

Let $m : Y \longrightarrow X$ be a morphism in $\mathcal{C}^\frown$. As above, the functor $j_m : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ induces three functors

$$j_m^* : \mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X) \longrightarrow \mathrm{Mod}_r(\mathcal{A}_Y, \mathcal{T}_Y), \qquad j_{m*}, j_{m!} : \mathrm{Mod}_r(\mathcal{A}_Y, \mathcal{T}_Y) \longrightarrow \mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$$

such that j_{m*} and $j_{m!}$ are right and left adjoint to j_m^* respectively. Moreover, the functors j_m^* and j_{m*} commute with the same functors between the categories of sheaves of sets defined in 1.6 with respect to forgetful functors.

For each sheaf of $\mathcal{A}_X$-modules $\mathcal{F}$ on X and for any morphism $m : Y \longrightarrow X$ one can defined a space of section of the sheaf $\mathcal{F}$ on Y by the formula

$$\Gamma(Y, \mathcal{F}) := \mathrm{Hom}(\mathcal{A}_Y, j_m^* \mathcal{F}).$$

Proposition 2.1. *Let $\Phi = (\mathcal{C}, \mathcal{T})$ be a site ringed by a sheaf of rings $\mathcal{A}$. Consider the canonical map $m : X \longrightarrow \underline{a}X$ of associated sheaf for a presheaf $X \in \mathcal{C}^\frown$. Then the* *functors* j_m^* *and* j_{m*} *between the categories* $\mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$ *and* $\mathrm{Mod}_r(\mathcal{A}_{\underline{a}X}, \mathcal{T}_{\underline{a}X})$ *are quasi-inverse equivalences.*

It follows from Proposition 1.4.

2.3. Assume that a presheaf X is representable, i.e. $X \in \mathcal{C}$. Let M be a right $\mathcal{A}(X)$-module. Let us attach to it a presheaf of (right) $\mathcal{A}_X$-modules $\widetilde{M}$ on the category $\mathcal{C}/X$ defined by the rule

$$\widetilde{M}(S) := M \otimes_{\mathcal{A}(X)} \mathcal{A}(S)$$

for an object $S \to X$ from $\mathcal{C}/X$.

Now consider an arbitrary object $X \in \mathcal{C}^\frown$ and the corresponding category $\mathcal{C}/X$.

Definition 2.2. *A presheaf* $\mathcal{F}$ *of (right)* $\mathcal{A}_X$ *-modules on the category* $\mathcal{C}/X$ *will be called a (right)* quasi-coherent module *on* X *if for any* $m \in X(R)$ *the presheaf* $\widehat{j_m}^*(\mathcal{F})$ *on* $\mathcal{C}/R$ *with the canonical structure of (right)* $\mathcal{A}_R$ *-module has a form* $\widetilde{M}$ *, where* M *is a (right)* $\mathcal{A}(R)$ *-module. A quasi-coherent sheaf is called locally free if the module* M *is projective for any* $m \in X(R)$ *.*

Morphisms between quasi-coherent modules are, by definition, morphisms between them considered as presheaves of (right) $\mathcal{A}_X$-modules. Denoted by $\mathrm{Qcoh}_r(X)$ the category of (right) quasi-coherent modules on X , which is a full subcategory of the category of presheaves of (right) $\mathcal{A}_X$-modules.

Note that the definition of a quasi-coherent module on $X \in \mathcal{C}^\frown$ does not depend on a topology.

2.4. It immediately follows from Definition 2.2 that each quasi-coherent module on a representable object $R \in \mathcal{C}$ is isomorphic to $\widetilde{M}$ for some $\mathcal{A}(R)$-module M , i.e in this case the category $\mathrm{Qcoh}_r(R)$ is equivalent to the category of (right) $\mathcal{A}(R)$-modules. Hence, a presheaf $\mathcal{F}$ of (right) $\mathcal{A}_X$-modules is a quasi-coherent module on $X \in \mathcal{C}^\frown$ if and only if for each $(R \stackrel{s}{\longrightarrow} X) \in \mathcal{C}/X$ the presheaf $\widehat{j_s}^*\mathcal{F}$ on R is quasi-coherent.

2.5. For every $X \in \mathcal{C}^\frown$ the category $\mathrm{Qcoh}_r(X)$ has all cokernels and all direct sums. Hence, it has all small colimits (see [4] I.5).

Let $m : Y \longrightarrow X$ be a morphism in $\mathcal{C}^\frown$. The functor $\widehat{j_m}^* : (\mathcal{C}/X)^\frown \longrightarrow (\mathcal{C}/Y)^\frown$ induces the inverse image functor from $\mathrm{Qcoh}_r(X)$ to $\mathrm{Qcoh}_r(Y)$ which will be denoted as m^* . The functor m^* is right exact, this means that it preserves cokernels. For a quasi-coherent module $\mathcal{F}$ on X and for any morphism $mY \longrightarrow X$ one can define a space of section of $\mathcal{F}$ on Y by the formula

$$\Gamma(Y, \mathcal{F}) := \mathrm{Hom}(\mathcal{A}_Y, m^*\mathcal{F}).$$

2.6. Assume that the topology $\mathcal{T}$ on $\mathcal{C}$ satisfies the following condition: every quasi-coherent module on an object $X \in \mathcal{C}^{\frown}$ is a sheaf for the topology $\mathcal{T}_X$ on $\mathcal{C}/X$. In this case the category $\mathrm{Qcoh}(X)$ is a full subcategory of $\mathrm{Mod}(\mathcal{A}_X, \mathcal{T}_X)$. Denote by φ_X the inclusion functor $\mathrm{Qcoh}(X) \hookrightarrow \mathrm{Mod}(\mathcal{A}_X, \mathcal{T}_X)$.

Proposition 2.3. *Let* $\Phi = (\mathcal{C}, \mathcal{T})$ *be a site ringed by a sheaf of rings* $\mathcal{A}$. *Suppose all quasi-coherent modules on any object* X *are sheaves for the topology* $\mathcal{T}_X$ *on* $\mathcal{C}/X$. *Let* $m : X \longrightarrow \underline{a}X$ *be the canonical map of associated sheaf for a presheaf* $X \in \mathcal{C}^{\frown}$. *Then the inverse image functor* $m^* : \mathrm{Qcoh}(\underline{a}X) \longrightarrow \mathrm{Qcoh}(X)$ *is fully faithful.*

Proof. There is the commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Qcoh}(X) & \overset{\varphi_X}{\hookrightarrow} & \mathrm{Mod}(\mathcal{O}_X, can_X) \\ {\scriptstyle m^*}\uparrow & & \uparrow{\scriptstyle j_m^*} \\ \mathrm{Qcoh}(\underline{c}X) & \overset{\varphi_{\underline{c}X}}{\hookrightarrow} & \mathrm{Mod}(\mathcal{O}_{\underline{c}X}, can_{\underline{c}X}) \end{array}$$

where the horizontal arrows are full embeddings. By Proposition 2.1, the functor j_m^* is an equivalence. Hence, the functor m^* is fully faithful. □

2.7. Suppose that the inclusion functor $\varphi_X : \mathrm{Qcoh}_r(X) \longrightarrow \mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$ has a right adjoint functor Q_X . The functor Q_X is called by the coherator. Since φ_X is fully faithful, the composition $Q_X \varphi_X$ is isomorphic to the identity functor on $\mathrm{Qcoh}_r(X)$. If $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is a morphism in $\mathrm{Qcoh}_r(X)$ then it has a kernel $Ker(f)$ which can be defined as $Q_X Ker(\varphi_X(f))$.

It is easy to check that if the coherator exists then the category $\mathrm{Qcoh}_r(X)$ is abelian.

For a representable object $X \in \mathcal{C}$ the coherator Q_X exists and it is the functor taking a sheaf of $\mathcal{A}_X$ -modules $\mathcal{F}$ to the quasi-coherent module $\widetilde{\mathcal{F}(X)}$.

Let $m : Y \longrightarrow X$ be a morphism in $\mathcal{C}^{\frown}$. If there is a coherator Q_X then the functor m^* has a right adjoint functor m_* which is defined as the composition $Q_X j_{m*} \varphi_Y$.

With any presheaf $X \in \mathcal{C}^{\frown}$ we associated the category of (right) quasi-coherent modules $\mathrm{Qcoh}_r(X)$. This construction can be extended on a larger class of object by two ways.

2.8. First, let us consider an immersion $i : Y \hookrightarrow X$. One can define a category of pair $\mathrm{Qcoh}_r(X, Y)$ as a full subcategory of $\mathrm{Qcoh}_r(X)$ consisting of such modules $\mathcal{F}$ for which $i^*(\mathcal{F}) \equiv 0$.

Any immersion $i : Y \hookrightarrow X$ induces the factor object X/Y . It is a pointed object and it is defined as the coproduct

$$\begin{array}{ccc} Y & \overset{i}{\hookrightarrow} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \hookrightarrow & X/Y. \end{array}$$

We can compare the categories $\mathrm{Qcoh}_r(X/Y, *)$ and $\mathrm{Qcoh}_r(X, Y)$.

Proposition 2.4. *The natural functor* $p^* : \mathrm{Qcoh}(X/Y, *) \longrightarrow \mathrm{Qcoh}(X, Y)$ *is an equivalence.*

Proof. Let us construct a quasiinverse functor $p_* : \mathrm{Qcoh}_r(X,Y) \longrightarrow \mathrm{Qcoh}_r(X/Y,*)$. For any $R \in \mathcal{C}$ there is an isomorphism of the sets $X/Y(R) = X(R)/Y(R)$. Let $s \in X/Y(R)$ be a section. Either there is a unique pull back of s to the section $s' \in X(R)$ or it goes through $* \to X/Y$. If now $\mathcal{F}$ is a quasi-coherent module on X such that $i^*(\mathcal{F}) \equiv 0$, then we put $p_*(\mathcal{F})(s) \cong \mathcal{F}(s')$ or put it by zero if it is pointed section. It is easy to see that this gives us a functor that is quasi-inverse to p^* . □

2.9. Second, we can consider not only presheaves of sets but presheaves of groupoids (or even presheaves of categories). By groupoid we mean a small category in which every morphism is invertible. All small groupoids form a 2–category. If we identify all functors between groupoids which are equivalent then we obtain the 1–category. Denote this 1–category of all small groupoids by Grd . Any set can be considered as a groupoid, objects of which are elements of this set and morphisms are only identities. To each groupoid G we can attach two sets: the first $Ob(G)$ is the set of all objects of G , the second $\pi_0(G)$ is the set of connected components of G . There are two natural morphisms of the groupoids $Ob(G) \longrightarrow G$ and $\pi_0 : G \longrightarrow \pi_0(G)$.

Denote as $\mathcal{C}^{\frown}_{gr}$ the category of presheaves of groupoids on $\mathcal{C}$, i.e the category of all contravariant functors from $\mathcal{C}$ to the category of small groupoids Grd . It contains the category $\mathcal{C}^{\frown}$ as a full subcategory. Any presheaf $X \in \mathcal{C}^{\frown}_{gr}$ defines two presheaves of sets $Ob(X)$ and $\pi_0(X)$ given as

$$Ob(X)(R) = Ob(X(R)), \quad \pi_0(X)(R) = \pi_0(X(R)), \qquad R \in \mathcal{C}.$$

There are natural morphisms $ob_X : Ob(X) \longrightarrow X$ and $\pi_{0X} : X \longrightarrow \pi_0(X)$ between the presheaves of groupoids. These correspondences give two functors Ob and π_0 from $\mathcal{C}^{\frown}_{gr}$ to $\mathcal{C}^{\frown}$.

The morphism $\pi_{0X} : X \longrightarrow \pi_0(X)$ splits for any $X \in \mathcal{C}^{\frown}_{gr}$.(Actually, choosing for each connected component some representative, we obtain a splitting). The functor $\pi_0 : \mathcal{C}^{\frown}_{gr} \longrightarrow \mathcal{C}^{\frown}$ is simultaneously right and left adjoint to the canonical immersion $\mathcal{C}^{\frown} \longrightarrow \mathcal{C}^{\frown}_{gr}$.

To any site $\Phi = (\mathcal{C}, \mathcal{T})$ one can attach the category of sheaves of groupoids $Shv_{gr}(\mathcal{C}, \mathcal{T})$ on it. The presheaf $X \in \mathcal{C}^{\frown}_{gr}$ is called a sheaf of groupoids if for any groupoid $G \in \mathrm{Grd}$ the presheaf of sets:

$$R \mapsto \mathrm{Hom}_{\mathrm{Grd}}(G, X(R))$$

is a sheaf.

2.10. Consider a site $\Phi = (\mathcal{C}, \mathcal{T})$ ringed by a sheaf $\mathcal{A}$. For any presheaf of categories X (in particular a presheaf of groupoids) define a category $\mathcal{C}/X$. The set of objects of $\mathcal{C}/X$ is the set of all $s \in Ob(X(S)),\ S \in \mathcal{C}$. Morphisms from $s \in Ob(X(S))$ to $r \in Ob(X(R))$ are pairs (α, f) , where $f : S \longrightarrow R$ in $\mathcal{C}$ and α is an isomorphism from s to $X(f)(r)$ in $X(S)$. There is a canonical functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$. Inducing the topology, we obtain the site $\Phi_X = (\mathcal{C}/X, \mathcal{T}_X)$ ringed by the sheaf $\mathcal{A}_X = j_X^* \mathcal{A}$. One can define the category of sheaves of $\mathcal{A}_X$ -modules $\mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$. Furthermore, repeating Definition

2.2, we obtain the category $\mathrm{Qcoh}_r(X)$ of quasi-coherent modules on X . The morphisms ob_X and π_{0X} induce the inverse image functors $ob_X^* : \mathrm{Qcoh}(X) \longrightarrow \mathrm{Qcoh}(Ob(X))$ and $\pi_{0X}^* : \mathrm{Qcoh}(\pi_0(X)) \longrightarrow \mathrm{Qcoh}(X)$ respectively.

3. Sheaves in commutative geometry.

3.1. Fix a commutative ring k . Denote as $Comalg/k$ the category of commutative algebras over k and as Aff/k denote the category of affine schemes that is opposite to $Comalg/k$. Note that the category of affine schemes has all fiber products, which correspond to the tensor products in the category $Comalg/k$. The canonical embedding of the category affine schemes to the category of all schemes Sch/k preserves all fiber products and finite disjoint unions. But it does not preserve coproducts in general. For example, the infinite disjoint union of $\{Spec(R_\alpha)\}_{\alpha\in I}$ in Sch/k does not coincide with $Spec(\prod\limits_{\alpha\in I} R_\alpha)$.

3.2. In this section we mean by $\mathcal{C}$ the category of all affine schemes Aff/k . The canonical contravariant functor $Aff/k \longrightarrow Comalg/k$ gives the presheaf of rings $\mathcal{O}$ on Aff/k with

$$\mathcal{O}(Spec(R)) := R$$

for any k-algebra R . This presheaf is represented by the ring scheme $Spec\, k[t]$. Therefore, it is a sheaf for any subcannonical topology on $\mathcal{C}$.

Thus, for any object $X \in \mathcal{C}^\frown$ the category $\mathcal{C}/X$ is ringed by the presheaf $\mathcal{O}_X$ which is a sheaf for any subcanonical topology.

3.3. Denote as can the canonical topology on $\mathcal{C} = Aff/k$ and as $\underline{c} : \mathcal{C}^\frown \longrightarrow Shv(\mathcal{C}, can)$ denote the associated sheaf functor defined in 1.4, which is a left adjoint to the inclusion functor $i : Shv(\mathcal{C}, can) \longrightarrow \mathcal{C}^\frown$. The covering families for the canonical topology are the universal effectively epimorphic families defined in 1.7. In the case of the category of affine schemes there is a simple but useful description of the universal effectively epimorphic families in terms of modules.

Proposition 3.1. *A family* $(f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha\in I}$ *is universal effectively epimorphic in* Aff/k *if and only if for any* R*-module* M *the following diagram*

$$M \longrightarrow \prod_{\alpha\in I} M \otimes_R R_\alpha \rightrightarrows \prod_{\alpha,\beta\in I} M \otimes_R (R_\alpha \otimes R_\beta) \tag{3}$$

is an equalizer.

Proof. $\Rightarrow$ Take an R-module M and consider $S = R \oplus M$ as R-algebra with multiplication low given by the rule $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_2 r_1)$. Denote as S_α the algebra $S \otimes_R R_\alpha$. Consider the family $(g_\alpha : Spec(S_\alpha) \to Spec(S))_{\alpha\in I}$, where $g_\alpha = 1_S \otimes f_\alpha$. This family is effectively epimorphic. Therefore, calculating morphisms to the scheme $Spec(k[t])$, we obtain that

$$S \longrightarrow \prod_{\alpha\in I} S \otimes_R R_\alpha \rightrightarrows \prod_{\alpha,\beta\in I} S \otimes_R (R_\alpha \otimes R_\beta) \tag{4}$$

is an equalizer. And hence, this is fulfilled for module M as for a direct summand as well.
$\Leftarrow$ Let $(f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha\in I}$ be a family. We know that for any morphism $Spec(S) \longrightarrow Spec(R)$ the sequence (4) is exact. Hence, for any algebra A the sequence of $\mathrm{Hom}'s$ from A to the sequence (4) in the category of commutative algebras is exact too. Thus, $(f_\alpha)_{\alpha\in I}$ is an universal effectively epimorphic family. □

As above, for any presheaf $X \in \mathcal{C}^\frown$ we denote by $\mathrm{Qcoh}(X)$ the category of quasi-coherent modules on X and denote by $\mathrm{Mod}(X, can_X)$ the category of sheaves of $\mathcal{O}_X$-modules for the canonical topology on $\mathcal{C}/X$.

Corollary 3.2. *Let X be a presheaf on the category $\mathcal{C} = Aff/k$. Any quasi-coherent module on X is a sheaf in the canonical topology on the category $\mathcal{C}/X$. Therefore, the category $\mathrm{Qcoh}(X)$ is a full subcategory of $\mathrm{Mod}(X, can_X)$.*

This corollary follows from the previous proposition and Proposition 1.5.

Denote by φ_X the inclusion $\mathrm{Qcoh}(X) \hookrightarrow \mathrm{Mod}(X, can_X)$.

3.4. The category of all schemes Sch/k is embedded as a full subcategory into $\mathcal{C}^\frown = (Aff/k)^\frown$ and for any scheme $X \in Sch/k$ quasi-coherent modules in our definition one-to-one correspond to the quasi-coherent sheaves in the usual definition. Actually, to any quasi-cohernet sheaf $\mathcal{F}$ on the scheme X one can construct a quasi-coherent module F on $\mathcal{C}/X$ by the rule

$$F(Spec(R)) = \Gamma(Spec(R), m^*\mathcal{F})$$

for each $m : Spec(R) \longrightarrow X$. This gives an equivalence, a quasi-inverse for which one can define in the following way. For a quasi-coherent module F on $\mathcal{C}/X$ one determines a quasi-coherent sheaf $\mathcal{F}$ on the scheme X by the formula

$$\mathcal{F}(U) = \Gamma(U, m^*F) = \mathrm{Hom}(\mathcal{O}_U, m^*F)$$

for any inclusion $m : U \hookrightarrow X$ of Zariski open set U . Some consideration, which we leave, are required to show that $\mathcal{F}$ is a sheaf in Zariski topology.

If the scheme X is quasi-compact and quasi-separated then there is a coherator $Q_X : \mathrm{Mod}(X, can_X) \longrightarrow \mathrm{Qcoh}(X)$. For the category of sheaves of $\mathcal{O}_X$-modules on the small Zariski site it was proved in [2], II.3. In our case it is also fulfilled by the same reasons. Actually, under given condition on a scheme the category $\mathrm{Qcoh}(X)$ has a set of generators an all colimits. Since the inclusin functor φ_X preserves colimits, the adjoint functor theorem ensures the existence of the right adjoint functor Q_X .

Proposition 3.3. *Let X be a presheaf on the category $\mathcal{C} = Aff/k$ and $m : X \longrightarrow \underline{c}X$ be a morphism of associated sheaf in the canonical topology. Then the inverse image functor $m^* : \mathrm{Qcoh}(\underline{c}X) \longrightarrow \mathrm{Qcoh}(X)$ is fully faithful.*

It immediately follows from Propositions 2.3 and Corollary 3.2.

Proposition 3.4. *Let* $T \overset{i}{\hookrightarrow} Spec(R)$ *be a sieve on an object* $Spec(R) \in \mathcal{C}$ *. The sieve* T *is covering in the canonical topology if and only if the functor* $i^* : \mathrm{Qcoh}(Spec(R)) \longrightarrow \mathrm{Qcoh}(T)$ *is fully faithful.*

Proof. We can assume that the sieve T is generated by a family $(f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha\in I}$. It was explained in 2.7 that the functor i^* has a right adjoint functor i_* that takes a quasi-coherent module $\mathcal{F}$ on T to $\widetilde{M}$ on $Spec(R)$, where the module M is an equalizer of

$$\prod_{\alpha\in I} \mathcal{F}(Spec(R_\alpha)) \;\rightrightarrows\; \prod_{\alpha,\beta\in I} \mathcal{F}(Spec(R_\alpha \otimes R_\beta)).$$

Assume that i^* is fully faithful. Then $i_* i^*(\widetilde{M}) \cong \widetilde{M}$ for any R-module M . Hence the sequence (3) is exact, and the sieve T is covering in the canonical topology.

Inverse statement direct follows from Proposition 3.3. □

3.5. On the category Aff/k there is a very important subcanonical topology. It is a well-known flat topology. It is generated by universal effectively epimorphic families $(f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha\in I}$ for which all morphisms f_α are flat, i.e. R_α are flat R-algebras for all α . The flat topology on the category Aff/k will be denoted as ft .

3.6. Now we introduce another subcanonical topology which will be called the effective descent topology (edt).

Recall briefly the descent theory for modules over commutative rings. Let $f : R \longrightarrow S$ be a homomorphism of commutative rings. Let M be an S-module. Consider two $S \otimes_R S$-modules $M \otimes_R S$ and $S \otimes_R M$. Any $S \otimes_R S$-homomorphism $\phi : M \otimes_R S \to S \otimes_R M$ defines three $S \otimes_R S \otimes_R S$-homomorphisms

$$\begin{array}{rlcl} \phi_1 : & S \otimes_R M \otimes_R S & \longrightarrow & S \otimes_R S \otimes_R M \\ \phi_2 : & M \otimes_R S \otimes_R S & \longrightarrow & S \otimes_R S \otimes_R M \\ \phi_3 : & M \otimes_R S \otimes_R S & \longrightarrow & S \otimes_R M \otimes_R S \end{array}$$

by tensoring ϕ with 1_S in the first, second and third positions respectively. Moreover, under the composition with the multiplication $\mu : S \otimes_R M \to M$, ϕ induces a homomorphism $\bar{\phi} : M \to M$ of S-modules:

$$\bar{\phi} : M \longrightarrow M \otimes_R S \xrightarrow{\phi} S \otimes_R M \xrightarrow{\mu} M.$$

By a descent datum on an S-module M we mean an $S \otimes_R S$-homomorphism $\phi : M \otimes_R S \to S \otimes_R M$ such that $\phi_2 = \phi_1 \phi_3$ and $\bar{\phi}$ is the identity on M .

If (M, ϕ) and (M', ϕ') are two descent data, then a morphism $g : (M, \phi) \to (M', \phi')$ between them is an S-homomorphism $g : M \to M'$ for which $(1_S \otimes g)\phi = \phi'(g \otimes 1_S)$.

This way for any homomorphism of commutative rings $f : R \longrightarrow S$ one obtains the category of descent data. Denote it as $Dd(f)$. There is a functor $d_f : \mathrm{Mod}(R) \longrightarrow Dd(f)$ that takes an R-module N to the pair $(N \otimes_R S, \phi_N)$, where ϕ_N defined by the rule $\phi_N(n \otimes s_1 \otimes s_2) = s_1 \otimes n \otimes s_2$. (An R-morphism $h : N \longrightarrow N'$ goes to $h \otimes 1_S$).

A homomorphism $f : R \longrightarrow S$ of commutative rings is called an effective descent morphism if the functor $d_f : \mathrm{Mod}(R) \longrightarrow Dd(f)$ is an equivalence.

Denote by $(T_f \stackrel{i}{\hookrightarrow} Spec(R)) \in \mathcal{C}^\frown$ the sieve on $Spec(R)$ consisting of all maps $Spec(B) \longrightarrow Spec(R)$ which go through the map $f^{op} : Spec(S) \longrightarrow Spec(R)$. As any sieve T_f can be considered as a subpresheaf of sets of $Spec(R)$. It is not hard to see that the category $Dd(f)$ is nothing else than the category $\mathrm{Qcoh}(T_f)$ of quasi-coherent modules on T_f and the functor d_f coincides with the inverse image functor i^* .

Now we say that $F = (f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha \in I}$ is an effective descent family if the functor $i^* : \mathrm{Qcoh}(Spec(R)) \longrightarrow \mathrm{Qcoh}(T_F)$ is an equivalence, where $T_F \stackrel{i}{\hookrightarrow} Spec(R) \in \mathcal{C}^\frown$ is the sieve generated by the family F .

Proposition 3.5. *Effective descent families satisfy the conditions P1)–3) of Definition 1.3, i.e. such families form a pretopology on* $\mathcal{C} = Aff/k$.

Proof. By Corollary 3.4 each effective descent family is an universal effectively epimorphic family. Let $F = (Spec(R_\alpha) \to Spec(R))_{\alpha \in I}$ be some effective descent family and let $Spec(S) \longrightarrow Spec(R)$ be a morphism. We have to show that the family $\bar{F} = (Spec(R_\alpha \otimes_R S) \to Spec(S))_{\alpha \in I}$ is also effective descent family. Take a quasi-coherent module $\bar{\mathcal{F}}$ on the sieve $T_{\bar{F}}$. Define a quasi-coherent module $\mathcal{F}$ on T_F by the rule $\mathcal{F}(\cdot) := \bar{\mathcal{F}}(\cdot \times_{Spec(R)} Spec(S))$ (this is the direct image of $\bar{\mathcal{F}}$ with respect to the canonical map from $T_{\bar{F}}$ to T_F). Since F is an effective descent family there is an R-module M such that $\mathcal{F} = i_F^* \widetilde{M}$ and the module M is an equalizer of the pair

$$\prod_{\alpha \in I} \bar{\mathcal{F}}(Spec(S \otimes_R R_\alpha)) \rightrightarrows \prod_{\alpha,\beta \in I} \bar{\mathcal{F}}(Spec(S \otimes_R R_\alpha \otimes_R R_\beta)).$$

This implies that the R-module M has a structure of S-module and $\bar{\mathcal{F}} = i_{\bar{F}}^* \widetilde{M}$. Therefore, $\bar{F}$ is an effective descent family.

Let $F = (Spec(R_\alpha) \to Spec(R))_{\alpha \in I}$ be an effective descent family. Futhermore, fix effective descent families $G_\alpha = (Spec(R_{\beta_\alpha}) \to Spec(R_\alpha))_{\beta_\alpha \in J_\alpha}$ for each $\alpha \in I$. Let G be some family $(Spec(R_\gamma) \to Spec(R))_{\gamma \in \coprod_{\alpha \in I} J_\alpha}$. Consider the covering sieves $i_F : T_F \hookrightarrow Spec(R)$, $i_{G_\alpha} : T_{G_\alpha} \hookrightarrow Spec(R_\alpha)$ and the sieve $i_G : T_G \hookrightarrow Spec(R)$ with a map $p : T_G \hookrightarrow T_F$.

Denote by $f_\alpha : Spec(R_\alpha) \longrightarrow T_F$ and $g_\alpha : T_{G_\alpha} \to T_G$ the canonical morphisms in the category $\mathcal{C}^\frown$. Consider the following diagram of functors

$$\begin{array}{ccc} \mathrm{Mod}(T_{G_\alpha}, can) & \xleftarrow{j^*_{g_\alpha}} & \mathrm{Mod}(T_G, can) \\ \wr \Big\uparrow j^*_{i_{G_\alpha}} & & \wr \Big\uparrow j^*_p \\ \mathrm{Mod}(Spec(R_\alpha), can) & \xleftarrow{j^*_{f_\alpha}} & \mathrm{Mod}(T_F, can). \end{array}$$

The both vertical arrows are equivalence because all effective descent families are covering in the canonical topology.

Let $\mathcal{F}$ be a quasi-coherent module on T_G. By Corollary 3.2 it is a sheaf of $\mathcal{O}_{T_G}$-modules in the canonical topology. Consider the sheaves of modules $\mathcal{F}_\alpha := j^*_{f_\alpha} j_{p*}(\mathcal{F})$. Since there is an isomorphism $j^*_{i_{G_\alpha}}(\mathcal{F}_\alpha) \cong j^*_{g_\alpha}(\mathcal{F})$ for each α and G_α is effective descent family, we get that the sheaves of modules $\mathcal{F}_\alpha := j^*_{f_\alpha} j_{p*}(\mathcal{F})$ are quasi-coherent. In respect that the sieve T_F is generated by the family $(f_\alpha)_{\alpha\in I}$ we conclude that the sheaf of modules $j_{p*}(\mathcal{F})$ is also quasi-coherent. This implies that any quasi-coherent module $\mathcal{F}$ on T_G is the pull back of the quasi-coherent module $j_{p*}(\mathcal{F})$. Hence, the functor $p^* : \mathrm{Qcoh}(T_F) \longrightarrow \mathrm{Qcoh}(T_G)$ is not only fully faithful but is an equivalence as well. Therefore, the functor $i^*_G : \mathrm{Qcoh}(Spec(R)) \longrightarrow \mathrm{Qcoh}(T_G)$ is also an equivalence and the family G is an effective descent family. □

3.7. The effective descent topology on the category $\mathcal{C} = Aff/k$ (edt) is a topology for which covering sieves are all sieves generated by the effective descent families. It follows from Proposition 3.4 that edt is subcanonical. Moreover it can be shown that the effective descent topology is finer than the flat topology, i.e. any covering in the flat topology is a covering in the effective descent topology. It is proved in many papers that a faithfully flat homomorphism $f : R \longrightarrow S$ of commutative ring is an effective descent morphism (see [3], [6], [7]). These proofs can be extended onto flat families as well.

Denote by $\underline{e} : \mathcal{C}^\frown \longrightarrow Shv(\mathcal{C}, edt)$ the associated sheaf functor for the effective descent topology.

Theorem 3.6. *Let* $X \in \mathcal{C}^\frown$ *be a presheaf and* $m : X \longrightarrow \underline{e}X$ *be the associated sheaf morphism in the effective descent topology. Then the inverse image functor* $m^* : \mathrm{Qcoh}(\underline{e}X) \longrightarrow \mathrm{Qcoh}(X)$ *is an equivalence.*

Proof. The functor $i\underline{e}$ is isomorphic to the composition $L \circ L$, where the functor L is defined by formula (1) in 1.4, i.e. the morphism m is the composition

$$X \xrightarrow{l_X} LX \xrightarrow{l_{LX}} \underline{e}X.$$

Hence it is sufficient to prove that the functor $l^*_X : \mathrm{Qcoh}(LX) \longrightarrow \mathrm{Qcoh}(X)$ is an equivalence for all $X \in \mathcal{C}^\frown$.

By Corollary 3.2, the category $\mathrm{Qcoh}(X)$ is a full subcategory of $\mathrm{Mod}(X, can_X)$. The functor l^*_X is induced by the functor $j^*_{l_X} : \mathrm{Mod}(LX, can_{LX}) \longrightarrow \mathrm{Mod}(X, can_X)$ which is an equivalence. Hence, to prove the theorem we must show that for any quasi-coherent module $\mathcal{F}$ on X the sheaf of $\mathcal{O}_{LX}$-modules $\mathcal{F}' = j_{l_X*}\mathcal{F}$ is quasi-coherent. By 2.4 we need to check that for any morphism $s : Spec(R) \longrightarrow LX$ the sheaf $j^*_s\mathcal{F}'$ is quasi-coherent. By definition of the functor L we have

$$LX(R) = \varinjlim_{T\in J(R)} \mathrm{Hom}_{\mathcal{C}^\frown}(T, X).$$

Therefore, there is a covering sieve $i : T \hookrightarrow Spec(R)$ such that the following diagram commutes

$$\begin{array}{ccc} T & \xrightarrow{s'} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle l_X} \\ Spec(R) & \xrightarrow{s} & LX. \end{array}$$

Since $j^*_{l_X}$ and $j_{l_X *}$ are quasi-inverse, by Proposition 2.1 there is an isomorphism $j^*_i j^*_s \mathcal{F}' \cong j^*_{s'}\mathcal{F}$ of quasi-coherent modules on T . As T is a covering sieve the functor j^*_i induces an equivalence $i^* : \mathrm{Qcoh}(Spec(R)) \xrightarrow{\sim} \mathrm{Qcoh}(T)$. Therefore, the sheaf $j^*_s\mathcal{F}'$ is quasi-coherent on $Spec(R)$ for all $s : Spec(R) \longrightarrow LX$. Thus, $\mathcal{F}' = j_{l_X *}\mathcal{F}$ is a quasi-coherent module on LX . □

4. Sheaves in noncommutative geometry.

4.1. Fix a base commutative ring k . Denote as Alg/k the category of algebras over base ring k . If $f_i : R \longrightarrow S_i,\ i = 1, 2$ be two homomorphisms of algebras then the coproduct $S_1 \coprod_R S_2$ always exists. This is the free product of R-algebras which will be denotes as $S_1 \star_R S_2$.

As $NAff/k$ denote the category that is opposite to Alg/k . We will call it by the category of noncommutative affine schemes. The object of $NAff/k$, which corresponds to an algebra R , will be denoted as $Sp(R)$. The category of noncommutative affine schemes has all fiber products and the product $Sp(S_1) \times_{Sp(R)} Sp(S_2)$ is isomorphic to $Sp(S_1 \star_R S_2)$.

4.2. Let $N : Aff/k \longrightarrow NAff/k$ be the natural inclusion functor that takes $Spec(R)$ to $Sp(R)$. The functor N is fully faithful and has a right adjoint functor $Com : NAff/k \longrightarrow Aff/k$. The functor Com sends an object $Sp(R)$ to $Spec(R_c)$. Here R_c is the maximal commutative factor-algebra of R , i.e. $R_c = R/I$, where I is the ideal generated by all relations of the form $< ab - ba >$.

There is a functor $\widehat{N}_! : (Aff/k)^{\widehat{}} \longrightarrow N(Aff/k)^{\widehat{}}$ that is an extension of N onto the category of presheaves (see 1.2). By 1.2 the functor $\widehat{N}_!$ has the following description

$$\widehat{N}_!(X)(Sp(S)) = \varinjlim_{Spec(A)\to X} \mathrm{Hom}(Sp(S), Sp(A)).$$

The functor $\widehat{N}_!$ is also fully faithful by 1.2 and it has a right adjoint functor $\widehat{N}^*$, restriction of which on $NAff/k$ coincides with the functor Com .

We mean the functor $\widehat{N}_!$ as a "course" embedding of the commutative geometry into the noncommutative geometry. To get a "fine" embedding we need to consider presheaves of groupoids.

4.3. For each object $Sp(S) \in NAff/k$ we define a category I_S . The objects of I_S are couples $(Spec(A), m)$ with $Spec(A) \in Aff/k$ and $m : Sp(S) \longrightarrow Sp(A)$. Morphisms from $(Spec(A), m)$ to $(Spec(A'), m')$ are morphisms $\xi : Spec(A) \to Spec(A')$ such that $m' = \xi m$.

Any morphism $f : Sp(S') \longrightarrow Sp(S)$ defines a functor $I_f : I_S \longrightarrow I_{S'}$. Moreover, there is a functor $pr_S : I_S \longrightarrow Aff/k$ that takes an object $(Spec(A), m)$ to $Spec(A)$. It is note hard to check that the functor $\widehat{N}_!$ can be given by the following formula

$$\widehat{N}_!(X)(Sp(S)) = \varinjlim_{I_S} X(pr_S(\cdot)). \tag{5}$$

4.4. Consider the category I_S/X of objects over X . Its objects are all triples of form $(Spec(A), m, a)$ with $m : Sp(S) \to Sp(A)$ and $a : Spec(A) \to X$. Its morphisms are all morphisms in I_S , which are compatible with maps to X . Denote by $Gr(I_S/X)$ the groupoid associated to the category I_S (i.e. we invert all arrows). Every morphism $g : Sp(S') \longrightarrow Sp(S)$ gives a functor $Gr(I_S/X) \longrightarrow Gr(I_{S'}/X)$. On the other hand, each $f : X' \longrightarrow X$ in $(Aff/k)^{\widehat{}}$ induces a functor $Gr(I_S/X') \longrightarrow Gr(I_S/X)$.

For an object $X \in (Aff/k)^{\widehat{}}$ let us define a presheaf of groupoids $\overline{N}X \in N(Aff/k)^{\widehat{}}_{gr}$ by the rule

$$\overline{N}X(Sp(S)) := Gr(I_S/X). \tag{6}$$

This correspondence gives a functor $\overline{N}$ from $(Aff/k)^{\widehat{}}$ to the category of presheaves of groupoids $N(Aff/k)^{\widehat{}}_{gr}$. It immediately follows from formula (5) that $\pi_0(\overline{N}X)$ is isomorphic to $\widetilde{N}_!X$. Hence, there is the commutative diagram of functors

$$\begin{array}{ccc}
(Aff/k)^{\widehat{}} & \xrightarrow{\overline{N}} & (NAff/k)^{\widehat{}}_{gr} \\
 & \underset{\widehat{N}_!}{\searrow} & \downarrow{\scriptstyle \pi_0} \\
 & & (NAff/k)^{\widehat{}}
\end{array}$$

Proposition 4.1. *The functor* $\overline{N} : (Aff/k)^{\widehat{}} \longrightarrow N(Aff/k)^{\widehat{}}_{gr}$ *has the following properties:*

1) *The presheaf of groupoids* $\overline{N}(Spec(A))$ *is isomorphic to the sheaf of sets* $N(Spec(A)) = Sp(A)$ *, i.e. the restriction of the functor* $\overline{N}$ *to the category* Aff/k *is isomorphic to* N *;*
2) *For any object* $Spec(A) \in Aff/k$ *the groupoid* $\overline{N}X(Sp(A))$ *is equivalent to the set* $X(Spec(A))$ *;*
3) *The functor* $\overline{N}$ *is fully faithful.*

Proof. The groupoid associated to a category with final or initial object is equivalent to the one-point set. This implies that $Gr(I_S/Spec(A))$ is equivalent to the set $\mathrm{Hom}_{NAff}(Sp(S), Sp(A))$ for any algebra S and commutative algebra A . Consequently, $\overline{N}(Spec(A)) \cong Sp(A)$ and property 1) holds. By the same reason, the groupoid $Gr(I_A/X)$ is equivalent to the set $X(Spec(A)$ for any commutative algebra A . This proves the property 2).

3) Since the functor $\widehat{N}_! = \pi_0 \overline{N}$ is fully faithful by 1.2 the functor $\overline{N}$ is faithful too. We know that the composition $\widehat{N}^* \pi_0 \overline{N}$ is isomorphic to the identity functor on $(Aff/k)^{\widehat{}}$.

To prove the fullness one has to check that any morphism $f : \overline{N}Y \longrightarrow \overline{N}X$ belongs to the image of the functor $\overline{N}$. The morphism f is defined by given the morphisms $f_S : Gr(I_S/Y) \longrightarrow Gr(I_S/X)$ for all $S \in Alg/k$ that are compatible together. Let $t = (Spec(A), m, a)$ be an element of $Gr(I_S/Y)$, where $m : Sp(S) \longrightarrow Sp(A)$ and $a \in Y(Spec(A))$. The morphism f_S takes this element to $t' = (Spec(A), m, a') \in Gr(I_S/X)$, where $a' \in X(Spec(A)$. Since f_S and f_A are compatible one obtains that $a' = f_A(a)$. Hence, knowing the morphism f_A we can find the element a' . Thus, any morphism f_S is uniquely defined by the set of morphisms f_A with commutative A . This implies that the morphism f coincides with the morphism $\overline{N}\widehat{N}^*\pi_0(f)$ and, consequently, it belongs to the image of $\overline{N}$. Hence the functor $\overline{N}$ is fully faithful. □

4.5. Let X be an object of $(Aff/k)^\frown$. One can attach to it two categories Aff/X and $NAff/\overline{N}X$ (the latter is defined in 2.10). There is a functor from Aff/X to $NAff/\overline{N}X$ that sends an element $a \in X(Spec(A))$ to the element $(Spec(A), id, a) \in Gr(I_{Spec(A)}/X)$ (see 4.4). Denote this functor as $\overline{n}_X$. It induces the inverse image functor $\overline{n}_X^* : \mathrm{Qcoh}_r(\overline{N}X) \longrightarrow \mathrm{Qcoh}(X)$.

The correspondence to a presheaf of sets $X \in (Aff/k)^\frown$ the presheaf of groupoids $\overline{N}X \in N(Aff/k)^\frown_{gr}$ is considered by us as a "fine" embedding of the commutative geometry into the noncommutative one, because the following condition holds.

Theorem 4.2. *Let X be an object from $(Aff/k)^\frown$. The functor $\overline{n}_X^* : \mathrm{Qcoh}_r(\overline{N}X) \longrightarrow \mathrm{Qcoh}(X)$, induced by the natural functor $\overline{n}_X : Aff/X \longrightarrow NAff/\overline{N}X$, is an equivalence.*

Proof. We construct a quasi-inverse functor for $\overline{n}_X$. For any object $X \in Aff/k$ define a presheaf of categories CX on $NAff/k$ (i.e. contravariant functor from $NAff/k$ to the category of all small categories) by the rule:

$$CX(Sp(S)) := I_S/X$$

Consider the category $NAff/CX$ defined in 2.10. Besides canonical functor from Aff/X to $NAff/CX$ there is a functor κ_X from $NAff/CX$ to Aff/X that sends an element $t = (Spec(A), m, a) \in I_S/X$ to the element $a \in X(Spec(A))$, where $m : Sp(S) \longrightarrow Spec(A)$.

Let $\mathcal{F}$ be a quasi-coherent module on X . We can define a presheaf $\mathcal{G}$ on the category $NAff/CX$ by the rule

$$\mathcal{G}(t) = \mathcal{F}(a) \otimes_{(A,m)} S$$

More formally the functor κ_X induces the functor $\hat{\kappa}_X^* : (Aff/X)^\frown \longrightarrow (NAff/CX)^\frown$ (see 1.2). Then the presheaf $\mathcal{G}$ can be defined as $\hat{\kappa}_X^*(\mathcal{F}) \otimes_{\hat{\kappa}_X^*(\mathcal{O}_X)} \mathcal{O}_{CX}$.

Further, a morphism $\alpha : t \to t'$ in I_S/X is, by definition, a morphism $\alpha : Spec(A) \longrightarrow Spec(A')$ such that $\alpha m = m'$ and $a'\alpha = a$. Any such morphism gives an isomorphism between $\mathcal{G}(t)$ and $\mathcal{G}(t')$. Indeed

$$\mathcal{G}(t) = \mathcal{F}(a) \otimes_{(A,m)} S \cong \mathcal{F}(a') \otimes_{(A',\alpha)} A \otimes_{(A,m)} S \cong \mathcal{F}(a') \otimes_{(A',m')} S = \mathcal{G}(t').$$

Therefore, this presheaf $\mathcal{G}$ is induced by some presheaf $\mathcal{H}$ on $\overline{N}X$ with respect to the canonical morphism $NAff/CX \longrightarrow NAff/\overline{N}X$. Moreover, it is clear that $\mathcal{H}$ is a quasi-coherent module. Analogously one defines this map on morphisms between quasi-coherent modules. Finally, we get a functor $\overline{n}_{*X}$ from $\mathrm{Qcoh}(X)$ to $\mathrm{Qcoh}_r(\overline{N}X)$ which is quasi-inverse of $\overline{n}^*_X$. □

4.6. Further, we will write $\mathcal{C}$ instead of $NAff/k$. Denote as $\Phi = (\mathcal{C}, can)$ the site ringed by the sheaf $\mathcal{O}$. Let X be an object of $\mathcal{C}^\frown$ and $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$ is the canonical functor. As above, denote by can_X the canonical topology on $\mathcal{C}/X$. It is induced by the canonical topology on $\mathcal{C}$ by 1.5. Consider the ringed site $\Phi_X = (\mathcal{C}/X, can_X)$ with the ring sheaf $\mathcal{O}_X$. In the commutative case we know that any quasi-coherent module on X is a sheaf for the canonical topology on $\mathcal{C}/X$ (see 3.2). The same statement in the noncommutative case is also true, but the proof is not so direct.

Theorem 4.3. *Let X be a presheaf of sets on the category $\mathcal{C} = NAff/k$. Then each quasi-coherent module $\mathcal{M}$ on X is a sheaf for the canonical topology on $NAff/X$.*

Proof. Let A be a k -algebra and let $(f_\alpha : A \longrightarrow B_\alpha),\ \alpha \in I$ be a family of morphisms of algebras such that the opposite family of maps f_α^{op} in opposite category $NAff/k$ is covering in the canonical topology. To prove the proposition we have to show that for any (right) A -module M a homomorphism ϵ_M in the diagram

$$M \xrightarrow{\epsilon_M} \prod_{\alpha\in I} M \otimes_A B_\alpha \rightrightarrows \prod_{\alpha,\beta\in I} M \otimes_A (B_\alpha \star_A B_\beta)$$

is an equalizer.

Denote as N an A-bimodule $A \otimes_{\mathbb{Z}} M$. Define a ring S as the space $A \oplus N$ with multiplication low $(a_1, n_1)(a_2, n_2) = (a_1a_2, a_1n_2 + n_1a_2)$. By Lemma 4.4 the canonical map $\phi : M \longrightarrow N$ is an embedding. Denote by i the embedding N to S and by ψ_α the canonical maps $M \otimes_A B_\alpha \longrightarrow S \star_A B_\alpha$ for any $\alpha \in I$. Let ψ be the product of ψ_α by all $\alpha \in I$.

Consider the following commutative diagram

$$(7)\qquad \begin{array}{ccccc} M & \xrightarrow{\epsilon_M} & \prod_{\alpha\in I} M \otimes_A B_\alpha & \rightrightarrows & \prod_{\alpha,\beta\in I} M \otimes_A (B_\alpha \star_A B_\beta) \\ {\scriptstyle i\phi}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow \\ S & \xrightarrow{e_S} & \prod_{\alpha\in I} S \star_A B_\alpha & \rightrightarrows & \prod_{\alpha,\beta\in I} (S \star_A B_\alpha) \star_S (S \star_A B_\beta) \end{array}$$

Since the family (f_α^{op}) is covering, the map e_S is an equalizer. Moreover, the maps ϕ and e_S are embeddings, hence the map ϵ_M is an embedding too. This implies that any quasi-coherent module is a separated presheaf.

Assume now that the following conditions hold

1). the maps ψ_α are embeddings for any α ,

2). intersection of $e_S(S)$ and $\psi(\prod_\alpha M \otimes_A B_\alpha)$ coincides with $e_S i\phi(M)$.

In this case we immediately get that ϵ_M is an equalizer.

Thus, to finish the proof we must to check that the properties 1) and 2) are fulfilled.

1) Denote an A -bimodule B_α/A as $\overline{B}_\alpha$. For any $\alpha \in I$ consider a graded ring R_α defined as

$$R_\alpha = B_\alpha \oplus (B_\alpha \otimes_A N \otimes_A B_\alpha) \oplus (B_\alpha \otimes_A N \otimes_A \overline{B}_\alpha \otimes_A N \otimes_A B_\alpha) \oplus \cdots$$

with composition low

$$(b_0 \otimes n_1 \otimes \bar{b}_1 \otimes \cdots \otimes \bar{b}_{i-1} \otimes n_i \otimes b_i)(b'_i \otimes n_{i+1} \otimes \bar{b}_{i+1} \otimes \cdots \otimes \bar{b}_{i+j-1} \otimes n_{i+j} \otimes b_{i+j}) = \\ (b_0 \otimes n_1 \otimes \bar{b}_1 \otimes \cdots \otimes \bar{b}_{i-1} \otimes n_i \otimes \overline{b_i b'_i} \otimes n_{i+1} \otimes \bar{b}_{i+1} \otimes \cdots \otimes \bar{b}_{i+j-1} \otimes n_{i+j} \otimes b_{i+j}).$$

There are the ring homomorphisms $B_\alpha \longrightarrow R_\alpha$ and $S \longrightarrow R_\alpha$. Therefore, there is the canonical ring homomorphism $S \star_A B_\alpha \longrightarrow R_\alpha$. By Lemma 4.4 the map $M \otimes_A B_\alpha \longrightarrow B_\alpha \otimes_A N \otimes_A B_\alpha \cong B_\alpha \otimes_{\mathbb{Z}} M \otimes_A B_\alpha$ is an embedding. Hence, the map $M \otimes_A B_\alpha \longrightarrow R_\alpha$ is an embedding too. It is the composition of ψ_α with the canonical map $S \star_A B_\alpha \longrightarrow R_\alpha$. Thus, we obtain that ψ_α is an embedding for any $\alpha \in I$.

2) The map i is the embedding of N into S . To check the second condition it is sufficient to show that intersection of $e_S i(N)$ and $\psi(\prod_{\alpha \in I} M \otimes_A B_\alpha)$ coincides with $e_S i \phi(M)$. Note that the maps $e_S i$ and ψ go through the canonical map $\prod_{\alpha \in I} B_\alpha \otimes_A N \otimes_A B_\alpha \longrightarrow \prod_{\alpha \in I} S \star_A B_\alpha$ On the other hand, there is a composition map

$$\prod_{\alpha \in I} S \star_A B_\alpha \longrightarrow \prod_{\alpha \in I} R_\alpha \longrightarrow \prod_{\alpha \in I} B_\alpha \otimes_A N \otimes_A B_\alpha$$

which splits previous one. Hence, $\prod_{\alpha \in I} B_\alpha \otimes_A N \otimes_A B_\alpha$ is a direct summand of $\prod_{\alpha \in I} S \star_A B_\alpha$.

Denote by ρ and θ the maps $N \longrightarrow \prod_{\alpha \in I} B_\alpha \otimes_A N \otimes_A B_\alpha$ and $\prod_{\alpha \in I} M \otimes_A B_\alpha \longrightarrow \prod_{\alpha \in I} B_\alpha \otimes_A N \otimes_A B_\alpha$ respectively. There is the commutative diagram

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M & \xrightarrow{\epsilon_M} & \prod\limits_{\alpha \in I} M \otimes_A B_\alpha \\ & & \phi\downarrow & & \downarrow\theta \\ 0 & \longrightarrow & N = A \otimes_{\mathbb{Z}} M & \xrightarrow{\rho} & \prod\limits_{\alpha \in I} B_\alpha \otimes_{\mathbb{Z}} M \otimes_A B_\alpha \end{array}$$

By Lemma 4.4 the embeddings ϕ and θ are split as maps of abelian groups. Moreover, it is easy to see that these splits are commute. Therefore, the intersection of $\rho(N)$ and $\theta(\prod_{\alpha \in I} M \otimes_A B_\alpha)$ coincides with $\rho\phi(M)$. This proves property 2) and the proposition as well. □

Lemma 4.4. *Let* R *be a ring and* M *be a right* R *-module. Then the canonical map* $M \longrightarrow R \otimes_{\mathbb{Z}} M$ *is an embedding of right* R *-modules and it is split as a morphism of abelian groups.*

Proof. It is clear that the composition of maps of abelian groups

$$M \longrightarrow R \otimes_{\mathbb{Z}} M \xrightarrow{\sim} M \otimes_{\mathbb{Z}} R \longrightarrow M$$

is the identity. (Here the last map is the action of the ring R on the right module M .) Hence, $M \longrightarrow R \otimes_{\mathbb{Z}} M$ is an embedding of abelian groups and, consequently, it is an embedding of right R -modules. □

Corollary 4.5. *Let X be a presheaf of sets on the category $\mathcal{C} = NAff/k$. Then the category of quasi-coherent modules $\mathrm{Qcoh}_r(X)$ is a full subcategory of the category of sheaves for the canonical topology $\mathrm{Mod}_r(X, can_X)$. If $m : X \longrightarrow \underline{c}X$ is the morphism of associated sheaf in the canonical topology, then the inverse image functor $m^* : \mathrm{Qcoh}_r(\underline{c}X) \longrightarrow \mathrm{Qcoh}_r(X)$ is fully faithful.*

This corollary follows from Theorem 4.3 and Proposition 2.1.

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